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Infinite families of Artin–Schreier function fields with any prescribed class group rank

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Abstract. We study the Galois module structure of the class groups of the Artin–Schreier extensions K over k of extension degree p, where $k := \mathbb{F}_q(T)$ is the rational function field and p is a prime number. The structure of the p-part $Cl_K(p)$ of the ideal class group of K as a finite G-module is determined by the invariant λ_n, where $G := \mathrm{Gal}(K/k) = \langle \sigma \rangle$ is the Galois group of K over k, and $\lambda_n = \dim_{\mathbb{F}_p}(Cl_K(p)^{(\sigma-1)^{n-1}}/Cl_K(p)^{(\sigma-1)^n})$. We find infinite families of the Artin–Schreier extensions over k whose ideal class groups have guaranteed prescribed λ_n-rank for $1 \le n \le 3$. We find an algorithm for computing λ_3-rank of $Cl_K(p)$. Using this algorithm, for a given integer $t \ge 2$, we get infinite families of the Artin–Schreier extensions over k whose λ_1-rank is t, λ_2-rank is $t-1$, and λ_3-rank is $t-2$. In particular, in the case where $p = 2$, for a given positive integer $t \ge 2$, we obtain an infinite family of the Artin–Schreier quadratic extensions over k whose 2-class group rank (resp. 2^2-class group rank and 2^3-class group rank) is *exactly* t (resp. $t-1$ and $t-2$). Furthermore, we also obtain a similar result on the 2^n-ranks of the divisor class groups of the Artin–Schreier quadratic extensions over k.

1 Introduction

There have been active studies on the structure of the class groups of number fields and function fields; for instance, we refer to [1–5, 6, 8, 10, 11, 13–16, 19–25]. For studying the structure of class groups, the following methods have been used: *genus theory* [1, 3, 6], *Rédei matrix* [2, 15, 23], and *Conner and Hurrelbrink's exact hexagon* [5, 13].

The Galois module structure of the class groups of cyclic extensions over the rational function field $k := \mathbb{F}_q(T)$ has been studied in [2, 8, 14, 19], where $\mathbb{F}_q$ is a finite field of order q. We need to introduce the following definitions for description of the previous developments. Let K be a cyclic extension over k of extension degree prime p. We denote the *ideal class group* of K by Cl_K and that of *divisor class group* by J_K. Let $G := \mathrm{Gal}(K/k)$ be the Galois group of K over k. Then Cl_K and J_K are finite G-modules. Let σ be a generator of G and $\mathbb{Z}_p$ the ring of p-adic integer. The

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structures of $Cl_K(p)$ and $J_K(p)$ as finite modules over the discrete valuation ring $\mathbb{Z}_p[\sigma]/(1+\sigma+\cdots+\sigma^{p-1}) \simeq \mathbb{Z}_p[\zeta_p]$ are determined by the following ranks:

$$\lambda_n := \dim_{\mathbb{F}_p}(Cl_K(p)^{(\sigma-1)^{n-1}}/Cl_K(p)^{(\sigma-1)^n}) \quad \text{and}$$
$$\mu_n := \dim_{\mathbb{F}_p}(J_K(p)^{(\sigma-1)^{n-1}}/J_K(p)^{(\sigma-1)^n}),$$

where $Cl_K(p)$ (resp. $J_K(p)$) is the p-Sylow subgroup of Cl_K (resp. J_K) and ζ_p is a primitive pth root of unity.

We point out that in particular, when $p=2$, the rank λ_n of Cl_K is exactly equal to the 2^n-rank of Cl_K and the rank μ_n of J_K gives the 2^n-rank of J_K, where the 2^n-rank of Cl_K is defined as $\dim_{\mathbb{F}_2}(Cl_K^{2^{n-1}}/Cl_K^{2^n})$ and similarly for J_K. This is because σ acts -1 on Cl_K, which implies that the rank λ_n of the finite module Cl_K over $\mathbb{Z}[\zeta_2]=\mathbb{Z}$ is exactly the 2^n-rank of Cl_K, and similarly it also holds for J_K.

There are exactly two kinds of cyclic extensions of prime extension degree over the rational function field k: *Kummer extension* and *Artin–Schreier extension*. For Kummer extensions L over k, Anglés and Jaulent [1] (resp. Wittmann [19]) studied the λ_1-rank (resp. λ_2-rank) of the ideal class groups of L and the authors of this paper [22] studied the λ_3-rank of the ideal class groups of L. Furthermore, for Artin–Schreier extensions over k, there also have been some studies on the computation of λ_1 and λ_2 for their ideal class groups [2, 8]. However, there has been no result yet on finding infinite families of Artin–Schreier extensions over k whose ideal class groups have guaranteed prescribed λ_n-rank of the ideal class group of Artin–Schreier extension for $1\le n\le 3$. This is one of the motivations of our paper.

In this paper, we study the Galois module structure of the class groups of the Artin–Schreier extensions K over k of extension degree p, where $k:=\mathbb{F}_q(T)$ is the rational function field of characteristic p and p is a prime number. The structure of the p-part $Cl_K(p)$ of the ideal class group of K as a finite G-module is determined by the invariant λ_n, where $G:=\mathrm{Gal}(K/k)=\langle\sigma\rangle$. In detail, first of all, for a given positive integer t, we obtain infinite families of K over k whose λ_1-rank of Cl_K is t and λ_n-rank of Cl_K is zero for $n\ge 2$, depending on the ramification behavior of the infinite place ∞ of k (Theorems 3.2–3.4). We then find infinite families of the Artin–Schreier extensions over k whose ideal class groups have guaranteed prescribed λ_n-rank for n up to 3. We find an algorithm for computing λ_3-rank of $Cl_K(p)$. Using this algorithm, for a given integer $t\ge 2$, we get infinite families of the Artin–Schreier extensions over k whose λ_1-rank is t, λ_2-rank is $t-1$, and λ_3-rank is $t-2$ (Theorem 5.1). In particular, in the case where $p=2$, for a given positive integer $t\ge 2$, we obtain an infinite family of the Artin–Schreier quadratic extensions over k which have 2-class group rank *exactly* t, 2^2-class group rank $t-1$, and 2^3-class group rank $t-2$ (Corollary 5.3). Furthermore, we also obtain a similar result on the 2^n-ranks of the divisor class groups of the Artin–Schreier quadratic extensions over k for n up to 3 (Corollary 5.4). Finally, in Tables 1 and 2, we give some implementation results for explicit infinite families using Theorems 3.2–3.4 and 5.1. These implementation results are done by MAGMA.

We remark that as a main tool for computation of λ_3, we use an analogue of *Rédei matrix*. We emphasize that there is no number field analogue for the Artin–Schreier extensions over k, while there is a number field analogue for Kummer extensions over k.

2 Preliminaries

Let q be a power of a prime number p, and let $k := \mathbb{F}_q(T)$ be the *rational function field*. The prime divisor of k corresponding to $(1/T)$ is called the *infinite place* and denoted by ∞. Let K/k be a cyclic extension of degree p. Then K/k is an *Artin–Schreier extension*: that is, $K = k(\alpha)$, where $\alpha^p - \alpha = D$, $D \in k$, and that D cannot be written as $x^p - x$ for any $x \in k$. Conversely, for any $D \in k$ and D cannot be written as $x^p - x$ for any $x \in k$, $k(\alpha)/k$ is a cyclic extension of degree p, where $\alpha^p - \alpha = D$.

For $D, D' \in k$, let $K_1 := k(\alpha)$ and $K_2 := k(\beta)$ be two Artin–Schreier extensions over k with $\alpha^p - \alpha = D$ and $\beta^p - \beta = D'$, respectively. Two Artin–Schreier extensions K_1 and K_2 are equal if and only if they satisfy the following relations [8, p. 256]:

$$\begin{aligned} \alpha &\quad \rightarrow \quad x\alpha + B_0 = \beta, \\ D &\quad \rightarrow \quad xD + (B_0^p - B_0) = D', \\ x &\in \mathbb{F}_p^\times, \; B_0 \in k. \end{aligned}$$

Thus, D can be *normalized* to satisfy the following conditions:

$$\begin{aligned} D &= \sum_{i=1}^{m} \frac{Q_i}{P_i^{r_i}} + f(T), \\ &(P_i, Q_i) = 1, \; p \nmid r_i \text{ for } 1 \le i \le t, \\ &p \nmid \deg f(T) \;\text{ if } \deg f(T) \ge 1, \text{ and} \\ &f(T) = 0 \;\text{ if } f(T) \in \mathbb{F}_q \text{ with } \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(f) = 0, \end{aligned} \tag{2.1}$$

where P_i is a monic irreducible polynomial in $\mathbb{F}_q[T]$, $Q_i, f(T) \in \mathbb{F}_q[T]$, and $\deg Q_i < \deg P_i^{r_i}$ for $1 \le i \le t$; the last condition follows from noting that if $f(T) = c$ in $\mathbb{F}_q^\times$ with $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(c) = 0$, then there exists $b \in \mathbb{F}_q^\times$ such that $b^p - b = c$.

Throughout this paper, let $K := k(\alpha_{D_m})$ be the Artin–Schreier extension over k of extension degree p, where $x^p - x = D_m$ has no root in k, α_{D_m} is a root of $x^p - x = D_m$, and the normalized D_m satisfies (2.1). We note that all the finite places of k which are totally ramified in K are $P_1, \ldots, P_t$. In the following lemma, we state the ramification behavior of the infinite place ∞ of k in K.

Lemma 2.1 [8, p. 256] *Let $K = k(\alpha_{D_m})$ be the Artin–Schreier extension over k of extension degree p, where $\alpha_{D_m}^p - \alpha_{D_m} = D_m$ and D_m is defined in (2.1). Then we have the followings.*

(i) *The infinite place ∞ of k is totally ramified in K if and only if* $\deg f(T) \ge 1$.
(ii) *The infinite place ∞ of k is inert in K if and only if $f(T) = c \in \mathbb{F}_q^\times$, where $x^p - x - c$ is irreducible over $\mathbb{F}_q$.*
(iii) *The infinite place ∞ of k splits completely in K if and only if $f(T) = 0$.*

For descriptions of λ_1 and λ_2, we use the notion of *the Hasse symbol* which is first introduced in [7].

