VORTICITY AVERAGES

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CONTENTS

1. Introduction. Recent studies of turbulent fluid motions have drawn attention to the transfer of vorticity. There have been some attempts to study turbulence by plane models, but these have been criticized justly for failing to reveal the true nature of a phenomenon which depends essentially on three-dimensional convection. I have thought it worthwhile to eschew current conjectures regarding turbulence, turning in preference to the method of a master of the theory of vortices and applying it to the discovery of certain average properties of the continuous rotational motion of any medium whatever. This memoir therefore makes no attempt to deal with the problem of turbulence, but it is possible that the theorems presented here, being exact consequences of the kinematical equations, may nevertheless enjoy a certain relevance.

This method was introduced originally to obtain conservation theorems valid in a class of plane motions of an inviscid fluid or of a viscous incompressible fluid. Some time ago the results were substantially generalized in a purely kinematical form, but still subject to the apparently essential but hardly desirable restriction to plane isochoric motion. In this paper I shall show that a somewhat different sequence of conservation theorems, expressed in terms of

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symmetrical vector moments of the vorticity vector, is appropriate to the three-dimensional motion of an arbitrary continuous medium.

It is interesting to discover that in one respect at least three-dimensional motions are simpler in their average properties than are plane motions. The reason lies partly in the different and apparently more complicated mechanism of convection in the three-dimensional case, but more essentially in the fact that *only in a space of three dimensions is the vorticity a vector*. So as to make use of the various cross operations which exist only in three-dimensional space, and which are essential in our subject, I shall employ the suggestive and elegant polyadic notations of Gibbs[1] [5].

In Parts I and II, the former of which is devoted to plane and the latter to three-dimensional motions, are collected all the vorticity integrals known up to the present time. These are presented in a generalized kinematical form, and compared and contrasted with one another. Part III contains two new general conservation theorems [2]. I hope that the relative shortness of this last portion will not be taken as indicative of its relative value, for in the present subject the generality and applicability of a result seems to vary in inverse ratio to the length and complexity of its proof.

Part I. Known Vorticity Integrals for Plane Motions

2. Poincaré's theory of vortices in plane flows of inviscid fluids. In the dynamics of systems of mass-points or of rigid bodies the construction of an integral of the motion represents substantial progress towards a complete solution of the problem. In the dynamics of continuous media, endowed with an infinite number of degrees of freedom, even an infinite number of integrals usually does not suffice for the solution of any specific problem, yet from the earliest days of continuum mechanics such integrals have been sought, sometimes because they display kinematical or physical properties of the medium affording some insight and grasp, sometimes because they enable the transformation of one problem into another or the construction of analogies between different disciplines, and sometimes for their mathematical elegance. The second of these possibilities suggested to Poincaré [18, §§65, 75, 123] that since the vorticity

$$w \equiv \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \tag{2.1}$$

of a plane motion of an inviscid incompressible fluid subject to conservative extraneous force remains constant for each particle, it might well be taken as the analogue of the mass-density in rigid dynamics. Introducing the "mass"

[1]My only departure from Gibbsian notations is to replace ∇, $\nabla\cdot$, and $\nabla\times$, by grad, div, curl, respectively, and to use boldface letters according to the convention customary in works in mathematical physics.

[2]The first of these has been announced in [25].

(2.2) $$\mathfrak{W} \equiv \int_{\mathfrak{S}} d\mathfrak{S}w,$$

the "centre of mass" $\mathbf{r}_w$:

(2.3) $$\mathfrak{W}\mathbf{r}_w \equiv \int_{\mathfrak{S}} d\mathfrak{S}w\mathbf{r},$$

where $\mathbf{r}$ is the radius vector, and the "moment of inertia"

(2.4) $$\mathfrak{J} \equiv \int_{\mathfrak{S}} d\mathfrak{S}wr^2,$$

Poincaré proved that, subject to suitable conditions of regularity of the velocity field, when the area occupied by the fluid is the whole plane these quantities are constant in time, and thus provide integrals of the motion. In the special case of an irrotational motion induced by vortices the integrals are to be replaced by finite sums over the vortices, and the conservation theorems are still valid, but now Poincaré obtained for the motion of the vortices equations of the same form as Hamilton's equations for mass-points, so that the integrals represent substantial progress towards the complete integration, and indeed when there are but three vortices present the problem is reduced to quadratures [18, §77].

3. Hamel's integrals. There are two ways in which one might seek to generalize Poincaré's results, still considering only plane motions. First, one might try to extend the validity of his integrals to more general media. Second, one might attempt to form other quantities

(3.1) $$\mathfrak{K} \equiv \int_{\mathfrak{S}} d\mathfrak{S}w\sum a_{ij}x^iy^j$$

of polynomial moment type, or perhaps still more general expressions, which are constant in time. Poincaré himself followed the first course, investigating the behaviour of his three integrals in motions of homogeneous viscous incompressible fluids filling the whole plane. He showed that the "mass" and "centre of mass" remain constant, while the "moment of inertia" satisfies the equation

(3.2) $$\frac{d\mathfrak{J}}{dt} = 4\nu\mathfrak{W},$$

where ν is the kinematic viscosity [18, §§154-156]. A result in accord with the second course was obtained incidentally in a different sort of investigation by Hamel[3] [6]; it may be expressed in the following way:

[3]I learned of (3.3) through its rediscovery by Kampé de Feriet [8, 9]. For the reference to Hamel's work I am indebted to Professor Weinstein and to the referee; the former has called my attention also to a paper by Zaremba [29], where the result is apparently employed without explicit statement, and to the proofs of the more difficult converse by Weyl [28] and himself [27]. All these authors state the result in terms of potential theory, and only Kampé de Feriet observes that it can be interpreted as the vorticity theorem above.

Let w be the vorticity of a continuously differentiable plane isochoric[4] *motion such that upon the boundary $\mathfrak{C}$ of a finite domain $\mathfrak{S}$ the material adheres without slipping; then if ϕ be a function harmonic in $\mathfrak{S}$,*

$$\int_{\mathfrak{S}} d\mathfrak{S} w\phi = 0. \tag{3.3}$$

4. Comparison and contrast of the integrals of Poincaré and Hamel: their limitations. In the special cases $\phi = 1, x, y$ Hamel's integrals reduce to Poincaré's "mass" and to the products of the "mass" by the components of the "centre of mass", respectively, but $\phi = x^2 + y^2$ is not a harmonic function, and from Hamel's theorem we obtain

$$\int_{\mathfrak{S}} d\mathfrak{S} w(x^2 - y^2) = 0, \qquad \int_{\mathfrak{S}} d\mathfrak{S} wxy = 0, \tag{4.1}$$

in place of Poincaré's "moment of inertia" integral. In fact Kampé de Feriet [8] notes that Hamel's theorem yields two linear relations connecting the various special moments of type (3.1) with $i + j = n$, for any n.

Poincaré's results apply to motions filling the whole plane while Hamel's apply to motions within a finite area. This distinction is not essential, since the results of either author are easily generalized to motions bounded in part by finite walls to which the material adheres without slipping and in part unbounded, subject to certain conditions on the manner in which the motion must vanish at infinity. The second apparent difference, that Poincaré's results concern the constancy of certain integrals while Hamel's concern the vanishing of certain integrals, is also inessential, for the method of both authors—indeed it is the method of Poincaré's whole theory of vortices—is to transform a surface integral into a line integral and then state conditions strong enough to secure the vanishing of this latter quantity. Now if such a transformation exist for a given integral, there is also necessarily a similar transformation for its time rate of change, which consequently can be shown to vanish under somewhat weaker conditions.

While Hamel's theorem is most elegant, I cannot help feeling that somehow it fails to reveal the essential average properties of vorticity because it is restricted to isochoric motions[5], which represent an extreme idealization. Now in view of Ampère's transformation

$$\int_{\mathfrak{S}} d\mathfrak{S} w = \oint_{\mathfrak{C}} (v_x dx + v_y dy), \tag{4.2}$$

the special case $\phi = 1$ is not so restricted, but is valid for any continuously differentiable motion. In the course of this investigation I was unable to rid

[4]An isochoric motion is one in which material volumes are preserved: div $\mathbf{v} = 0$.

[5]In Hamel's treatment this restriction arises through the consideration of the Laplacian of the stream-function, which is proportional to the vorticity only in isochoric motions, and indeed a single stream-function does not necessarily exist in the more general case.

these theorems of this unfortunate limitation, so long as plane motion be the subject of enquiry.

This point has now been clarified by Synge [31], who has shown that Hamel's theorem is a special case of a *characterization* of plane fields defined within and vanishing upon a single closed boundary, the general result no longer yielding a simple vorticity theorem since it depends (as would be expected) upon the expansion θ as well. I have constructed two analogous characterizations [32, 33] for the three-dimensional case, again in terms of integrals whose integrands contain both $\mathbf{w}$ and θ, integrals which in the special case $\theta = 0$ reduce to the theorems IV and I of Berker, respectively, given as formulae (9.14) and (6.10) below. The present memoir does not seek characterizations, aiming rather to formulate conditions sufficient that certain quantities be integrals of the motion.

Part II. Known Vorticity Integrals for Three-dimensional Motions

5. Plan of the analysis of three-dimensional motions. In a three-dimensional motion the vorticity

$$\mathbf{w} \equiv \operatorname{curl} \mathbf{v} \tag{5.1}$$

is an axial vector. Before Poincaré's study of the vorticity of plane motions, Lamb had demonstrated the vanishing of a volume integral of $\mathbf{w}$, but it involves a product of $\mathbf{w}$ by $\mathbf{v}$ rather than by $\mathbf{r}$, and it applies only to isochroric motions. Poincaré himself exhibited a second integral of this type. Both these results, and their recent generalization by Berker, are discussed in §**6**. If we were to proceed by analogy to Poincaré's treatment of the plane case we might introduce and analyze such moments as

$$\int_{\mathfrak{B}} \mathbf{w}\, d\mathfrak{B}, \qquad \int_{\mathfrak{B}} \mathbf{r} \cdot \mathbf{w}\, d\mathfrak{B}, \qquad \int_{\mathfrak{B}} \mathbf{r} \times \mathbf{w}\, d\mathfrak{B}, \qquad \int_{\mathfrak{B}} r^2 \mathbf{w}\, d\mathfrak{B}.$$

From results of Moreau published a few months ago it is shown in §**8** that the third and fourth of these moments are integrals of the motion only in case it is circulation-preserving and isochoric. The first, however, was shown to be an integral under rather general circumstances by A. Föppl and Jaffé, and a year ago Berker showed that the second also is a possible integral; these results are outlined in §**9**.

The foregoing rather limited gleaning might suggest that as in plane motion so also in the three-dimensional case the search for vorticity integrals quickly reaches a point of diminishing returns. In fact, however, all previous investigators of this subject have cultivated an unwilling soil. By planting the fresh seed of *symmetrical vector moments* of a vector we shall find in §§**11-12** that Poincaré's method of reduction—in this case, reduction of a volume integral to a surface integral, which vanishes subject to certain simple conditions —is always applicable, and will enable us to reap an unexpectedly rich harvest

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of integrals. Thus it will appear that Berker's scalar moment may be an integral because under similar circumstances the symmetrical vector moment

$$\int_{\mathfrak{V}}(\mathbf{rw} + \mathbf{wr})d\mathfrak{V},$$

of which it is the scalar, may be an integral. On the other hand, Moreau's second scalar moment cannot be expected to be an integral in general, since the symmetrical second vector moment

$$\int_{\mathfrak{V}}(\mathbf{rrw} + \mathbf{rwr} + \mathbf{wrr})d\mathfrak{V}$$

may well be an integral, whence by contraction arises not Moreau's moment but rather

$$\int_{\mathfrak{V}}(r^2\mathbf{w} + 2\mathbf{r}\cdot\mathbf{wr})d\mathfrak{V}.$$