Definition 2.1 [8, p. 257] Let $K = k(\alpha_{D_m})$ be the Artin–Schreier extension over k of extension degree p, where $\alpha_{D_m}^p - \alpha_{D_m} = D_m$ for some $D_m \in k$. Let P be a finite place of k which is unramified in K, and let $\left(\frac{K/k}{P}\right)$ be the *Artin symbol* of P. Then $\left(\frac{K/k}{P}\right)\alpha_{D_m} =$

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$\alpha_{D_m} + \left\{\frac{D_m}{P}\right\}$, where $\left\{\frac{D_m}{P}\right\}$ is defined as follows:

$$\left\{\frac{D_m}{P}\right\} = \mathrm{Tr}_{(\mathcal{O}_K/P)/\mathbb{F}_p}(D_m \bmod P);$$

$\mathrm{Tr}_{(\mathcal{O}_K/P)/\mathbb{F}_p}$ denotes the *trace* function from $\mathcal{O}_K/P$ to $\mathbb{F}_p$ and $\mathcal{O}_K$ is the integral closure of K. We call $\left\{\frac{\cdot}{\cdot}\right\}$ *the Hasse symbol.*

Lemma 2.2 [8] *Let $K = k(\alpha_{D_m})$ be the Artin–Schreier extension over k of extension degree p, where $\alpha_{D_m}^p - \alpha_{D_m} = \sum_{i=1}^m \frac{Q_i}{P_i^{r_i}} + f(T)$, which is defined in (2.1). Then we have the followings.*

(i) $$\lambda_1 = \begin{cases} m & \text{if } \deg f(T) \geq 1 \text{ or} \\ & \quad f(T) = c \in \mathbb{F}_q^\times, \text{ where } x^p - x = c \in \mathbb{F}_q^\times \text{ is irreducible over } \mathbb{F}_q, \\ m-1 & \text{if } f(T) = 0. \end{cases}$$

(ii) *We have $\lambda_2 = \lambda_1 - \mathrm{rank}(R)$, where $R = [r_{ij}]$ is a matrix over $\mathbb{F}_p$ defined by*

$$r_{ij} = \begin{cases} \left\{\frac{Q_j/P_j^{r_j}}{P_i}\right\}, & \text{for } 1 \leq i \neq j \leq m, \\ -\left(\sum_{j=1, i\neq j}^m r_{ij} + \left\{\frac{f}{P_i}\right\}\right), & \text{for } 1 \leq i = j \leq m. \end{cases}$$

We call such matrix R as the Rédei matrix.

We recall that the *Hilbert class field H_K* of K is the maximal unramified abelian extension of K where the infinite places of k split completely in K. The *genus field $\mathcal{G}_K$ of K* is the maximal subextension $K \subseteq \mathcal{G}_K \subseteq H_K$ which is abelian over k. In Lemma 2.3, we state a description of the genus field of the Artin–Schreier extension.

Lemma 2.3 [8, Theorem 4.1] *Let $K = k(\alpha_{D_m})$ be the Artin–Schreier extension over k of extension degree p, where D_m is defined in (2.1) and α_{D_m} is a root of $x^p - x = D_m$. Let α_i (resp. β) be a root of $x^p - x = Q_i/P_i^{r_i}$ for $1 \leq i \leq m$ (resp. $x^p - x = f(T)$) in $\overline{k}$. Then the genus field $\mathcal{G}_K$ of K is $\mathcal{G}_K = k(\alpha_1, \ldots, \alpha_m, \beta)$.*

We now introduce explicit criteria for determining whether a place of k is totally ramified or not in the Artin–Schreier extension K.

Lemma 2.4 [18, Proposition 3.7.8] *Let $K = k(y)$ be the Artin–Schreier extension over k of extension degree p, where $y^p - y = u$ for some $u \in k$. For a place P of k, we define the integer m_P by*

$$m_P := \begin{cases} m, & \text{if there is an element } z \in k \text{ satisfying} \\ & v_P(u - (z^p - z)) = -m < 0 \text{ and } m \not\equiv 0 \pmod p, \\ -1, & \text{if } v_P(u - (z^p - z)) \geq 0 \text{ for some } z \in k. \end{cases}$$

Then we have the followings.

(i) *P is totally ramified in K/k if and only if $m_P > 0$.*
(ii) *P is unramified in K/k if and only if $m_P = -1$.*

Lemma 2.5 [17, Proposition 14.1] *Let K be a function field over the rational function field $k = \mathbb{F}_q(T)$, and let ∞ be the infinite place of k. Denote the ideal class group (resp.*

the divisor class group) of K by Cl_K (resp. J_K) and S be a set of places of K lying over ∞. Then

$$0 \to \mathcal{D}_K^0(S)/\mathcal{P}_K(S) \to J_K \to Cl_K \to \mathbb{Z}/d\mathbb{Z} \to 0$$

is an exact sequence, where $\mathcal{D}_K^0(S)$ is the divisor group with support only in S whose degree is zero, $\mathcal{P}_K(S)$ is a principal divisor with support only in S, and d is the greatest common divisor of the elements in $\{\deg P : P \in S\}$.

Using Lemma 2.5, we can easily obtain the following corollary, which gives relation between the ideal class group of K and the divisor class group of K, where K is the Artin–Schreier function field over k.

Lemma 2.6 *Let K be the Artin–Schreier extension over k with extension degree p, and let all the notations be the same as in Lemma 2.5. Then we have the following.*

(i) *If ∞ is totally ramified in K, then $\mathcal{D}_K^0(S)$ is trivial and $d = 1$; thus,*

$$0 \to J_K \to Cl_K \to 0$$

is exact.

(ii) *If ∞ is inert in K, then $\mathcal{D}_K^0(S)$ is trivial and $d = p$; therefore,*

$$0 \to J_K \to Cl_K \to \mathbb{Z}/p\mathbb{Z} \to 0$$

is an exact sequence.

(iii) *If ∞ splits completely in K, then $d = 1$; thus,*

$$0 \to \mathcal{D}_K^0(S)/\mathcal{P}_K(S) \to J_K \to Cl_K \to 0$$

is exact.

3 Infinite families of Artin–Schreier function fields with any prescribed class group λ-rank

In this section, for any positive integer t, we find infinite families of Artin–Schreier function fields K over k whose λ-rank of the ideal class group Cl_K of K is t and λ_n-rank of Cl_K is zero for $n \geq 2$, depending on the ramification behavior of the infinite place ∞ of k. Theorem 3.2 deals with the case where the infinite place ∞ of k is totally ramified in K and Theorem 3.3 (resp. Theorem 3.4) treats the case where the infinite place ∞ of k splits completely (resp. ∞ is inert) in K.

We first give the following lemma, which shows the property of the trace over finite fields. This lemma plays a key role in the proofs of Theorems 3.2–3.4.

Lemma 3.1 *Let h be a monic irreducible polynomial in $\mathbb{F}_q[T]$ and $\mathfrak{h} := q^{\deg h}$. Let g be a nonzero element in $\mathbb{F}_q[T]$, and let $\tilde{g} \in \mathbb{F}_{\mathfrak{h}}$ be $\phi \circ \pi(g)$, where*

$$g \in \mathbb{F}_q[T] \xrightarrow{\pi} \pi(g) \in \mathbb{F}_q[T]/\langle h \rangle \xrightarrow{\phi} \mathbb{F}_{\mathfrak{h}}.$$

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Then we have $\mathrm{Tr}_{\mathbb{F}_{\mathfrak{h}}/\mathbb{F}_q} \tilde{g} = 0$ *if and only if the following holds:*

(i) *If* $\deg g = 0$, *then* $q \mid \deg h$.
(ii) *If* $\deg g \geq 1$, *then* $g \equiv b(T)^q - b(T) \pmod{h}$ *for some* $b(T) \in \mathbb{F}_q[T]$.

Proof We note that $\mathbb{F}_{\mathfrak{h}} \simeq \mathbb{F}_q[T]/\langle h \rangle$ since h is an irreducible polynomial over $\mathbb{F}_q$.

First, assume that $\deg g = 0$: that is, g is an element of $\mathbb{F}_q^{\times}$, and so $g = \tilde{g}$. Then we have the following:

$$\mathrm{Tr}_{\mathbb{F}_{\mathfrak{h}}/\mathbb{F}_q} \tilde{g} = 0 \text{ if and only if } q \mid \deg h;$$

this is because $\mathrm{Tr}_{\mathbb{F}_{\mathfrak{h}}/\mathbb{F}_q} \tilde{g} = \tilde{g} \cdot \deg h$ in $\mathbb{F}_q$.

Now, we consider the case where $\deg g \geq 1$. Assume that $g \equiv b(T)^q - b(T) \pmod{h}$. Then we have

$$\tilde{g} = \phi \circ \pi(g) = \phi((b(T))^q - (b(T))) = \phi(b(T))^q - \phi(b(T)) = \tilde{b}^q - \tilde{b},$$

where $\tilde{b} := \phi(b(T)) \in \mathbb{F}_{\mathfrak{h}}$. Therefore, the result follows immediately by [12, Theorem 2.25]. Conversely, now assume that $\mathrm{Tr}_{\mathbb{F}_{\mathfrak{h}}/\mathbb{F}_q}(\tilde{g}) = 0$: that is, there exists some $\tilde{b} \in \mathbb{F}_{\mathfrak{h}}$ such that $\tilde{g} = \tilde{b}^q - \tilde{b}$. Let $b(T) := \phi^{-1}(\tilde{b})$; there exists such $b(T) \in \mathbb{F}_q[T]$ since ϕ is isomorphism. Thus, we get

$$g = \pi^{-1} \circ \phi^{-1}(\tilde{g}) = \pi^{-1} \circ \phi^{-1}(\tilde{b}^q - \tilde{b}) = \pi^{-1}((b(T))^q - (b(T)));$$

this implies that $g \equiv b(T)^q - b(T) \pmod{h}$. ∎

Theorem 3.2 *Let t be a positive integer. Let* $K = k(\alpha_{D_t})$ *be the Artin–Schreier extension over the rational function field* $k = \mathbb{F}_q(T)$ *of extension degree p, where*

$$\alpha_{D_t}^p - \alpha_{D_t} = \sum_{i=1}^{t} \frac{Q_i}{P_i^{r_i}} + f(T)$$

satisfies (2.1). Assume that the infinite place ∞ *of k is totally ramified in K; equivalently,* $\deg f(T) \geq 1$ *with* $p \nmid \deg f(T)$. *We further assume that the followings hold:*

(i) $p \nmid \deg P_i$ *for any i with* $1 \leq i \leq t$.
(ii) $f(T) \equiv \mathfrak{c}_i \pmod{P_i}$, *where* $\mathfrak{c}_i \in \mathbb{F}_q^{\times}$ *such that* $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{c}_i) \neq 0$ *for any i with* $1 \leq i \leq t$.
(iii) $Q_j \equiv P_j^{r_j}(b_i(T)^q - b_i(T)) \pmod{P_i}$ *for any i with* $1 \leq i \neq j \leq t$, *where* $b_i(T)$ *is a polynomial in* $\mathbb{F}_q[T]$.

Then the λ_1*-rank of the ideal class group* Cl_K *of K and* μ_1*-rank of the divisor class group* J_K *of K are t. Moreover, for* $n \geq 2$, *the* λ_n*-rank of* Cl_K *and the* μ_n*-rank of* J_K *are zero.*

In particular, for the case when $p = 2$, *the 2-class groups* $Cl_K(2)$ *and* $J_K(2)$ *are elementary abelian 2-groups: that is, isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^t$.

Proof We note that by Lemma 2.6, the ideal class group of K and the divisor class group of K are isomorphic; thus, $\lambda_n = \mu_n$ for $n \geq 1$. Since λ_n is a decreasing sequence as n grows (λ_{n-1} and λ_n may have the same value), it suffices to show the following:

$$\lambda_1 = t \quad \text{and} \quad \lambda_2 = 0. \tag{3.1}$$

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By Lemma 2.2, we can easily get $\lambda_1 = t$. Thus, we will show that the rank of R is t, where R is the Rédei matrix over $\mathbb{F}_p$ which is defined in Lemma 2.2.