6. The integrals of Lamb, Poincaré, and Berker for isochoric motions. The analysis of this section rests upon two easily demonstrated forms of Green's transformation[6]:

$$\oint_{\mathfrak{S}}[d\mathfrak{S}\cdot(\mathbf{bc} + \mathbf{cb}) - d\mathfrak{S}\mathbf{b}\cdot\mathbf{c}]$$

$$= \int_{\mathfrak{V}}[\mathrm{curl}\mathbf{c} \times \mathbf{b} + \mathrm{curl}\mathbf{b} \times \mathbf{c} + \mathbf{b}\,\mathrm{div}\,\mathbf{c} + \mathbf{c}\,\mathrm{div}\,\mathbf{b}]d\mathfrak{V}, \tag{6.1}$$

$$\oint_{\mathfrak{S}}[d\mathfrak{S}\cdot(\mathbf{bc} + \mathbf{cb})\mathbf{r} - d\mathfrak{S}\mathbf{b}\cdot\mathbf{cr}]$$

$$= \int_{\mathfrak{V}}[\mathbf{bc}+\mathbf{cb}-\mathbf{c}\cdot\mathbf{bI}+(\mathrm{curl}\mathbf{c}\times\mathbf{b}+\mathrm{curl}\mathbf{b}\times\mathbf{c}+\mathbf{b}\,\mathrm{div}\,\mathbf{c}+\mathbf{c}\,\mathrm{div}\,\mathbf{b})\mathbf{r}]d\mathfrak{V}, \tag{6.2}$$

for whose validity it is sufficient that $\mathbf{b}$ and $\mathbf{c}$ be continuously differentiable within $\mathfrak{V}$ and continuous upon $\mathfrak{S}$.

In (6.1) put $\mathbf{b} = \mathbf{c} = \mathbf{v}$ and suppose $\mathrm{div}\,\mathbf{v} = 0$; there results

$$\oint_{\mathfrak{S}}[d\mathfrak{S}\cdot\mathbf{vv} - d\mathfrak{S}\tfrac{1}{2}v^2] = \int_{\mathfrak{V}}\mathbf{w} \times \mathbf{v}d\mathfrak{V}. \tag{6.3}$$

By formulating conditions under which the surface integral vanishes, we obtain the integral of Lamb [12, 13]:

In a continuously differentiable isochoric motion, if the material adhere without slipping ($\mathbf{v} = 0$) *upon all finite boundaries, while in any portion of the material extending to infinity the condition $rv \to 0$ as $r \to \infty$ be satisfied, then*

$$\int_{\mathfrak{V}}\mathbf{w} \times \mathbf{v}d\mathfrak{V} = 0. \tag{6.4}$$

[6]In [25, 26] it is shown that these two transformations are the cases $n = 0$ and $n = 1$ of a new general form of Green's transformation containing an arbitrary positive integer n.

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Notice that while this theorem applies to a wide class of motions of viscous incompressible fluids, including all regular motions in a bounded domain of any connectivity, since the adherence condition is not satisfied in the absence of friction the result can hold for motions of incompressible perfect fluids only in case they fill all of space.

In (6.2) put $\mathbf{b} = \mathbf{c} = \mathbf{v}$ and suppose div $\mathbf{v} = 0$; there results

$$\oint_{\mathfrak{S}}[d\mathfrak{S}\cdot\mathbf{vvr} - d\mathfrak{S}\mathbf{r}\tfrac{1}{2}v^2] = \int_{\mathfrak{V}}[\mathbf{vv} - \tfrac{1}{2}v^2\mathbf{I} + \mathbf{w}\times\mathbf{vr}]d\mathfrak{V}. \tag{6.5}$$

Taking the scalar of this result yields the kinetic energy formula of Lamb[7] [12, 13] and J. J. Thomson [21]:

$$\tfrac{1}{2}\int_{\mathfrak{V}} v^2 d\mathfrak{V} = \oint_{\mathfrak{S}} d\mathfrak{S}\cdot(\tfrac{1}{2}v^2\mathbf{r} - \mathbf{vv}\cdot\mathbf{r}) + \int_{\mathfrak{V}}\mathbf{r}\cdot\mathbf{w}\times\mathbf{v}d\mathfrak{V}, \tag{6.6}$$

while taking the vector yields

$$\oint_{\mathfrak{S}}[d\mathfrak{S}\cdot\mathbf{vr}\times\mathbf{v} + d\mathfrak{S}\times\mathbf{r}\tfrac{1}{2}v^2] = \int_{\mathfrak{V}}\mathbf{r}\times(\mathbf{w}\times\mathbf{v})d\mathfrak{V}, \tag{6.7}$$

whence we obtain the integral of Poincaré [18, §115]:

In a continuously differentiable isochoric motion, if the material adhere without slipping ($\mathbf{v} = 0$) *upon all finite boundaries, while in any portion of the material extending to infinity the condition* $r^3v^2 \to 0$ *as* $r \to \infty$ *be satisfied, then*

$$\int_{\mathfrak{V}}\mathbf{r}\times(\mathbf{w}\times\mathbf{v})d\mathfrak{V} = 0. \tag{6.8}$$

Notice the conditions for the validity of Poincaré's integral insure the simultaneous validity of Lamb's integral.

In (6.1) put $\mathbf{c} = \mathbf{v}$ and suppose div $\mathbf{v} = 0$, and suppose further that curl $\mathbf{b} = 0$, div $\mathbf{b} = 0$ (i.e. $\mathbf{b} = \text{grad}\,\phi$, where $\nabla^2\phi = 0$); there results

$$\oint_{\mathfrak{S}}[d\mathfrak{S}\cdot(\mathbf{bv} + \mathbf{vb}) - d\mathfrak{S}\mathbf{v}\cdot\mathbf{b}] = \int_{\mathfrak{V}}\mathbf{w}\times\mathbf{b}d\mathfrak{V}, \tag{6.9}$$

whence we obtain the theorem IV of Berker [2, 30]:

Given a continuously differentiable isochoric motion such that upon all finite boundaries the material adheres without slipping, then for any continuously differentiable harmonic vector $\mathbf{b}$ *we have*

$$\int_{\mathfrak{V}}\mathbf{w}\times\mathbf{b}d\mathfrak{V} = 0, \tag{6.10}$$

provided that in any portion of the material extending to infinity the condition $bvr^2 \to 0$ *be satisfied.*

[7]Notice that the conservation of the actual kinetic energy in a plane motion is equivalent to the conservation of the "moment of momentum" in the analogy; of [18, §76].