Let $f(T)$ be a polynomial in $\mathbb{F}_q[T]$ which satisfies condition (ii). For convenience, let $\delta_i := \deg P_i$ for $1 \le i \le t$. Then we have the following:

$$\mathrm{Tr}_{\mathbb{F}_{q^{\delta_i}}/\mathbb{F}_q}(f \pmod{P_i}) = \mathrm{Tr}_{\mathbb{F}_{q^{\delta_i}}/\mathbb{F}_q}\, \mathfrak{c}_i = \delta_i \mathfrak{c}_i;$$

the last equality follows from the fact that $\mathfrak{c}_i \in \mathbb{F}_q^{\times}$. Thus, by the definition of the Hasse symbol, we obtain

(3.2)
$$\left\{\frac{f(T)}{P_i}\right\} = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}_{\mathbb{F}_{q^{\delta_i}}/\mathbb{F}_q}(f \pmod{P_i})) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\delta_i \mathfrak{c}_i) = \delta_i\, \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\, \mathfrak{c}_i \neq 0;$$

for the last equality, we use conditions (i) and (ii).

Now, let Q_j $(1 \le j \le t)$ be a polynomial in $\mathbb{F}_q[T]$ which satisfies condition (iii). Then, for $1 \le i \neq j \le t$, we have

$$Q_j \overline{P_j}^{r_j} \equiv b_i(T)^q - b_i(T) \pmod{P_i},$$

where $P_j \overline{P_j} \equiv 1 \pmod{P_i}$. We note that $\overline{P_j}$ always exist since P_i and P_j are relative prime in $\mathbb{F}_q[T]$. Then, by Lemma 3.1, we obtain $\mathrm{Tr}_{\mathbb{F}_{\delta_i}/\mathbb{F}_q}(Q_j \overline{P_j}^{r_j} \pmod{P_i}) = 0$, where $\delta_i := \deg P_i$. Thus, we obtain

$$\left\{\frac{Q_j/P_j^{r_j}}{P_i}\right\} = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}_{\mathbb{F}_{q^{\delta_i}}/\mathbb{F}_q}(Q_j \overline{P_j}^{r_j} \pmod{P_i})) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\, 0 = 0. \tag{3.3}$$

Therefore, we get a $t \times t$ Rédei matrix $R = [r_{ij}]$ over $\mathbb{F}_p$ as follows:

$$R = \begin{bmatrix} r_{11} & 0 & \cdots & 0 \\ 0 & r_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{tt} \end{bmatrix}, \tag{3.4}$$

where $r_{ii} = \left\{\frac{f(T)}{P_i}\right\} \neq 0$ in $\mathbb{F}_p$ for every $1 \le i \le t$. We can easily check that the rank of R is t; therefore, we get $\lambda_2 = \lambda_1 - \mathrm{rank}(R) = 0$.

For the case where $p = 2$, the 2^n-rank of Cl_K and that of J_K are exactly λ_n and μ_n, respectively; therefore, $Cl_K(2) \simeq J_K(2) \simeq (\mathbb{Z}/2\mathbb{Z})^t$. ∎

Theorem 3.3 *Let t be a positive integer. Let $K = k(\alpha_{D_{t+1}})$ be the Artin–Schreier extension over the rational function field $k = \mathbb{F}_q(T)$ of extension degree p, where*

$$\alpha_{D_{t+1}}^p - \alpha_{D_{t+1}} = \sum_{i=1}^{t+1} \frac{Q_i}{P_i^{r_i}} + f(T)$$

satisfies (2.1). Assume that the infinite place ∞ splits completely in K; equivalently, $f(T) = 0$. We further assume that the followings hold:

(i) *$p \nmid \deg P_i$ for any i with $1 \le i \le t+1$.*
(ii) *$Q_t \equiv \mathfrak{c}_i P_t^{r_t} \pmod{P_i}$, where $\mathfrak{c}_i \in \mathbb{F}_q^{\times}$ such that $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathfrak{c}_i) \neq 0$ for any i with $1 \le i \le t$.*

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(iii) $Q_j \equiv P_j^{r_j}(b_i(T)^q - b_i(T)) \pmod{P_i}$ *for any* $1 \le i \le t+1$, $1 \le j \le t$, $i \ne j$, *where* $b_i(T) \in \mathbb{F}_q[T]$.

Then the λ_1*-rank of the ideal class group* Cl_K *of* K *is* t. *Moreover, for* $n \ge 2$, *the* λ_n*-rank of* Cl_K *is zero.*

In particular, for the case when $p = 2$, *the* 2*-class group* $Cl_K(2)$ *is an elementary abelian* 2*-group: that is, isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^t$.

Proof As in Theorem 3.2, we will show (3.1). The fact that $\lambda_1 = t$ comes immediately from Lemma 2.2. Thus, it is sufficient to show that $\lambda_2 = 0$: that is, $\operatorname{rank}(R) = \lambda_1 = t$, where R is the Rédei matrix of K defined in Lemma 2.2.

Let $D_i := \frac{Q_i}{P_i^{r_i}}$ for $1 \le i \le t+1$. Using the same reasoning as in Theorem 3.2, we get $\{D_t/P_i\} \ne 0$ for every $1 \le i \le t$; we note that we use conditions (i) and (ii). Thus, the $i(t+1)$th entry of R is nonzero for $1 \le i \le t$. By condition (iii), we obtain $\{D_j/P_i\} = 0$ from Lemma 3.1; this implies that the ijth entries of R are all zero for $1 \le i \le t+1$ and $1 \le j \le t$ with $i \ne j$.

Therefore, we obtain a $(t+1)\times(t+1)$ matrix $R = [r_{ij}]$ over $\mathbb{F}_p$ as follows:

$$R = \begin{bmatrix} -r_{1,t+1} & 0 & \cdots & 0 & r_{1,t+1} \\ 0 & -r_{2,t+1} & \cdots & 0 & r_{2,t+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -r_{t,t+1} & r_{t,t+1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $r_{i,t+1} \ne 0$ in $\mathbb{F}_p$ for every $1 \le i \le t$. Thus, the result follows immediately.

For the case where $p = 2$, since λ_n gives the full 2^n-rank of Cl_K, we obtain that $Cl_K(2) \simeq (\mathbb{Z}/2\mathbb{Z})^t$. ∎

Theorem 3.4 *Let* t *be a positive integer. Let* $K = k(\alpha_{D_t})$ *be the Artin–Schreier extension over the rational function field* $k = \mathbb{F}_q(T)$ *of extension degree* p, *where*

$$\alpha_{D_t}^p - \alpha_{D_t} = \sum_{i=1}^{t} \frac{Q_i}{P_i^{r_i}} + f(T)$$

satisfies (2.1). Assume that ∞ *is inert in* K; *equivalently,* $f(T) = c \in \mathbb{F}_q^\times$, *where* $x^p - x - c$ *is irreducible over* $\mathbb{F}_q$. *We further assume that the followings hold: for some* $\mathfrak{c} \in \mathbb{F}_q$,

(i) $p \nmid \deg P_i$ *for every* $1 \le i \le t$.

(ii) $Q_j \equiv P_j^{r_j}(b_i(T)^q - b_i(T))$ *for any* i *with* $1 \le i \ne j \le t$, *where* $b_i(T) \in \mathbb{F}_q[T]$.

Then the λ_1*-rank of the ideal class group* Cl_K *of* K *is* t. *Moreover, for* $n \ge 2$, *the* λ_n*-rank of* Cl_K *is zero.*

In particular, for the case when $p = 2$, *then* $Cl_K(2)$ *is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^t$ *and* $J_K(2)$ *is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^{t-1}$.

Proof We can simply get $\lambda_1 = t$ by Lemma 2.2; we now show that $\lambda_2 = 0$, which implies that the rank of the Rédei matrix R is t. As usual, set $D_i := \frac{Q_i}{P_i^{r_i}}$. Using Lemma 3.1, we obtain $\{D_j/P_i\} = 0$ for every $1 \le i \ne j \le t$. Now, we compute $\{c/P_i\}$ for $1 \le i \le t$, where $c \in \mathbb{F}_q^\times$. Let δ_i be the degree of P_i. By the definition of Hasse norm, we have

$$\left\{\frac{c}{P_i}\right\} = \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p} \operatorname{Tr}_{\mathbb{F}_q^{\delta_i}/\mathbb{F}_q}(c \pmod{P_i}) = \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\delta_i c) = \delta_i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(c). \tag{3.5}$$

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We note that $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(c) \neq 0$ since $x^p - x - c$ is irreducible over $\mathbb{F}_q$. Therefore, (3.5) is nonzero; we use condition (i). Using the definition of the Rédei matrix R in Lemma 2.2, we get a $t \times t$ matrix $R = [r_{ij}]$ over $\mathbb{F}_p$ which is given in (3.4). Hence, the desired result follows.

For the case where $p = 2$, the 2-class group of Cl_K is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^t$ by the fact that λ_n gives the full 2^n-rank of Cl_K. By Lemma 2.6, the remaining result follows. ∎

4 Computing the λ_3-rank of class groups of Artin–Schreier function fields

In this section, Algorithm 1 presents an explicit method for computing the λ_3-rank of the ideal class groups of Artin–Schreier extensions K over k. In Theorem 4.3, we provide a proof for Algorithm 1. In particular, we obtain an explicit method for determining the exact 2^3-rank of the ideal class groups of Artin–Schreier quadratic extensions over k (Corollary 4.4).

The following lemma plays a crucial role for the proof of Theorem 4.3.

Lemma 4.1 *Let $K = k(\alpha_{D_m})$ be the Artin–Schreier extension over k of extension degree p, where $D_m(T) = \sum_{i=1}^m \frac{Q_i}{P_i^{r_i}} + f(T)$ is defined as (2.1) and α_{D_m} is a root of $x^p - x = D_m$. For $1 \leq i \leq m$, let α_i be a root of $x^p - x = D_i := Q_i/P_i^{r_i}$ and let γ_i be a root of the following equation in $\overline{k}$:*

$$\mathbf{X}^p - \mathbf{X} = \mathcal{D}_i := \frac{{\alpha_i}^2 P_i^{r_i}}{Q_i}.$$

Then $k(\alpha_i, \gamma_i)/k(\alpha_i)$ is unramified, where all the infinite places of $k(\alpha_i)$ split completely in $k(\alpha_i, \gamma_i)$.

Proof We first show that $k(\alpha_i, \gamma_i)/k(\alpha_i)$ is an unramified extension. Let $\mathfrak{p}_i \in k(\alpha_i)$ be a place which lies above a finite place P of k. We note that it suffices to show the following by Lemma 2.4:

$$\nu_{\mathfrak{p}_i}(\mathcal{D}_i) = 2\nu_{\mathfrak{p}_i}(\alpha_i) + \nu_{\mathfrak{p}_i}(P_i^{r_i}) - \nu_{\mathfrak{p}_i}(Q_i) \geq 0. \tag{4.1}$$

We consider the following three possible cases: $P = P_i$ for $1 \leq i \leq m$, P divides $Q_i \in \mathbb{F}_q[T]$, and $(P, P_i) = (P, Q_i) = 1$. Using a valuation property, we can easily show the following, where n is a positive integer.

$$\text{If } \nu_{\mathfrak{p}_i}(\alpha_i^n - \alpha_i) < 0, \text{ then } \nu_{\mathfrak{p}_i}(\alpha_i^n - \alpha_i) = n\nu_{\mathfrak{p}_i}(\alpha_i) < 0. \tag{4.2}$$

$$\text{If } \nu_{\mathfrak{p}_i}(\alpha_i^n - \alpha_i) > 0, \text{ then } \nu_{\mathfrak{p}_i}(\alpha_i) \geq 0. \tag{4.3}$$

$$\text{If } \nu_{\mathfrak{p}_i}(\alpha_i^n - \alpha_i) = 0, \text{ then } \nu_{\mathfrak{p}_i}(\alpha_i^n - \alpha_i) = \nu_{\mathfrak{p}_i}(\alpha_i) = 0. \tag{4.4}$$

We denote *the ramification index of $\mathfrak{p}_i$ over P in $k(\alpha_i)/k$ by $e(\mathfrak{p}_i|P)$ and the residue class field degree of $\mathfrak{p}_i$ over P* by $f(\mathfrak{p}_i|P)$.

(i) Suppose that $P = P_i$. Then we have $e(\mathfrak{p}_i|P) = e(\mathfrak{p}_i|P_i) = p$ since P_i is the only totally ramified finite place for $k(\alpha_i)/k$. Therefore, we have $\nu_{\mathfrak{p}_i}(\alpha_i^p - \alpha_i) =$

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$v_{\mathfrak{p}_i}(Q_i/P_i^{r_i}) = -pr_i < 0$; this implies that $v_{\mathfrak{p}_i}(\alpha_i) = -r_i$ by (4.2). Therefore, (4.1) holds true.

(ii) Suppose that P divides Q_i in $\mathbb{F}_q[T]$. Under the given assumption, we have $e(\mathfrak{p}_i|P) = 1$; this is because $(P, P_i) = 1$ as $(P_i, Q_i) = 1$ and P_i is the only totally ramified finite place for $k(\alpha_i)/k$. Consequently, we have

$$v_{\mathfrak{p}_i}(\alpha_i^p - \alpha_i) = v_{\mathfrak{p}_i}(Q_i/P_i^{r_i}) = v_P(Q_i/P_i^{r_i}) = v_P(Q_i) > 0;$$

thus, $v_{\mathfrak{p}_i}(\alpha_i) \geq 0$ by (4.3). Assuming that $v_{\mathfrak{p}_i}(\alpha_i) = 0$, we obtain

$$v_P(N_{k(\alpha_i)/k}(\alpha_i)) = f(\mathfrak{p}_i|P)v_{\mathfrak{p}_i}(\alpha_i) = 0. \tag{4.5}$$

However, since $v_{\mathfrak{p}_i}(N_{k(\alpha_i)/k}(\alpha_i)) = v_{\mathfrak{p}_i}(Q_i/P_i^{r_i}) > 0$ (4.5) cannot happen. Therefore, we have $v_{\mathfrak{p}_i}(\mathcal{D}_i) = 2v_P(Q_i) - v_P(Q_i) > 0$ and (4.1) follows; we use the fact that $v_{\mathfrak{p}_i}(\alpha_i) = v_P(Q_i) > 0$. As a result, $\mathfrak{p}_i$ is unramified in $k(\alpha_i, \gamma_i)$.

(iii) Suppose that $(P, P_i) = (P, Q_i) = 1$. In this case, we get $v_{\mathfrak{p}_i}(\alpha_i) = 0$ by (4.4) since $v_{\mathfrak{p}_i}(\alpha_i^p - \alpha_i) = 0$. Therefore, (4.1) follows immediately.

Now, it remains to show that all the infinite places of $k(\alpha_i)$ split completely in $k(\alpha_i, \gamma_i)$. Let $\mathfrak{p}_\infty$ (resp. $\mathfrak{P}_\infty$) be a place of $k(\alpha_i)$ (resp. $k(\alpha_i, \gamma_i)$) lying above the infinite place ∞ of k (resp. $\mathfrak{p}_\infty$). We first note that $v_{\mathfrak{p}_\infty}(\alpha_i^p - \alpha_i) = v_{\mathfrak{p}_\infty}(Q_i/P_i^{r_i}) > 0$; thus, $v_{\mathfrak{p}_\infty}(\alpha_i) \geq 0$ by (4.3). By a similar computation method as in (4.5), we obtain $v_{\mathfrak{p}_\infty}(\alpha_i) > 0$, and therefore $v_{\mathfrak{p}_\infty}(\alpha_i) = v_{\mathfrak{p}_\infty}(\alpha_i^p - \alpha_i) = \deg P_i^{r_i} - \deg Q_i$. Hence, we get

$$v_{\mathfrak{p}_\infty}(\mathcal{D}_i) = 2v_{\mathfrak{p}_\infty}(\alpha_i) + v_{\mathfrak{p}_\infty}(P_i^{r_i}) - v_{\mathfrak{p}_\infty}(Q_i) = 2(\deg P_i^{r_i} - \deg Q_i) - \deg P_i^{r_i} + \deg Q_i > 0;$$

from this fact and by Lemma 2.4, we can conclude that $\mathfrak{p}_\infty$ is unramified in $k(\alpha_i, \gamma_i)/k(\alpha_i)$.

Now, it is enough to show that $f(\mathfrak{P}_\infty|\mathfrak{p}_\infty)$ is 1. For the proof, we assume that $f(\mathfrak{P}_\infty|\mathfrak{p}_\infty) = p$. We first note that

$$N_{k(\alpha_i,\gamma_i)/k(\alpha_i)}(\gamma_i) = \gamma_i^p - \gamma_i = \alpha_i^2 P_i^{r_i}/Q_i. \tag{4.6}$$

On the other hand, we have

$$v_{\mathfrak{p}_\infty}(N_{k(\alpha_i,\gamma_i)/k(\alpha_i)}(\gamma_i)) = f(\mathfrak{P}_\infty|\mathfrak{p}_\infty)v_{\mathfrak{P}_\infty}(\gamma_i) = pv_{\mathfrak{P}_\infty}(\gamma_i). \tag{4.7}$$

Also, we can obtain

$$pv_{\mathfrak{P}_\infty}(\gamma_i) = v_{\mathfrak{p}_\infty}(\gamma_i^p - \gamma_i) = v_{\mathfrak{P}_\infty}(\gamma_i^p - \gamma_i), \tag{4.8}$$

by combining (4.6) with (4.7). Furthermore, since $v_{\mathfrak{p}_\infty}(\gamma_i^p - \gamma_i) = pv_{\mathfrak{P}_\infty}(\gamma_i) > 0$, we have

$$pv_{\mathfrak{P}_\infty}(\gamma_i) = \min\{pv_{\mathfrak{P}_\infty}(\gamma_i), v_{\mathfrak{P}_\infty}(\gamma_i)\} = v_{\mathfrak{P}_\infty}(\gamma_i), \tag{4.9}$$

which is a contradiction. Therefore, the infinite place of $k(\alpha_i)$ splits completely in $k(\alpha_i, \gamma_i)$. ∎

Lemma 4.2 *Let K be the Artin–Schreier extension over k of extension degree p. Let H_K be the Hilbert class field of K, and let $\mathcal{G}_K$ be the genus field of K. Let $\mathcal{H}$ be*

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Algorithm 1 (Computation of λ_3 for the Artin–Schreier function field K)

Input:

- q : a power of a prime p
- $D_m(T) := \sum_{i=1}^{m} \frac{Q_i}{P_i^{r_i}} + f(T)$ defined by (2.1)
- $K = k(\alpha_{D_m})$ with α_{D_m} defined in (2.1)

Output: the λ_3-rank of the ideal class group of K

(1) Find λ_1 of K, and compute a Rédei matrix R over $\mathbb{F}_p$ using Lemma 2.2.
(2) Compute $\lambda_2 = \lambda_1 - \text{rank}(R)$.
(3) **If** $\lambda_2 = 0$, then **Stop**.
(4) **Else**
  (4.1) **If** $\lambda_2 < \lambda_1$, then let
  $\mathcal{I} := \{1 \le i \le m \mid \text{the } i\text{th row vector of } R \text{ is zero}\} = \{s_1, \ldots, s_{\lambda_2}\}$ with $s_i < s_j$ for $1 \le i < j \le \lambda_2$.
  (4.2) **Else** let $\mathcal{I} := \{1, \ldots, \lambda_2\} = \{s_1, \ldots, s_{\lambda_2}\}$ with $i = s_i$ for $1 \le i \le \lambda_2$.
(5) **For** $1 \le i \le \lambda_2$,
  (5.1) set $\mathcal{P}_i := P_{s_i}$ and $\mathcal{F}_i := Q_{s_i}/P_{s_i}^{r_{s_i}}$.
  (5.2) let $\mathfrak{a}_i$ be a root of $x^p - x = \mathcal{F}_i$ in $\overline{k}$, and set $\mathcal{D}_i = \mathfrak{a}_i^2/\mathcal{F}_i$.
(6) **For** $1 \le i, j \le \lambda_2$,
  find a $\lambda_2 \times \lambda_2$-matrix $\mathcal{R} = [\mathfrak{r}_{ij}]$ over $\mathbb{F}_p$, where $\mathfrak{r}_{ij}$ is defined as $\mathfrak{r}_{ij} = \left\{\frac{\mathcal{D}_j}{\mathcal{P}_i}\right\}$.
(7) Compute $\lambda_3 = \lambda_2 - \text{rank}(\mathcal{R})$.

a fixed field of a subgroup of $\text{Gal}(H_K/\mathcal{G}_K)$ *which is isomorphic to* $Cl_K^{(\sigma-1)^2}$. *Then* $Cl_K(p)^{(\sigma-1)}/Cl_K(p)^{(\sigma-1)^2}$ *is isomorphic to* $\text{Gal}(\mathcal{H}/\mathcal{G}_K)$*; thus, we can define the following composite map:*

$$\Psi : Cl_K(p)^G \cap Cl_K(p)^{(\sigma-1)} \to Cl_K(p)^{(\sigma-1)}/Cl_K(p)^{(\sigma-1)^2} \xrightarrow{\simeq} \text{Gal}(\mathcal{H}/\mathcal{G}_K), \tag{4.10}$$

where the first map is induced by the inclusion map.

Then λ_3 *is equal to* $\lambda_2 - \text{rank}(\mathcal{R})$*, where* $\mathcal{R}$ *is a matrix representing* Ψ *over* $\mathbb{F}_p$ *and* λ_2 *is obtained by Lemma 2.2.*

Proof We note that $\text{Gal}(H_K/K) \simeq Cl_K$ and $\text{Gal}(\mathcal{G}_K/K) \simeq Cl_K(p)/Cl_K(p)^{(\sigma-1)} \simeq Cl_K/Cl_K^{(\sigma-1)}$ [19, pp. 328–329]; therefore, $\text{Gal}(H_K/\mathcal{G}_K) \simeq Cl_K^{(\sigma-1)}$. By the Galois correspondence, we have isomorphisms $\text{Gal}(\mathcal{H}/\mathcal{G}_K) \simeq Cl_K^{(\sigma-1)}/Cl_K^{(\sigma-1)^2}$ and $Cl_K^{(\sigma-1)}/Cl_K^{(\sigma-1)^2} \simeq Cl_K(p)^{(\sigma-1)}/Cl_K(p)^{(\sigma-1)^2}$; thus, we have the isomorphism $Cl_K(p)^{(\sigma-1)}/Cl_K(p)^{(\sigma-1)^2} \xrightarrow{\simeq} \text{Gal}(\mathcal{H}/\mathcal{G}_K)$.