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7. Three basic formulae of kinematics. It was shown by Euler [3] that $D \log d\mathfrak{V}/Dt = \operatorname{div} \mathbf{v}$. Hence if $\mathfrak{V}$ be a finite material volume and if Σ be any polyadic, subject to the usual conditions for differentiating an integral we have

$$\frac{D}{Dt}\int_{\mathfrak{V}} \Sigma\, d\mathfrak{V} = \int_{\mathfrak{V}} \left(\frac{D\Sigma}{Dt} + \Sigma \operatorname{div} \mathbf{v} \right) d\mathfrak{V}. \tag{7.1}$$

By an application of Green's transformation to (7.1) we obtain the *transport theorem* of Reynolds[8] [19]:

$$\frac{D}{Dt}\int_{\mathfrak{V}} \Sigma\, d\mathfrak{V} = \frac{\partial}{\partial t}\int_{\mathfrak{V}} \Sigma\, d\mathfrak{V} + \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{v}\Sigma. \tag{7.2}$$

By calculating the curl of the acceleration it is easy to deduce the vorticity equation of Lagrange and Beltrami[9]:

$$\frac{D\mathbf{w}}{Dt} = \operatorname{curl} \mathbf{a} + \mathbf{w} \cdot \operatorname{grad} \mathbf{v} - \mathbf{w} \operatorname{div} \mathbf{v}. \tag{7.3}$$

In the results of the succeeding sections two boundary conditions are employed. First, upon any stationary boundary

$$d\mathfrak{S} \cdot \mathbf{v} = 0. \tag{7.4}$$

Second, upon a stationary boundary to which the material adheres without slipping ($\mathbf{v} = 0$), the vorticity is wholly tangential [14]:

$$d\mathfrak{S} \cdot \mathbf{w} = 0. \tag{7.5}$$

In what follows let b_r denote the radial component of $\mathbf{b}$, let $\mathbf{b}_t$ denote the projection of $\mathbf{b}$ onto a plane perpendicular to $\mathbf{r}$, and let b_t denote the magnitude of $\mathbf{b}_t$. Similarly, for a dyadic Σ, by Σ_r we shall mean its projection in the radial direction, and by $|\Sigma_r|$ the magnitude of that projection.

8. The integrals of Moreau for isochoric circulation-preserving motions. Let us decompose the acceleration $\mathbf{a}$ into a "conservative" portion grad W and a "non-conservative" portion $\mathbf{a}'$:

$$\mathbf{a} = \operatorname{grad} W + \mathbf{a}'. \tag{8.1}$$

Such a resolution is possible in an infinite number of ways, and of itself has no kinematical significance; in a dynamical application, however, some particular decomposition of this type often has special meaning. In any case curl $\mathbf{a}$ =

[8]Curiously enough Reynolds saw fit to lay down this easily demonstrated transformation as a postulate in his theory of matter; still more curiously, in the literature of physics it is often applied tacitly in special cases as if it were "obvious" (i.e., too difficult to prove). The only text book of vector analysis in which I have found it mentioned is [20].

[9]The case curl $\mathbf{a} = 0$, which in the still more special case div $\mathbf{v} = 0$ is generally called "Helmholtz's equation", was first derived by Lagrange [10]. The equation (7.3) is given by Beltrami [1].

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curl $\mathbf{a}'$. If there exist such a decomposition with $\mathbf{a}' = 0$, Kelvin's theorem of circulation holds and we may call the motion a *circulation-preserving motion.* Moreau [15, 16] has recently discussed the dynamics of isochoric motions by a method equivalent to finding the resultant and vector moment of $\mathbf{a}'$, which we shall now derive by a purely kinematical analysis.

By combining the three vectorial identities

$$\operatorname{div} \mathbf{r}\mathbf{a}' = 3\mathbf{a}' + \mathbf{r}\cdot\operatorname{grad} \mathbf{a}', \tag{8.2}$$

$$\operatorname{grad} (\mathbf{r}\cdot\mathbf{a}') = \mathbf{r}\cdot\operatorname{grad} \mathbf{a}' + \mathbf{a}' + \mathbf{r} \times \operatorname{curl} \mathbf{a}', \tag{8.3}$$

$$\operatorname{div} (\mathbf{w}\mathbf{r} \times \mathbf{v}) = \mathbf{w} \times \mathbf{v} + \mathbf{r} \times (\mathbf{w}\cdot\operatorname{grad} \mathbf{v}), \tag{8.4}$$

for any motion we have

$$\begin{aligned} &\mathbf{r} \times [\operatorname{curl} \mathbf{a}' + \mathbf{w}\cdot\operatorname{grad} \mathbf{v}] \\ &\quad = \mathbf{v} \times \mathbf{w} + 2\mathbf{a}' + \operatorname{grad} (\mathbf{r}\cdot\mathbf{a}') + \operatorname{div} (\mathbf{w}\mathbf{r} \times \mathbf{v} - \mathbf{r}\mathbf{a}'). \end{aligned} \tag{8.5}$$

Since curl $\mathbf{a}'$ = curl $\mathbf{a}$, by Euler's transformation (7.1), the Lagrange-Beltrami equation (7.3), and Green's transformation we thus obtain

$$\begin{aligned} \frac{D}{Dt}\int_{\mathfrak{V}} \mathbf{r} \times \mathbf{w}\, d\mathfrak{V} &= \int_{\mathfrak{V}} \left[\mathbf{v} \times \mathbf{w} + \mathbf{r} \times \left(\frac{D\mathbf{w}}{Dt} + \mathbf{w} \operatorname{div} \mathbf{v}\right)\right] d\mathfrak{V}, \\ &= \int_{\mathfrak{V}} [\mathbf{v} \times \mathbf{w} + \mathbf{r} \times (\operatorname{curl} \mathbf{a}' + \mathbf{w}\cdot\operatorname{grad} \mathbf{v})]\, d\mathfrak{V}, \\ &= \int_{\mathfrak{V}} [2\mathbf{v} \times \mathbf{w} + 2\mathbf{a}']\, d\mathfrak{V} + \oint_{\mathfrak{S}} [d\mathfrak{S}\mathbf{r}\cdot\mathbf{a}' + d\mathfrak{S}\cdot(\mathbf{w}\mathbf{r} \times \mathbf{v} - \mathbf{r}\mathbf{a}')]. \end{aligned} \tag{8.6}$$