Let Ψ be the map defined as in (4.10). Then we have

$$|\,\text{Ker}(\Psi)| = |Cl_K(p)^G \cap Cl_K(p)^{(\sigma-1)^2}|.$$

We claim that for any positive integer n,

$$|Cl_K(p)^G \cap Cl_K(p)^{(\sigma-1)^{n-1}}| = |Cl_K(p)^{(\sigma-1)^{n-1}}/Cl_K(p)^{(\sigma-1)^n}|. \tag{4.11}$$

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We consider a short exact sequence

$$0 \to Cl_K(p)^G \cap Cl_K(p)^{(\sigma-1)^{n-1}} \xrightarrow{\iota} Cl_K(p)^{(\sigma-1)^{n-1}} \xrightarrow{\sigma-1} Cl_K(p)^{(\sigma-1)^n} \to 0,$$

where ι denotes an inclusion map. Then $Cl_K(p)^{(\sigma-1)^n}$ is isomorphic to

$$Cl_K(p)^{(\sigma-1)^{n-1}}/\mathrm{Im}(\iota) = Cl_K(p)^{(\sigma-1)^{n-1}}/Cl_K(p)^G \cap Cl_K(p)^{(\sigma-1)^{n-1}}.$$

Therefore, we have the following:

$$|Cl_K(p)^{(\sigma-1)^n}| = \frac{|Cl_K(p)^{(\sigma-1)^{n-1}}|}{|Cl_K(p)^G \cap Cl_K(p)^{(\sigma-1)^{n-1}}|}.$$

We can rewrite this as

$$|Cl_K(p)^G \cap Cl_K(p)^{(\sigma-1)^{n-1}}| = \frac{|Cl_K(p)^{(\sigma-1)^{n-1}}|}{|Cl_K(p)^{(\sigma-1)^n}|} = |Cl_K(p)^{(\sigma-1)^{n-1}}/Cl_K(p)^{(\sigma-1)^n}|;$$

hence, (4.11) follows.

Therefore, we compute as follows:

$$\begin{aligned}\lambda_3 &= \dim_{\mathbb{F}_p}(Cl_K(p)^{(\sigma-1)^2}/Cl_K(p)^{(\sigma-1)^3}) = \dim_{\mathbb{F}_p}(Cl_K(p)^G/Cl_K(p)^{(\sigma-1)^2})\\ &= \dim_{\mathbb{F}_p}(\mathrm{Ker}(\Psi)) = \dim_{\mathbb{F}_p}(Cl_K(p)^G \cap Cl_K(p)^{(\sigma-1)}) - \dim_{\mathbb{F}_p}(\mathrm{Im}(\Psi))\\ &= \dim_{\mathbb{F}_p}(Cl_K(p)^{(\sigma-1)}/Cl_K(p)^{(\sigma-1)^2}) - \dim_{\mathbb{F}_p}(\mathrm{Im}(\Psi)) = \lambda_2 - \dim_{\mathbb{F}_p}(\mathrm{Im}(\Psi))\\ &= \lambda_2 - \mathrm{rank}(\mathcal{R}),\end{aligned}$$

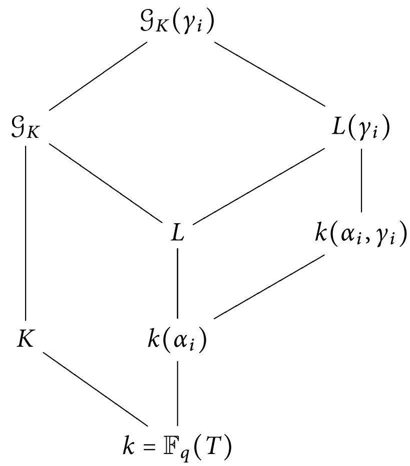
where $\mathcal{R}$ is a matrix representing Ψ over $\mathbb{F}_p$ and λ_2 is obtained by Lemma 2.2. We note that the second equality and the fifth one hold by (4.11) with $n = 3$ and 2, respectively. ■

Theorem 4.3 *Let K be the Artin–Schreier extension over the rational function field k of extension degree p. Then the λ_3-rank of the ideal class group of K can be computed by Algorithm 1.*

Proof By Lemma 4.2, we have $\lambda_3 = \lambda_2 - \mathrm{rank}(\mathcal{R})$, where $\mathcal{R}$ is a matrix representing Ψ which is defined as in (4.10). Therefore, it is sufficient to compute the matrix $\mathcal{R}$ in an explicit way for computation of λ_3. We describe how to compute the matrix $\mathcal{R}$ as follows.

Let $\mathcal{I} := \{1 \le i \le m \mid \text{the } i\text{th row vector of } R \text{ is zero}\} = \{s_1, \dots, s_{\lambda_2}\}$, where $s_i < s_j$ for $1 \le i < j \le \lambda_2$. For simplicity, we set $\mathcal{P}_i := P_{s_i}$ and $\mathcal{F}_i = Q_{s_i}/P_{s_i}^{r_{s_i}}$ for $1 \le i \le \lambda_2$. Let $\mathcal{D}_i := \mathfrak{a}_i{}^2/\mathcal{F}_i$, and let γ_i be a root of $\mathbf{X}^p - \mathbf{X} = \mathcal{D}_i$ in $\overline{k}$, where $\overline{k}$ is the algebraic closure of k and $\mathfrak{a}_i$ is the root of $x^p - x = \mathcal{F}_i$ in $\overline{k}$.

Let $L := k(\alpha_1, \dots, \alpha_m)$ be a subfield of the genus field $\mathcal{G}_K$ defined as the following, where $\mathcal{G}_K$ is given in Lemma 2.3.



We now show that $\mathcal{G}_K(\gamma_i)$ is a subfield of H_K for $1 \leq i \leq \lambda_2$. We point out that $\mathcal{G}_K(\gamma_i)/\mathcal{G}_K$ is an abelian extension by the fact that it is the Artin–Schreier function field. It suffices to show that $\mathcal{G}_K(\gamma_i)/\mathcal{G}_K$ is an unramified extension and all the infinite places of $\mathcal{G}_K$ split completely in $\mathcal{G}_K(\gamma_i)$. By Lemma 4.1, $k(\alpha_i, \gamma_i)/k(\alpha_i)$ is an unramified extension and all the infinite places of $k(\alpha_i)$ split completely in $k(\alpha_i, \gamma_i)$. Thus, $L(\gamma_i)/L$ is an unramified extension; hence, $\mathcal{G}_K(\gamma_i)/\mathcal{G}_K$ is an unramified extension.

Now, we show that all the infinite places of $\mathcal{G}_K$ split completely in $\mathcal{G}_K(\gamma_i)$. Every infinite place of $k(\alpha_i)$ splits completely in $k(\alpha_i, \gamma_i)$ as shown above and all the infinite places of L split completely in $L(\gamma_i)$. Also, all the infinite places split completely in $L/k(\alpha_i)$ by Lemma 2.1. Consequently, all the infinite places of L split completely in the compositum $L(\gamma_i)$ of L and $k(\alpha_i, \gamma_i)$.

Let $\mathcal{P}_\infty$ be a place of L which lies above the infinite place ∞ of k and $\mathcal{P}'$ a place of $\mathcal{G}_K$ which lies above $\mathcal{P}_\infty$. We consider the following two possible cases: $\mathcal{P}_\infty$ splits completely in $\mathcal{G}_K$ or $\mathcal{P}_\infty$ is totally ramified or inert in $\mathcal{G}_K$. We note that the result follows immediately in the former case; thus, it is sufficient to consider the latter case where there is exactly one place lying above $\mathcal{P}_\infty$ in $\mathcal{G}_K$, the number of places in $\mathcal{G}_K(\gamma_i)$ which lie above $\mathcal{P}'$ is exactly p; this is because the infinite places split completely in $L(\gamma_i)/L$. Therefore, $\mathcal{P}'$ splits completely in $\mathcal{G}_K(\gamma_i)$, and the result holds.

We have $\mathcal{H} = \mathcal{G}_K(\gamma_1, \ldots, \gamma_{\lambda_2})$ since $\mathcal{G}_K(\gamma_i) \subseteq H_K$ and $[\mathcal{H} : \mathcal{G}_K] = p^{\lambda_2}$. We get

$$\left(\frac{\mathcal{H}/\mathcal{G}_K}{\mathfrak{p}_i}\right)(\gamma_j) = \gamma_j + \left\{\frac{\mathcal{D}_j}{\mathcal{P}_i}\right\},$$

where $\mathfrak{p}_i$ is a place of $\mathcal{G}_K$ lying above $\mathcal{P}_i$ for $1 \leq i \leq \lambda_2$ by the action of the Artin map in the Artin–Schreier function field. Therefore, we determine $\mathcal{R} = [\mathfrak{r}_{ij}] = \left\{\frac{\mathcal{D}_j}{\mathcal{P}_i}\right\}$.

This process is implemented in Algorithm 1. Steps (1) and (2) of Algorithm 1 give the process of computing λ_1, λ_2, and the Rédei matrix R. Step (3) explains the case where $\lambda_2 = 0$ and then the algorithm stops. If $0 < \lambda_2 < \lambda_1$, then we go to Step (4.1), and if $\lambda_2 = \lambda_1$, then we proceed with Step (4.2). Steps (5.1) and (5.2) explain the process of finding $\mathcal{D}_i$ for $1 \leq i \leq \lambda_2$. In Step (6), we determine a matrix $\mathcal{R}$ over $\mathbb{F}_p$, and finally we obtain $\lambda_3 = \lambda_2 - \text{rank}(\mathcal{R})$ in Step (7). ∎

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Corollary 4.4 *Let K be the Artin–Schreier quadratic extension over k, and let the λ_3-rank of Cl_K be computed by Algorithm 1. Then the 2^3-rank of Cl_K is exactly λ_3: that is, $Cl_K(2)$ has a subgroup isomorphic to $(\mathbb{Z}/2^3\mathbb{Z})^{\lambda_3}$.*

Proof This follows immediately from the fact that λ_n is exactly equal to the full 2^n-rank of Cl_K and Theorem 4.3. ∎

Remark 4.5 For readers, focusing on the case: $p = 2$, we first briefly explain the analogy between Rédei symbols (the 4-rank of the class groups) and the 8-rank of the class groups in the quadratic field case (for more details, see [9]). Then we describe the analogy between Artin–Schreier quadratic extensions over k and quadratic extensions over $\mathbb{Q}$ for computation of λ_3.

Let F be a quadratic extension over $\mathbb{Q}$, and let Cl_F be the ideal class group of F. Let r_4 (resp. r_8) be the 2^2-rank (resp. 2^3-rank) of Cl_F. Let H be the Hilbert class field of F, and let H_n be the unramified abelian subextension of H such that $\mathrm{Gal}(H_n/F) \simeq Cl_F/Cl_F^n$ for $n = 2, 4$.

Basically, a strategy for computing the 2^2-rank (resp. 2^3-rank) is explicitly finding a subextension H_2 (resp. H_4) of the Hilbert class field of F whose Galois group is isomorphic to $\mathrm{Gal}(Cl_F/Cl_F^2)$ (resp. $\mathrm{Gal}(Cl_F^2/Cl_F^4)$).

Define two maps as follows:

$$R_4 : \mathbb{F}_2^t \to Cl_F[2] \xrightarrow{\varphi} Cl_F/Cl_F^2 \xrightarrow{\simeq} \mathrm{Gal}(H_2/F) \to \mathrm{Gal}(H_2/\mathbb{Q}) = \prod_{i=1}^{t} \mathrm{Gal}(\mathbb{Q}(\sqrt{d_i})/\mathbb{Q}),$$

$$R_8 : \mathrm{Ker}\, R_4 \to Cl_F[2] \cap Cl_F^2 \xrightarrow{\psi} Cl_F^2/Cl_F^4 \xrightarrow{\simeq} \mathrm{Gal}(H_4/H_2) = \prod_{i=1}^{r_4} \mathrm{Gal}(H_2(\sqrt{\alpha_i})/H_2) \to \mathbb{F}_2^{r_4},$$

where t is the number of finite primes of $\mathbb{Q}$ which are ramified in F, $Cl_F[2]$ is the 2-torsion part of Cl_F, and the maps φ and ψ are induced by the inclusion maps. For computation of r_4 and r_8, we find appropriate d_i $(1 \le i \le t)$ and α_i $(1 \le i \le r_4)$. Then we have

$$r_4 = t - \dim_{\mathbb{F}_2} R_4 \quad \text{and} \quad r_8 = r_4 - \dim_{\mathbb{F}_2} R_8.$$

To show the analogy between Artin–Schreier quadratic extensions over k and quadratic extensions over $\mathbb{Q}$ for computation of λ_3 (2^3-rank), let K be the Artin–Schreier quadratic extension over k. Then the map R_8 corresponds to the map Ψ defined in (4.10):

$$\Psi : Cl_K(2)^G \cap Cl_K^2 \to Cl_K^2/Cl_K^4 \xrightarrow{\simeq} \mathrm{Gal}(\mathcal{H}/\mathcal{G}_K).$$

Then we have $\lambda_3 = \lambda_2 - \mathrm{rank}\,\mathcal{R}$, where $\mathcal{R}$ is a matrix over $\mathbb{F}_2$ representing the map Ψ. We recall that λ_3 is the 2^3-rank of Cl_K.

5 An infinite family of Artin–Schreier function fields with higher λ_n-rank

In this section, we find an infinite family of Artin–Schreier function fields which have *prescribed* λ_n-rank of the ideal class group for $1 \le n \le 3$. In Theorem 5.1, for any positive integer $t \ge 2$, we obtain an infinite family of Artin–Schreier extensions over k

whose λ_1-rank is t, λ_2-rank is $t-1$, and λ_3-rank is $t-2$. Then Corollary 5.3 shows the case where $p=2$, for a given positive integer $t \geq 2$, we obtain an infinite family of the Artin–Schreier quadratic extensions over k whose 2-class group rank (resp. 2^2-class group rank and 2^3-class group rank) is *exactly* t (resp. $t-1$ and $t-2$). Furthermore, we also obtain a similar result on the 2^n-ranks of the divisor class groups of the Artin–Schreier quadratic extensions over k in Corollary 5.4.

Throughout this section, we define D_m as follows.

Notation 1 *Let $D_m := \sum_{i=1}^{m} D_i + f(T)$ be defined in (2.1) with $D_i = Q_i/P_i^{r_i}$, where m, P_i, Q_i, and $f(T)$ satisfy one of the followings:*

(i) $m = \begin{cases} t, & \text{if } \deg f(T) \geq 1 \\ & \text{or } f(T) = c \in \mathbb{F}_q^{\times} \text{ such that } x^p - x = c \text{ is irreducible over } \mathbb{F}_q, \\ t+1, & \text{if } f(T) = 0. \end{cases}$

(ii) $Q_j \equiv P_j^{r_j}(b_i(T)^q - b_i(T)) \pmod{P_i}$ *for any* $1 \leq i \neq j \leq m$ *except* $(i,j) = (1,2)$, *where* $b_i(T) \in \mathbb{F}_q[T]$.

(iii) *If* $\deg f(T) \geq 1$, *then* $f(T) \equiv P_j^{r_j}(b_i(T)^q - b_i(T)) \pmod{P_i}$, *where* $b_i(T) \in \mathbb{F}_q[T]$ *for any* $1 \leq i \leq m$.

(iv) *If* $f(T) \in \mathbb{F}_q^{\times}$, *then* $q \mid \deg P_i$ *for any* i *with* $1 \leq i \leq m$.

(v) $Q_j^{-1} \equiv P_j^{r_j}(b_i(T)^q - b_i(T)) \pmod{P_i}$, *where* $b_i(T) \in \mathbb{F}_q[T]$ *and* Q_j^{-1} *denotes the inverse of* Q_j *modulo* P_i *for any* $1 \leq i \neq j \leq m$ *except* $(i,j) \neq (1,2)$.

Theorem 5.1 *For a given positive integer $t \geq 2$, there is an infinite family of Artin–Schreier extensions over k whose λ_1-rank is t, λ_2-rank is $t-1$, and λ_3-rank is $t-2$.*

Let $K = k(\alpha_{D_m})$ be the Artin–Schreier function field over k of extension degree p, where D_m is defined in Notation 1 and α_{D_m} is a root of $x^p - x = D_m$. Then the ideal class group Cl_K of K has $\lambda_1 = t$, $\lambda_2 = t-1$, and $\lambda_3 = t-2$.

Remark 5.2 Let $\mathbb{F}_q$ be a finite field of order q, t be a given integer, and $f(T) \in \mathbb{F}_q$. By condition (i), $m = t+1$. By condition (ii), we can choose monic irreducible polynomials $P_i \in \mathbb{F}_q[T]$ whose degrees are divisible by p. We note that conditions (iii) and (iv) can be interpreted as

$$\left\{\frac{D_j}{P_i}\right\} = \left\{\frac{Q_j^{-1}}{P_i}\right\} = 0; \tag{5.1}$$

by the surjectivity of the trace map, there always exist D_j and Q_j^{-1} which satisfy (5.1). Since our choice of P_i's are infinite, we have an infinite family of Artin–Schreier extensions which satisfy the conditions in Theorem 5.1.

Proof of Theorem 5.1 Recall that $\lambda_2 = \lambda_1 - \operatorname{rank}(R)$ and $\lambda_3 = \lambda_2 - \operatorname{rank}(\mathcal{R})$, where R (resp. $\mathcal{R}$) is a matrix over $\mathbb{F}_p$ defined in Lemma 2.2 (resp. Algorithm 1). We need to show that

$$\lambda_1 = t, \qquad \lambda_2 = t-1, \qquad \lambda_3 = t-2; \tag{5.2}$$

this is equivalent to $\operatorname{rank}(R) = \operatorname{rank}(\mathcal{R}) = 1$.

We divide into the following three cases: $\deg f(T) \geq 1$, $\deg f(T) = 0$, and $f(T) = c$, where $x^p - x - c$ is irreducible over $\mathbb{F}_q$.

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Case I. $\deg f(T) \geq 1$: that is, the infinite place of k is totally ramified in K.

Since $\deg f(T) \geq 1$, we have $m = t$ by condition (i); this implies that $\lambda_1 = m = t$ by Lemma 2.2. For computing λ_2, we compute every entry of the Rédei matrix R: that is, the Hasse norm $\{D_j/P_i\}$ and $\{f(T)/P_i\}$ for $1 \leq i \neq j \leq m$. Using Lemma 3.1 and condition (ii), we can easily obtain that $\left\{\frac{D_2}{P_1}\right\} \neq 0$ and $\left\{\frac{D_j}{P_i}\right\} = 0$ for any $1 \leq i \neq j \leq m$ except $(i,j) \neq (1,2)$. Furthermore, we get $\left\{\frac{f}{P_i}\right\} = 0$ for any $1 \leq i \leq m$ by condition (iii). Therefore, the Rédei matrix R can be written as $R = \begin{bmatrix} p-1 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$; thus, $\lambda_2 = \lambda_1 - \operatorname{rank}(R) = t-1$. Lastly, we compute λ_3 of K using Algorithm 1 and Theorem 4.3. Using the definition of a matrix $\mathcal{R}$ which is given in Algorithm 1, it suffices to compute $\left\{\frac{1/Q_j}{P_i}\right\}$ for $1 \leq i \neq j \leq m$. By the same reasoning as in the computation of R, we get $\lambda_3 = \lambda_2 - \operatorname{rank}(\mathcal{R}) = t-2$. Therefore, (5.2) follows.

Case II. $\deg f(T) = 0$: that is, the infinite place of k splits completely in K, which is a real extension.

We can easily obtain $\lambda_1 = t$ by using Lemma 2.2 and the condition $m = t+1$. For computing λ_2, we compute every entry of the Rédei matrix R: that is, the value of Hasse norm $\{D_j/P_i\}$ for $1 \leq i \neq j \leq m$. By the definition of Hasse norm which is defined in Definition 2.1, we get $\{D_2/P_1\} \neq 0$ and $\{D_j/P_i\} = 0$, where $1 \leq i \neq j \leq m$ except $(i,j) = (1,2)$. As in Case 1, the rank of Rédei matrix is one: that is, $\lambda_2 = \lambda_1 - \operatorname{rank}(R) = t-1$. Lastly, we compute λ_3 of K; by the same computation method as in Case I, we have $\lambda_3 = \lambda_2 - \operatorname{rank}(\mathcal{R}) = t-2$. Therefore, (5.2) follows.

Case III. $f(T) = c \in \mathbb{F}_q^\times$, where $x^p - x - c$ is irreducible over $\mathbb{F}_q$: that is, the infinite place of k is inert in K.

Under this assumption, K is an imaginary extension; so, $m = t$. We claim that (5.2) holds for this case. We can simply get $\lambda_1 = t$ by Lemma 2.2 and we also obtain $\{D_j/P_i\} = 0$ for every $1 \leq i \neq j \leq t = m$ except $(i,j) = (1,2)$ by using the same reasoning as in Case I. Now, we compute the value of $\{c/P_i\}$ for $1 \leq i \leq t = m$, where $c \in \mathbb{F}_q^\times$. We have

$$\left\{\frac{c}{P_i}\right\} = \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\operatorname{Tr}_{\mathbb{F}_\mathfrak{d}/\mathbb{F}_q} c) = \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(c \deg P_i) = \deg P_i(\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p} c);$$

the second equation holds since c is a nonzero element of $\mathbb{F}_q$ and the last equation holds by the property of a trace map over a finite field. We get $\deg P_i(\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p} c) = 0$ in $\mathbb{F}_p$ by Lemma 3.1 by the assumption that $q \mid \deg P_i$ for every $1 \leq i \leq m$; therefore, (3.5) is zero in $\mathbb{F}_p$. Hence, $\lambda_2 = t-1$. By the same reasoning as in Case I, $\lambda_3 = t-2$ and we have (5.2). ∎

Corollary 5.3 *Let $K = k(\alpha_{D_m})$ be the Artin–Schreier quadratic function field over k of extension degree 2, where D_m is defined in Notation 1 and α_{D_m} is a root of $x^2 - x = D_m$.*

For any positive integer $t \geq 2$, there is an infinite family of Artin–Schreier quadratic extensions over k whose 2-class group rank is exactly t, 2^2-class group rank is $t-1$, and 2^3-class group rank is $t-2$.