For an isochoric motion we may express the first term on the right as a surface integral by (6.3):

$$\begin{aligned} \frac{D}{Dt}\int_{\mathfrak{V}} \mathbf{r} \times \mathbf{w}\, d\mathfrak{V} = 2\int_{\mathfrak{V}} \mathbf{a}'\, d\mathfrak{V} + \oint_{\mathfrak{S}} [d\mathfrak{S}(\mathbf{r}\cdot\mathbf{a}' + v^2) \\ + d\mathfrak{S}\cdot(-2\mathbf{v}\mathbf{v} + \mathbf{w}\mathbf{r} \times \mathbf{v} - \mathbf{r}\mathbf{a}')], \end{aligned} \tag{8.7}$$

hence by the transport theorem (7.2) of Reynolds obtaining finally

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathfrak{V}} \mathbf{r} \times \mathbf{w}\, d\mathfrak{V} = 2\int_{\mathfrak{V}} \mathbf{a}'\, d\mathfrak{V} + \oint_{\mathfrak{S}} [d\mathfrak{S}(\mathbf{r}\cdot\mathbf{a}' + v^2) \\ + d\mathfrak{S}\cdot(-2\mathbf{v}\mathbf{v} + \mathbf{w}\mathbf{r} \times \mathbf{v} - \mathbf{v}\mathbf{r} \times \mathbf{w} - \mathbf{r}\mathbf{a}')]. \end{aligned} \tag{8.8}$$

By formulating conditions sufficient that the right-hand side of this formula shall vanish we derive Moreau's first integral:

In a continuously differentiable isochoric circulation-preserving motion all of whose boundaries are stationary, if $w = 0$ and $v =$ const. upon any finite boundary, while in any portion extending to infinity the conditions $vr \to 0$, $r^2|(\mathbf{w}\mathbf{r} \times \mathbf{v} - \mathbf{v}\mathbf{r} \times \mathbf{w})_r| \to 0$ be satisfied, then

$$\int_{\mathfrak{V}} \mathbf{r} \times \mathbf{w} d\mathfrak{V} = \text{const.} \tag{8.9}$$

Note that if the first limit condition $vr \to 0$ be satisfied, then for the second to be satisfied also it is sufficient that $r^2 w$ be bounded.

By combining the three vectorial identities

$$\operatorname{div} (r^2 \mathbf{w}\mathbf{v}) = 2\mathbf{v}\mathbf{r} \cdot \mathbf{w} + r^2 \mathbf{w} \cdot \operatorname{grad} \mathbf{v}, \tag{8.10}$$

$$\mathbf{r} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}\mathbf{r} \cdot \mathbf{w} - \mathbf{w}\mathbf{r} \cdot \mathbf{v}, \tag{8.11}$$

$$\operatorname{curl} (r^2 \mathbf{a}') = r^2 \operatorname{curl} \mathbf{a}' + 2\mathbf{r} \times \mathbf{a}', \tag{8.12}$$

for any motion we have

$$\begin{aligned} &2\mathbf{r} \cdot \mathbf{v}\mathbf{w} + r^2[\operatorname{curl} \mathbf{a}' + \mathbf{w} \cdot \operatorname{grad} \mathbf{v}] \\ &\qquad = -2\mathbf{r} \times (\mathbf{v} \times \mathbf{w}) - 2\mathbf{r} \times \mathbf{a}' + \operatorname{curl} (r^2 \mathbf{a}') + \operatorname{div} (r^2 \mathbf{w}\mathbf{v}). \end{aligned} \tag{8.13}$$

By Euler's transformation (7.1), the Lagrange-Beltrami equation (7.3), and Green's transformation we thus obtain

$$\begin{aligned} -\frac{D}{Dt}\int_{\mathfrak{V}} r^2 \mathbf{w} d\mathfrak{V} &= -\int_{\mathfrak{V}} \left[2\mathbf{r} \cdot \mathbf{v}\mathbf{w} + r^2\left(\frac{D\mathbf{w}}{Dt} + \mathbf{w} \operatorname{div} \mathbf{v}\right)\right] d\mathfrak{V}, \\ &= -\int_{\mathfrak{V}} [2\mathbf{r} \cdot \mathbf{v}\mathbf{w} + r^2(\operatorname{curl} \mathbf{a}' + \mathbf{w} \cdot \operatorname{grad} \mathbf{v})] d\mathfrak{V}, \\ &= \int_{\mathfrak{V}} [2\mathbf{r} \times (\mathbf{v} \times \mathbf{w}) + 2\mathbf{r} \times \mathbf{a}'] d\mathfrak{V} \\ &\qquad - \oint_{\mathfrak{S}} [d\mathfrak{S} \times r^2 \mathbf{a}' + d\mathfrak{S} \cdot r^2 \mathbf{w}\mathbf{v}]. \end{aligned} \tag{8.14}$$

For an isochoric motion we may express the first term on the right as a surface integral by (6.7):

$$\begin{aligned} -\frac{D}{Dt}\int_{\mathfrak{V}} r^2 \mathbf{w} d\mathfrak{V} = \int_{\mathfrak{V}} 2\mathbf{r} \times \mathbf{a}' d\mathfrak{V} - \int_{\mathfrak{S}} &[d\mathfrak{S} \times (\mathbf{r}v^2 + r^2 \mathbf{a}') \\ &+ d\mathfrak{S} \cdot (2\mathbf{v}\mathbf{r} \times \mathbf{v} + r^2 \mathbf{w}\mathbf{v})], \end{aligned} \tag{8.15}$$

hence by the transport theorem (7.2) of Reynolds obtaining finally

$$\begin{aligned} -\frac{\partial}{\partial t}\int_{\mathfrak{V}} r^2 \mathbf{w} d\mathfrak{V} = \int_{\mathfrak{V}} 2\mathbf{r} \times \mathbf{a}' d\mathfrak{V} - \oint_{\mathfrak{S}} &[d\mathfrak{S} \times (\mathbf{r}v^2 + r^2 \mathbf{a}') \\ &+ d\mathfrak{S} \cdot (-r^2[\mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v}] + 2\mathbf{v}\mathbf{r} \times \mathbf{v})]. \end{aligned} \tag{8.16}$$

Thus we have Moreau's second integral:

In a continuously differentiable isochoric circulation-preserving motion all of whose boundaries are stationary, if $w = 0$ and $v =$ const. upon any finite boundary, while in any portion extending to infinity the conditions $r^3v^2 \to 0$, $r^4|(\mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v})_r| \to 0$ be satisfied, then

(8.17) $$\int_{\mathfrak{V}} r^2 \mathbf{w} d\mathfrak{V} = \text{const.}$$

Note that if the first limit condition $r^3v^2 \to 0$ be satisfied, then for the second to be satisfied also it is sufficient that r^5w^2 be bounded. In any case the conditions sufficient for the validity of Moreau's second integral ensure the simultaneous validity of Moreau's first integral.

The kinematical conditions sufficient for the validity of Moreau's integrals are likely to be compatible with the equations of fluid dynamics (if at all) only in the case of a motion of an inviscid incompressible fluid subject to conservative extraneous force and filling all of space.

If all finite boundaries be stationary and if by some chance the non-conservative acceleration $\mathbf{a}'$ and the velocity $\mathbf{v}$ vanish upon them, then Moreau notes that (8.8) and (8.16) respectively reduce to

(8.18) $$\frac{\partial}{\partial t}\int_{\mathfrak{V}} \mathbf{r} \times \mathbf{w} d\mathfrak{V} = 2\int_{\mathfrak{V}} \mathbf{a}' d\mathfrak{V},$$

(8.19) $$-\frac{\partial}{\partial t}\int_{\mathfrak{V}} r^2 \mathbf{w} d\mathfrak{V} = 2\int_{\mathfrak{V}} \mathbf{r} \times \mathbf{a}' d\mathfrak{V}.$$

9. Kelvin's transformation: the general integrals of A. Föppl, Jaffé, and Berker. The integrals obtained by the foregoing analyses are few, and their validity is limited. Some integrals for wider classes of motions may be found by means of a simple transformation apparently first used by Kelvin [22]: Let $\mathbf{c}$ be any solenoidal vector (div $\mathbf{c} = 0$), and let b stand for either a vector or a scalar: then

(9.1) $$\operatorname{div} \mathbf{c} b = \mathbf{c} \cdot \operatorname{grad} b,$$

and hence

(9.2) $$\int_{\mathfrak{V}} \mathbf{c} \cdot \operatorname{grad} b d\mathfrak{V} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{c} b.$$

An immediate three-dimensional generalization of Poincaré's "mass" is the *total vorticity*

(9.3) $$\mathfrak{W} \equiv \int_{\mathfrak{V}} \mathbf{w} d\mathfrak{V},$$

introduced by A. Föppl[10] [4]. Putting $\mathbf{c} = \mathbf{w}$, $b = \mathbf{r}$ in (9.2) we obtain

(9.4) $$\mathfrak{W} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{w}\mathbf{r},$$

and hence Föppl's integral:

[10]The formula (9.4) was apparently known to Föppl, though he did not state it explicitly. It is equation (7) of [17].

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In any continuously differentiable motion such that

a) *upon all finite boundaries the normal component of vorticity vanish, and*

b) *in any portion of the material extending to infinity the condition* $r^3 w_r \to 0$ *be satisfied, then the total vorticity vanishes*:

$$\mathfrak{W} = 0. \tag{9.5}$$

As special cases we notice that the result holds for the material within a closed vortex-tube of any continuously differentiable motion; for the entire material of any continuously differentiable motion in a bounded domain upon whose boundaries $w = 0$; and, by (7.5), for any such motion within finite stationary boundaries to which the material adheres without slipping.

It was shown by Jaffé [7] that Poincaré's theorem of the conservation of "mass" can be extended to three-dimensional motions of viscous incompressible fluids filling all space and vanishing suitably at infinity; that is, the total vorticity of such motions is constant in time. In an earlier paper [23] I showed that Jaffé's theorem can be expressed in a purely kinematical form, valid for any continuous medium; a proof will be given incidentally in §**12** of this memoir.

Three other special cases of Kelvin's transformation have been published recently by Berker[11] [2, 30], as follows.

First, write $\mathbf{f} = \text{grad}\, b$ in (9.2):

$$\int_{\mathfrak{V}} \mathbf{c} \cdot \mathbf{f} d\mathfrak{V} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{c} b. \tag{9.6}$$

Hence we conclude a generalization of the theorem III of Berker:

Let $\mathbf{c}$ *be a continuously differentiable solenoidal field whose normal component upon any finite boundary of* $\mathfrak{V}$ *is zero, and let* $\mathbf{f} = \text{grad}\, b$ *be any continuous laminar field (vector or dyadic), and suppose further that in any portion of* $\mathfrak{V}$ *extending to infinity the condition* $r^2 b c_r \to 0$ *be satisfied; then*

$$\int_{\mathfrak{V}} \mathbf{c} \cdot \mathbf{f} d\mathfrak{V} = 0. \tag{9.7}$$

Second, since $\text{div}\, \mathbf{w} = 0$, we may put $\mathbf{w} = \mathbf{c}$ in (9.6) and obtain

$$\int_{\mathfrak{V}} \mathbf{w} \cdot \mathbf{f} d\mathfrak{V} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{w} b, \tag{9.8}$$

hence concluding a generalization of Berker's theorem II:

Let $\mathbf{w}$ *be any continuous vorticity field whose normal component upon any finite boundary of* $\mathfrak{V}$ *is zero, and let* $\mathbf{f} = \text{grad}\, \mathbf{b}$ *be any continuous laminar field (vector or dyadic), and suppose further that in any portion of* $\mathfrak{V}$ *extending to infinity the condition* $r^2 b w_r \to 0$ *be satisfied, then*

[11] A fourth result of Berker has been noted in §**6**.

$$\int_{\mathfrak{V}} \mathbf{w} \cdot \mathbf{f} d\mathfrak{V} = 0. \tag{9.9}$$

Notice the conditions of this theorem are satisfied by the material within a closed vortex-tube of any continuously differentiable motion; for the entire material of any continuously differentiable motion in a bounded domain upon whose boundaries $w = 0$; and, by (7.5), for any such motion within finite stationary boundaries to which the material adheres without slipping (Berker's case). Putting $\mathbf{f} = \mathbf{I} = \text{grad } \mathbf{r}$, where $\mathbf{I}$ is the unit dyadic, we obtain (9.5) as a special case. Putting $\mathbf{f} = \mathbf{r} = \text{grad } r^2/2$, we obtain

$$\int_{\mathfrak{V}} \mathbf{r} \cdot \mathbf{w} d\mathfrak{V} = 0. \tag{9.10}$$

Third, let $\mathbf{f}$ be a vector field such that $\text{div } \mathbf{f} = 0$, $\nabla^2 \mathbf{f} = 0$; equivalently, $\text{curl curl } \mathbf{f} = 0$, or $\text{curl } \mathbf{f} = \text{grad } b$, say. Let $\mathbf{c} = \mathbf{v}$, and suppose $\text{div } \mathbf{v} = 0$. Then, Kelvin's transformation (9.2) becomes

$$\int_{\mathfrak{V}} \text{curl } \mathbf{f} \cdot \mathbf{v} d\mathfrak{V} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{v} b. \tag{9.11}$$