In particular, $Cl_K(2)$ contains a subgroup isomorphic to $(\mathbb{Z}/2^n\mathbb{Z})^{t-n+1}$ for $1 \leq n \leq 3$.

Proof We note that λ_n is exactly equal to the full 2^n-rank ($1 \le n \le 3$) of the ideal class group Cl_K of K; therefore, the result follows immediately from Theorem 5.1. ∎

Corollary 5.4 *For a given positive integer t, let $K = k(\alpha_{D_m})$ be the Artin–Schreier quadratic function field over k, where $D_m = \sum_{i=1}^{m} Q_i/P_i^{r_i} + f(T)$ such that $P_i, Q_i, f(T)$, and m satisfy the conditions (i)–(v) in Notation 1. Let J_K be the divisor class group of K. Then we have the following infinite family of Artin–Schreier quadratic extensions.*

(i) *For $t \ge 2$, if $\deg f(T) \ge 1$ (equivalently, ∞ is totally ramified in K), then the 2^n-class group rank of J_K is exactly equal to $t + 1 - n$ for $1 \le n \le 3$.*
(ii) *For $t \ge 2$, if $f(T) = 0$ (equivalently, ∞ splits completely in K), then the 2^n-class group rank of J_K is exactly either $t + 1 - n$ or $t + 2 - n$ for $1 \le n \le 3$.*
(iii) *For $t \ge 3$, if $f(T) \in \mathbb{F}_q^\times$ (equivalently, ∞ is inert in K), then the 2^n-class group rank of J_K is exactly either $t + 1 - n$ or $t - n$ for $1 \le n \le 3$.*

Proof Since D_m satisfies the conditions (i)–(v) in Notation 1, the ideal class group Cl_K of K has λ_1-rank t, λ_2-rank $t - 1$, and λ_3-rank $t - 2$.

We first assume that $\deg f(T) \ge 1$: that is, the infinite place ∞ of k is totally ramified in K. Then the ideal class group Cl_K of K is isomorphic to the divisor class group J_K of K by Lemma 2.6. Thus, by Lemma 5.3, the 2^n-rank of the divisor class group J_K of K is $t + 1 - n$ for n up to 3; thus, (i) follows.

Next, suppose that $f(T) = 0$. This is the case where the infinite place ∞ of k splits completely in K. Then, by Lemma 2.6, we note that J_K/R is isomorphic to Cl_K, where R denotes the group $\mathcal{D}_K^0(S)/\mathcal{P}_K(S)$. By the fact the group R is a cyclic group, the 2^n-rank of the divisor class group J_K is either $t + 1 - n$ or $t + 2 - n$ for n up to 3.

Finally, we assume that $f(T) \in \mathbb{F}_q^\times$: the case where ∞ is inert in K. Then, by the exact sequence given in Lemma 2.6(ii), we get $|Cl_K| = 2|J_K|$. Since $Cl_K(2)$ contains a subgroup isomorphic to $(\mathbb{Z}/2^n\mathbb{Z})^{t-n+1}$ for $1 \le n \le 3$, $J_K(2)$ contains a subgroup isomorphic to $(\mathbb{Z}/2^n\mathbb{Z})^{t-n+1}$ or $(\mathbb{Z}/2^n\mathbb{Z})^{t-n}$ for $1 \le n \le 3$; therefore, (iii) holds. ∎

Remark 5.5 We briefly mention that the λ_2-rank is connected to the embedding problem. For instance, in the quadratic number field $F = \mathbb{Q}(\sqrt{d})$, the solvability of the conics $X^2 = aY^2 + \frac{d}{a}Z^2$ yields unramified cyclic quartic extensions of F. The solvability of this conic is related to the λ_2-rank of Cl_F, which is computed by the Rédei matrix in terms of Legendre symbols. Then the embedding problem for F is not solvable. On the other hand, in our context, the embedding problem for Artin–Schreier extensions K over k is solvable and every finite place of k is wildly ramified in K.

6 Implementation results

In this section, as implementation results, we explicitly present concrete infinite families of Artin–Schreier extensions over k whose ideal class groups have guaranteed prescribed λ_n-rank of the ideal class group for $1 \le n \le 3$. In Table 1, for a given positive integer t, we obtain explicit families of Artin–Schreier extensions K over k whose λ_1-rank of the ideal class group Cl_K is t and λ_n-rank is zero for $n \ge 2$, depending on the ramification behavior of the infinite place ∞ of k (Theorems 3.2–3.4). Furthermore, in Table 2, for a given integer $t \ge 2$, we get explicit families of Artin–Schreier extensions

t	p	q	$D = \sum Q_i/(P_i^{r_i}) + f$	Ideal class group	Divisor class group	∞
1	2	2	$\frac{1}{T} + T + \zeta$	$\mathbb{Z}_2$		
			$\frac{1}{T^3} + T + \zeta$	$\mathbb{Z}_2 \times \mathbb{Z}_{13}$		
			$\frac{1}{T^3+\zeta T^2+1} + T^3 + \zeta T^2 + \zeta^2$	$\mathbb{Z}_2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5$		
	3	3^2	$\frac{1}{T^2} + T + \zeta$	$(\mathbb{Z}_2)^4 \times \mathbb{Z}_3 \times \mathbb{Z}_{13}$		
			$\frac{1}{(T+\zeta)^2} + T^2 + T + 1$	$\mathbb{Z}_3 \times \mathbb{Z}_{13} \times \mathbb{Z}_{103}$		
			$\frac{T+\zeta^5}{T^2+\zeta^3 T+1} + T^4 + \zeta^3 T^3 + T^2 + \zeta$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_{79} \times \mathbb{Z}_{139}$		Totally
2	2	2^2	$\frac{T+1}{T^3} + \frac{T}{T^3+T+1} + T^3 + T + \zeta$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_5 \times \mathbb{Z}_{101}$		ramified
			$\frac{T+1}{T^3} + \frac{T}{T^3+T+1} + T^5 + T^3 + T^2 + \zeta$	$(\mathbb{Z}_2)^2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5 \times \mathbb{Z}_{5^2}$		
	3	3^2	$\frac{\zeta T+\zeta^3}{T^2} + \frac{\zeta T}{T^2+\zeta T+\zeta^3} + T^2 + \zeta T + \zeta^5$	$(\mathbb{Z}_2)^2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_{19} \times \mathbb{Z}_{9643}$		
			$\frac{T+\zeta^6}{T^2} + \frac{T}{T^2+T+\zeta^7} + T^2 + T + \zeta$	$(\mathbb{Z}_3)^3 \times \mathbb{Z}_{223} \times \mathbb{Z}_{10789}$		
1	2	2^2	$\frac{T+1}{T^3} + \frac{\zeta(T+1)}{T^3+T+1}$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5 \times \mathbb{Z}_7$	
			$\frac{\zeta T^2+T}{(T+1)^3} + \frac{1}{T^3+\zeta^2 T^2+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_{83}$	
			$\frac{\zeta T^2+T}{(T+1)^3} + \frac{\zeta}{T^3+\zeta^2 T^2+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_{71}$	
	3	3^2	$\frac{1}{T^2} + \frac{\zeta T+\zeta^6}{T^2+2T+\zeta}$	$\mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_{1069}$	
			$\frac{T^3+\zeta^5 T}{(T+\zeta)^4} + \frac{\zeta^5}{T+\zeta^2}$	$\mathbb{Z}_3$	$\mathbb{Z}_3 \times (\mathbb{Z}_{23})^2 \times \mathbb{Z}_{37}$	Splits
			$\frac{T+\zeta^5}{T^2+\zeta^3 T+1} + \frac{\zeta^3 T+\zeta^3}{(T+\zeta^3)^2}$	$(\mathbb{Z}_{2^2})^2 \times \mathbb{Z}_3 \times \mathbb{Z}_7$	$(\mathbb{Z}_{2^2})^2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_{37}$	completely

Table 1: Infinite families of Artin–Schreier extensions $K = k(\alpha_D)$ over k whose λ_1-rank of the ideal class groups is t and λ_n-rank is zero for $n \geq 2$, where $\alpha_D^p - \alpha_D = D$.

t	p	q	$D = \sum Q_i/(P_i^{r_i}) + f$	Ideal class group	Divisor class group	∞
2	2	2^2	$\frac{T+1}{T^3} + \frac{T}{T^3+T+1} + \frac{T^2+\zeta T+\zeta^2}{(T+\zeta)^5}$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^2 \times (\mathbb{Z}_{3^2}) \times \mathbb{Z}_{17} \times \mathbb{Z}_{37}$	
			$\frac{T+1}{T^3} + \frac{\zeta}{T^3+T+1} + \frac{T^2+\zeta T+\zeta^2}{(T+\zeta)^5}$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_7 \times \mathbb{Z}_{13} \times \mathbb{Z}_{17}$	
	3	3^2	$\frac{\zeta T+\zeta^3}{T^2} + \frac{\zeta T}{T^2+\zeta T+\zeta^3} + \frac{\zeta^5 T+\zeta^2}{(T+\zeta)^2}$	$(\mathbb{Z}_3)^2$	$(\mathbb{Z}_3)^2 \times \mathbb{Z}_{3434467}$	
			$\frac{\zeta T+\zeta^3}{T^2} + \frac{\zeta T}{T^2+\zeta T+\zeta^3} + \frac{\zeta^2}{(T+\zeta)^2}$	$(\mathbb{Z}_3)^2$	$(\mathbb{Z}_3)^2 \times \mathbb{Z}_{31} \times \mathbb{Z}_{139} \times \mathbb{Z}_{1279}$	
1	2	2^2	$\frac{1}{T} + \zeta$	$\mathbb{Z}_2$	Identity	
			$\frac{T^2+\zeta T+1}{T^3} + \zeta^2$	$\mathbb{Z}_2 \times \mathbb{Z}_5$	$\mathbb{Z}_5$	
			$\frac{1}{T^3+\zeta T^2+1} + \zeta$	$\mathbb{Z}_2 \times \mathbb{Z}_{17}$	$\mathbb{Z}_{17}$	
	3	3^2	$\frac{1}{T^2} + 1$	$\mathbb{Z}_3 \times \mathbb{Z}_7$	$\mathbb{Z}_7$	
			$\frac{T^3+\zeta^5 T}{(T+\zeta)^4} + 2$	$(\mathbb{Z}_{2^2})^2 \times \mathbb{Z}_3 \times (\mathbb{Z}_5)^2$	$(\mathbb{Z}_{2^2})^2 \times (\mathbb{Z}_5)^2$	
			$\frac{T+\zeta^5}{T^2+\zeta^3 T+1} + \zeta^7$	$\mathbb{Z}_3 \times \mathbb{Z}_{97}$	$\mathbb{Z}_{97}$	Inert
2	2	2^2	$\frac{T+1}{T^3} + \frac{T}{T^3+T+1} + \zeta$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_{113}$	$\mathbb{Z}_2 \times \mathbb{Z}_{113}$	
			$\frac{T+1}{T^5} + \frac{T}{T^3+T+1} + \zeta^2$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_{227}$	$\mathbb{Z}_2 \times \mathbb{Z}_{277}$	
	3	3^2	$\frac{\zeta T+\zeta^3}{T^2} + \frac{\zeta T}{T^2+\zeta T+\zeta^3} + \zeta^5$	$(\mathbb{Z}_3)^3 \times (\mathbb{Z}_{2^2})^2 \times \mathbb{Z}_{463}$	$(\mathbb{Z}_{2^2})^2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_{463}$	
			$\frac{T+\zeta^6}{T^2} + \frac{T}{T^2+T+\zeta^7} + \zeta^3$	$(\mathbb{Z}_3)^2 \times (\mathbb{Z}_5)^2 \times \mathbb{Z}_{151}$	$\mathbb{Z}_3 \times (\mathbb{Z}_5)^2 \times \mathbb{Z}_{151}$	

Table 1: Continued.

t	$p=q$	$D=\sum Q_i/(P_i^{r_i})+f$	Ideal class group	Divisor class group	∞
2	2	$\frac{T+1}{T^3}+\frac{T}{T^3+T+1}+T+1$		$\mathbb{Z}_2\times\mathbb{Z}_{2^2}\times\mathbb{Z}_5$	
		$\frac{T+1}{T^3}+\frac{T}{T^3+T+1}+T^5+T^2+T$		$\mathbb{Z}_2\times\mathbb{Z}_{2^2}\times\mathbb{Z}_{7^2}$	Totally
		$\frac{T^2}{(T+1)^3}+\frac{T+1}{T^2+T+1}+T^5+T^2+T+1$		$\mathbb{Z}_2\times\mathbb{Z}_{2^2}\times\mathbb{Z}_{17}$	ramified
	3	$\frac{1}{T^2+T+2}+\frac{1}{(T^2+1)^2}+T^2+2T+1$		$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{3^2}\times\mathbb{Z}_{7^2}\times\mathbb{Z}_{157}$	
		$\frac{1}{T^2+T+2}+\frac{1}{(T^2+1)^2}+2T^2+T+2$		$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{3^2}\times\mathbb{Z}_{751}$	
		$\frac{1}{T^2+T+2}+\frac{1}{(T^2+1)^2}+T^2+T+2$		$\mathbb{Z}_{2^2}\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{3^2}\times\mathbb{Z}_{127}$	
	2	$\frac{T+1}{T^3}+\frac{T+1}{T^2+T+1}+\frac{T}{T^3+T+1}$	$\mathbb{Z}_2\times\mathbb{Z}_{2^2}$	$\mathbb{Z}_2\times\mathbb{Z}_{2^3}\times\mathbb{Z}_5$	
		$\frac{T+1}{T^3}+\frac{T^5+T^3+1}{(T^2+T+1)^5}+\frac{T}{T^3+T+1}$		$\mathbb{Z}_2\times\mathbb{Z}_{2^2}\times\mathbb{Z}_5\times\mathbb{Z}_{29}$	Splits
		$\frac{T^4}{(T^2+T+1)^3}+\frac{T^2+T+1}{(T^4+T+1)^3}+\frac{T^5+1}{(T^3+T^2+1)^3}$		$\mathbb{Z}_2\times\mathbb{Z}_{2^2}\times\mathbb{Z}_{17}\times\mathbb{Z}_{8839}$	completely

Table 2: Infinite families of Artin–Schreier extensions $K=k(\alpha_D)$ over k whose λ_1-rank of the ideal class groups is t, λ_2-rank is $t-1$, and λ_3-rank is $t-2$, where $\alpha_D^p-\alpha_D=D$.

t	$p = q$	$D = \sum Q_i/(P_i^{r_i}) + f$	Ideal class group	Divisor class group	∞
	3	$\frac{T^3+T+1}{(T^2+T+2)^2} + \frac{T^4+T^2+1}{(T^3+2T^2+1)^2} + \frac{T^4+2T+2}{(T^3+2T+1)^2}$	$(\mathbb{Z}_2)^2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_{3^2}$	$(\mathbb{Z}_2)^2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_{3^2}$ $\times \mathbb{Z}_{13} \times \mathbb{Z}_{787} \times \mathbb{Z}_{1693}$	
		$\frac{T^3+T+1}{(T^2+T+2)^2} + \frac{T^4+T^3+T^2+1}{(T^3+2T^2+1)^2} + \frac{T^4+2T+2}{(T^3+2T+1)^2}$		$(\mathbb{Z}_2)^2 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_7$ $\times \mathbb{Z}_{13} \times \mathbb{Z}_{103} \times \mathbb{Z}_{84211}$	
		$\frac{T^3+T+1}{(T^2+T+2)^2} + \frac{T^4+T^2+1}{(T^3+2T^2+1)^2} + \frac{T^4+2T+2}{(T^3+T^2+2)^2}$		$(\mathbb{Z}_{2^2})^2 \times (\mathbb{Z}_3)^2 \times (\mathbb{Z}_3)^2$ $\times \mathbb{Z}_{61} \times \mathbb{Z}_{327667}$	
	2	$\frac{T+1}{T^2+T+1} + \frac{T^3}{T^4+T^3+T^2+T+1} + 1$	$\mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{5^2}$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_{5^2}$	
		$\frac{1}{T^2+T+1} + \frac{T^3}{T^4+T^3+T^2+T+1} + 1$	$\mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_7$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_7$	
		$\frac{T+1}{T^2+T+1} + \frac{T^3}{T^4+T+1} + 1$	$\mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_7$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_3 \times \mathbb{Z}_7$	
	3	$\frac{T+2}{T^3+2T+1} + \frac{T^2+1}{(T^3+2T^2+1)^2} + 1$	$(\mathbb{Z}_3)^2 \times (\mathbb{Z}_{3^2})^2 \times \mathbb{Z}_{13} \times \mathbb{Z}_{379}$	$\mathbb{Z}_3 \times (\mathbb{Z}_{3^2})^2 \times \mathbb{Z}_{13} \times \mathbb{Z}_{379}$	Inert
		$\frac{2T^2+2T+2}{T^3+2T+1} + \frac{T^2+1}{(T^3+2T^2+1)^2} + 1$	$(\mathbb{Z}_3)^3 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{19} \times \mathbb{Z}_{433}$	$(\mathbb{Z}_3)^4 \times \mathbb{Z}_{19} \times \mathbb{Z}_{433}$	
		$\frac{2T^2+2}{T^3+2T+1} + \frac{T^2+1}{(T^3+2T^2+1)^2} + 1$	$(\mathbb{Z}_3)^2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_7 \times \mathbb{Z}_{1303}$	$(\mathbb{Z}_3)^3 \times \mathbb{Z}_7 \times \mathbb{Z}_{1303}$	

Table 2: Continued.

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over k whose λ_1-rank of the ideal class groups is t, λ_2-rank is $t-1$, and λ_3-rank is $t-2$ (Theorem 5.1). In the tables, we denote $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}_m$ for a positive integer m.

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References

[1] B. Anglés and J. Jaulent, *Théorie des genres des corps globaux*. Manuscripta Math. **101**(2000), 513–532.
[2] S. Bae, S. Hu, and H. Jung, *The generalized Rédei-matrix for function fields*. Finite Fields Appl. **18**(2012), 760–780.
[3] S. Bae and J. K. Koo, *Genus theory for function fields*. J. Aust. Math. Soc. Ser. A **60**(1996), 301–310.
[4] V. Bautista-Ancona, J. Diaz-Vargas, and J. A. Lara Rodríguez, *Imaginary quadratic function fields with ideal class group of prime exponent*. J. Number Theory **173**(2017), 243–253.
[5] P. E. Conner and J. Hurrelbrink, *Class number parity*, World Scientific, Singapore, 1988.
[6] C. F. Gauss, *Disquisitiones arithmeticae*, Springer, New York, 1801.
[7] H. Hasse, *Theorie der relativ-zyklischen algebraischen Funktionenkörper, insbesondere bei endlichem Konstantenkörper*. J. Reine Angew. Math. **172**(1934), 37–54.
[8] S. Hu and Y. Li, *The genus fields of Artin–Schreier extensions*. Finite Fields Appl. **16**(2010), 255–264.
[9] A. Iadarola, *On the 8-rank of quadratic class groups*, Master's thesis, Universiteit Leiden, 2017.
[10] H. Ichimura, *Quadratic function fields whose class numbers are not divisible by three*. Acta Arith. **91**(1999), no. 2, 181–190.
[11] Y. Lee and J. Yoo, *Indivisibility of divisor class numbers of Kummer extensions over the rational function field*. J. Number Theory **192**(2018), 270–292.
[12] R. Lidl and H. Niederreiter, *Finite fields*, Cambridge University Press, Cambridge, 2003.
[13] G. Peng, *The ideal class group of function fields*. J. Sichuan Univ. **36**(1999), no. 4, 652–659.
[14] G. Peng, *The genus fields of Kummer function fields*. J. Number Theory **98**(2003), 221–227.
[15] L. Rédei and H. Reichardt, *Die anzahl der durch 4 teilbaren invarianten der Klassengruppe eines beliebigen quadratischen zahlkörpers*. J. Reine Angew. Math. **170**(1933), 69–74.
[16] M. Rosen, *Ambiguous divisor classes in function fields*. J. Number Theory **9**(1977), 160–174.
[17] M. Rosen, *Number theory in function fields*, Springer, New York, 2002.
[18] H. Stichtenoth, *Algebraic function fields and codes*, Springer, Berlin–Heidelberg, 2009.
[19] C. Wittmann, *l-class groups of cyclic function fields of degree l*. Finite Fields Appl. **13**(2007), 327–347.
[20] C. Wittmann, *l-parts of divisor class groups of cyclic function fields of degree l*. Int. J. Number Theory **3**(2007), no. 2, 171–190.
[21] J. Yoo, *Class groups of cyclic extensions over the rational function field*, Dissertation, Ewha Womans University, 2019.
[22] J. Yoo and Y. Lee, *Infinite families of cyclotomic function fields with any prescribed class group rank*. J. Pure Appl. Algebra **225**(2021), no. 9, Article no. 106658, 22 pp.
[23] Q. Yue, *The generalized Rédei-matrix*. Math. Z. **261**(2009), no. 1, 23–37.
[24] Z. Zhao, *3-class groups of cubic cyclic function fields*. Turk. J. Math. **42**(2018), no. 5, 2607–2620.
[25] Z. Zhao and W. Hu, *On l-class groups of global function fields*. Int. J. Number Theory **12**(2016), no. 2, 341–356.

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