Now

$$\text{curl } \mathbf{f} \cdot \mathbf{v} = \text{div } (\mathbf{f} \times \mathbf{v}) + \mathbf{f} \cdot \mathbf{w}; \tag{9.12}$$

hence (9.11) becomes

$$\int_{\mathfrak{V}} \mathbf{f} \cdot \mathbf{w} d\mathfrak{V} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot [\mathbf{v} b + \mathbf{v} \times \mathbf{f}], \tag{9.13}$$

whence follows a generalization of Berker's theorem I:

In any continuously differentiable isochoric motion such that upon any finite boundaries the material adheres without slipping, if $\mathbf{f}$ *be any twice continuously differentiable field such that* $\text{curl curl } \mathbf{f} = 0$, *and hence* $\text{curl } \mathbf{f} = \text{grad } b$, *and if further in any portion of the motion extending to infinity the conditions* $r^2 v_r b \to 0$, $r^2 v f \to 0$ *be satisfied, then*

$$\int_{\mathfrak{V}} \mathbf{f} \cdot \mathbf{w} d\mathfrak{V} = 0. \tag{9.14}$$

It is interesting to notice that the form of both the integrals (9.9) and (9.14) is the same, although the meaning is different. The former is not restricted to isochoric motions, as is the latter, but in the former the field $\mathbf{f}$ must satisfy $\text{curl } \mathbf{f} = 0$, while in the latter it need satisfy only $\text{curl curl } \mathbf{f} = 0$.

Part III. The General Conservation Theorems

In the following sections we shall show that a slight generalization of Kelvin's transformation enables the construction of an infinite sequence of moments of vorticity which themselves can be expressed as surface integrals, and whose derivatives may be so expressed, and thus we shall obtain two sequences of possible integrals, which include those discussed in §**9** as the simplest special cases.

10. Symmetrical moments of vorticity. For the polyadic nth power of a vector $\mathbf{b}$ let us introduce the notation $\mathbf{b}^n$:

$$\mathbf{b}^0 \equiv 1, \quad \mathbf{b}^1 \equiv \mathbf{b}, \quad \mathbf{b}^2 \equiv \mathbf{bb}, \quad \mathbf{b}^3 \equiv \mathbf{bbb}, \ldots \tag{10.1}$$

(In this paper the notation $\mathbf{r}^2$ will not be used in its usual sense of $\mathbf{r}\cdot\mathbf{r}$, which we shall denote instead by r^2.) Then let the notation $\{\mathbf{b}^n\mathbf{c}\}$ stand for the completely symmetrical polyadic

$$\{\mathbf{b}^n\mathbf{c}\} \equiv \mathbf{b}^n\mathbf{c} + \mathbf{b}^{n-1}\mathbf{c}\mathbf{b} + \mathbf{b}^{n-2}\mathbf{c}\mathbf{b}^2 + \ldots + \mathbf{c}\mathbf{b}^n; \tag{10.2}$$

thus *e.g.*

$$\{\mathbf{b}^0\mathbf{c}\} \equiv \mathbf{c}, \quad \{\mathbf{b}^1\mathbf{c}\} \equiv \mathbf{bc} + \mathbf{cb}, \quad \{\mathbf{b}^2\mathbf{c}\} \equiv \mathbf{bbc} + \mathbf{bcb} + \mathbf{cbb}. \tag{10.3}$$

We then define the *nth symmetrical moment of vorticity* $\mathfrak{W}_n$ by

$$\mathfrak{W}_n \equiv \int_{\mathfrak{V}} \{\mathbf{r}^n\mathbf{w}\}\, d\mathfrak{V}, \tag{10.4}$$

where $\mathfrak{V}$ is an arbitrary finite domain, within whose interior $\mathbf{w}$ is assumed to be continuously differentiable. Comparison of the first three moments

$$\mathfrak{W}_0 \equiv \int_{\mathfrak{V}} \mathbf{w}\, d\mathfrak{V}, \tag{10.5}$$

$$\mathfrak{W}_1 \equiv \int_{\mathfrak{V}} (\mathbf{rw} + \mathbf{wr})\, d\mathfrak{V},$$

$$\mathfrak{W}_2 \equiv \int_{\mathfrak{V}} (\mathbf{rrw} + \mathbf{rwr} + \mathbf{wrr})\, d\mathfrak{V},$$

with (9.3), (2.3), (9.10) and (2.4) shows that $\mathfrak{W}_0$ is Föppl's total vorticity $\mathfrak{W}$, that either the z-row or the z-column of $\mathfrak{W}_1$ taken over a cylinder of unit height in the case of a plane motion is a vector whose components are Poincaré's $\mathfrak{W}x_w$, $\mathfrak{W}y_w$, and $\mathfrak{W}$, that the scalar of $\mathfrak{W}_1$ is one of Berker's integrals, and that Poincaré's $\mathfrak{J}$ is the sum of two of the components of $\mathfrak{W}_2$ taken over a cylinder of unit height in the case of a plane motion.

11. The first general conservation theorem. Since div $\mathbf{w} = 0$, it is easy to see that

$$\operatorname{div}\, [\mathbf{w}\mathbf{r}^{n+1}] = \{\mathbf{r}^n\mathbf{w}\} \tag{11.1}$$

Putting this result into (10.4), by Green's transformation we obtain an expression for $\mathfrak{W}_n$ as a surface integral:

$$\mathfrak{W}_n = \int_{\mathfrak{S}} d\mathfrak{S}\cdot\mathbf{w}\mathbf{r}^{n+1}. \tag{11.2}$$

Hence *all the moments* $\mathfrak{W}_n$ *are independent of the motion at interior points, being completely determined by the normal component of vorticity upon the bounding* *surfaces*. This purely kinematical result illustrates the predominant effect of boundaries upon vorticity. It may be regarded as indicating that the familiar hydrodynamical theorem that vorticity cannot be generated in the interior of a homogeneous viscous liquid subject to conservative extraneous force, but must be diffused inward from the boundaries, continues to hold for arbitrary continuous media, provided it be expressed in terms of the average rather than the local vorticity. In particular, we conclude the following first general conservation theorem:

If upon any finite boundary of a continuously differentiable motion the normal component w_n of the vorticity be zero, while in any portion extending to infinity the radial component w_r satisfy the condition

$$r^{n+3}w_r \to 0, \tag{11.3}$$

then the first $n + 1$ symmetrical moments of vorticity vanish:

$$\mathfrak{W}_0 = \mathfrak{W}_1 = \ldots = \mathfrak{W}_n = 0. \tag{11.4}$$

As special cases we may notice that the result holds *for all n* for the material within a closed vortex-tube of any continuously differentiable motion; for the entire material of any continuously differentiable motion in a bounded domain upon whose boundaries $w = 0$; and, by (7.5), for any such motion within finite stationary boundaries to which the material adheres without slipping.

12. The second general conservation theorem. While the first conservation theorem yields an infinite number of integrals for many motions, including even so general a case as any continuously differentiable motion of a non-homogeneous viscous compressible fluid of variable viscosity in a finite domain, for motions extending to infinity the limit condition (11.3) may be satisfied only for the first few values of n. It is possible, however, that for some larger values of n the symmetrical moments of vorticity while not zero may nevertheless remain constant in time, as the following analysis shows.

First, let the boundary surface $\mathfrak{S}$ be a material surface. Then from (11.2) we have

$$\begin{aligned} \frac{D\mathfrak{W}_n}{Dt} &= \oint_{\mathfrak{S}} \frac{D}{Dt}\,(d\mathfrak{S}\cdot\mathbf{w}\mathbf{r}^{n+1}), \\ &= \oint_{\mathfrak{S}} \left[d\mathfrak{S}\cdot\left(\mathbf{w}\{\mathbf{r}^n\mathbf{v}\} + \frac{D\mathbf{w}}{Dt}\,\mathbf{r}^{n+1}\right) + \frac{Dd\mathfrak{S}}{Dt}\cdot\mathbf{w}\mathbf{r}^{n+1}\right]. \end{aligned} \tag{12.1}$$

But if we employ the Lagrange-Beltrami equation (7.3) and the formula

$$\frac{Dd\mathfrak{S}}{Dt} = -\operatorname{grad}\mathbf{v}\cdot d\mathfrak{S} + \operatorname{div}\mathbf{v}\,d\mathfrak{S}, \tag{12.2}$$

of Lamb [11], by a happy circumstance the convective portions of these two expressions cancel each other and (12.1) reduces simply to

$$\frac{D\mathfrak{W}_n}{Dt} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{w}\{\mathbf{r}^n\mathbf{v}\} + \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \operatorname{curl} \mathbf{a}'\mathbf{r}^{n+1}. \tag{12.3}$$

The former integral is the convective rate of change of $\mathfrak{W}_n$, the latter is the diffuse rate of change[12]. To obtain the Eulerian derivative of $\mathfrak{W}_n$ for a fixed volume, we need only employ the Reynolds transport theorem (7.2), thus obtaining

$$\frac{\partial\mathfrak{W}_n}{\partial t} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot [\mathbf{w}\{\mathbf{r}^n\mathbf{v}\} - \mathbf{v}\{\mathbf{r}^n\mathbf{w}\}] + \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \operatorname{curl} \mathbf{a}'\mathbf{r}^{n+1}. \tag{12.4}$$

Since we have identically

$$\begin{gathered} d\mathbf{b} \cdot \operatorname{curl} (\mathbf{r}^{n+1}\mathbf{c}) = -\{\mathbf{r}^n(d\mathbf{b} \times \mathbf{c})\} + d\mathbf{b} \cdot \operatorname{curl} \mathbf{c}\, \mathbf{r}^{n+1}, \\ \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \operatorname{curl} \Sigma = \int_{\mathfrak{B}} \operatorname{div} \operatorname{curl} \Sigma \, d\mathfrak{B} = 0, \end{gathered} \tag{12.5}$$

we may put the diffusive term of (12.4) into another form, obtaining finally the not inelegant relation

$$\frac{\partial\mathfrak{W}_n}{\partial t} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot [\mathbf{w}\{\mathbf{r}^n\mathbf{v}\} - \mathbf{v}\{\mathbf{r}^n\mathbf{w}\}] + \int_{\mathfrak{S}} \{\mathbf{r}^n(d\mathfrak{S} \times \mathbf{a}')\}. \tag{12.6}$$

The special case $n = 0$, generalizing a formula of Jaffé [7], I gave in an earlier paper [23]. From (12.6) follows the second general conservation theorem:

If all finite boundaries of a continuously differentiable motion be stationary, and if upon them the normal component w_n of vorticity be zero, and either or both of the conditions

$$a_t = 0, \quad a'_t = 0 \tag{12.7}$$

be satisfied, while in any portion of the material extending to infinity the condition

$$r^{n+2}(vw_r - wv_r - a'_t) \to 0 \tag{12.8}$$

be satisfied, then the first $n + 1$ symmetrical moments of vorticity are constant in time:

$$\mathfrak{W}_0 = \text{const.}, \quad \mathfrak{W}_1 = \text{const.}, \ldots, \mathfrak{W}_n = \text{const.} \tag{12.9}$$

Notice that in the case when all finite boundaries are stationary and the material adheres without slipping, we have $a_t = 0$ as well as (7.5), and hence the only condition of the theorem which remains to be considered is the limit condition (12.8) at infinity. Jaffé's theorem is included in the case $n = 0$.

13. Conclusion. To the best of my knowledge the foregoing theorems furnish the first instances of an infinite set of possible integrals for a class of three-dimensional motions of a continuum, even for a continuum of a special

[12]For the terms "convection" and "diffusion", see [24].

type. The analysis is purely kinematical, and thus the results as stated are independent of whatever dynamical characteristics the medium may possess. The dynamical response of a continuum is generally specified by giving a formula for the acceleration, and thus will determine for what values of n the condition of the theorems will be satisfied. In general it may be expected that a motion vanishing at infinity[13] will satisfy the conditions of the first theorem for a small value of n, and those of the second theorem for a somewhat larger n, so that the first few moments of vorticity will vanish and the next one or two will remain constant.

[13]Since a plane motion does not vanish at infinity, no plane motion can satisfy the conditions of our two general theorems. If we attempt to apply them to the region confined between the planes $z = 0$ and $z = 1$ in a plane motion, we find that the normal component of vorticity does not vanish upon these planes, and hence again the theorems are inapplicable. It is this fact which justifies the statement in §1 that three-dimensional motions are simpler in their average properties than are plane motions.

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