ANTISYMMETRICAL DIGRAPHS

W. T. TUTTE

To Professor H. S. M. Coxeter on his sixtieth birthday

Summary. We call a digraph "antisymmetrical" if there is an automorphism θ of its graph, of period 2, which reverses the direction of every edge and maps no edge or vertex onto itself. We construct a theory of flows invariant under θ for such a diagraph. This theory is analogous to the Max Flow Min Cut theory for ordinary flows in digraphs. It is found to include that part of the theory of undirected graphs which discusses the existence of spanning subgraphs with a specified valency at each vertex.

1. Skew paths. The graphs of this paper are finite. Each graph G is defined by a set of $V(G)$ of *vertices*, a set $E(G)$ of *edges*, and a relation of incidence which associates with each edge two vertices, possibly coincident, called its *ends*. An edge is a *link* or a *loop* according as its ends are distinct or coincident.

A *digraph D* is a graph in which one end of each edge is distinguished as the *positive* and the other as the *negative* end. If A is an edge with negative end x and positive end y, we say it is directed *from x to y*. If $x \in X \subseteq V(D)$ and $y \in Y \subseteq V(D)$, we say that *A joins X* to Y.

A subgraph of G is a graph H such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and each edge of H has the same ends in H as in G. A subgraph H of G is *detached* if no edge of G has one end in $V(H)$ and the other in $V(G) - V(H)$. A non-null detached subgraph H of G is a *component* of G if no other such subgraph of G is a subgraph of H. It is easy to show that each edge or vertex of G belongs to exactly one component of G. *Subdigraphs*, *detached subdigraphs*, and *components* of a digraph D are defined in the same way, with the additional proviso that each edge of a subdigraph K of D must have the same positive end and the same negative end in K as in D. A graph or digraph with only one component is *connected*.

A *path* in a digraph D is an existent sequence

$$(1) \qquad P = (a_0, A_1, a_1, A_2, \ldots, A_n, a_n)$$

which satisfies the following conditions:

(i) *The terms of P are alternately vertices a_i and edges A_j of D.*

(ii) *If $1 \leqslant j \leqslant n$, A_j is directed from a_{j-1} to a_j.*

Clearly the edges and vertices occurring in a path P are confined to a single component of D. We call P a path from a_0 to a_n. Its *origin* is a_0 and its *terminus*

Received March 13, 1967.

is a_n. The number n is the *length* $s(P)$ of P. We denote the sets of all vertices and edges of D occurring in P by $V(P)$ and $E(P)$ respectively. We speak of P as *passing through* the vertices of $V(P)$ and as *traversing* the edges of $E(P)$.

The path P is *re-entrant* if $a_0 = a_n$ and *simple* if no vertex occurs twice. If $s(P) = 0$, that is if P has only the one term a_0, then P is *degenerate*.

Let P and Q be paths in D from x to y and from y to z respectively. We can construct a path from x to z in D by taking the terms of P in order and continuing with the terms of Q, other than the first, in order. This path is the product PQ of P and Q, in that order. Multiplication of paths is clearly associative. We therefore write a product $(PQ)R$ or $P(QR)$ simply as PQR. We note the following rule.

1.1. *A path P of length n can be written as a product QR, where Q is the part of P extending from the first term to the kth vertex-term* $(1 \leqslant k \leqslant n + 1)$.

A *reverse-mapping* θ of D is defined by a 1–1 mapping $x \rightarrow x^* = \theta x$ of $V(D)$ onto itself, together with a 1–1 mapping $A \rightarrow A^* = \theta A$ of $E(D)$ onto itself, provided that the following conditions are satisfied:

(i) *If x is an edge or vertex of D, then $(x^*)^* = x$ and $x^* \neq x$.*

(ii) *If an edge A is directed from x to y, then A^* is directed from y^* to x^*.*

If a reverse-mapping of D exists, we say D is *antisymmetrical*. In this paper we suppose each antisymmetrical digraph to be taken with a fixed reverse-mapping θ. If two edges or vertices correspond under θ, we say that they are *conjugate*.

In the remainder of this section we suppose D to be antisymmetrical. If S is a subset of $V(D)$ or $E(D)$, the class of conjugates of the members of S is the *conjugate* set S^* or θS of S. If K is a subdigraph of D, there is a *conjugate* subdigraph K^* or θK of D whose vertices and edges are the conjugates of the vertices and edges respectively of K. A subset of $V(D)$ or $E(D)$, or a subdigraph of D, may be self-conjugate.

If $P = (a_0, A_1, a_1, A_2, \ldots, A_n, a_n)$ is a path in D from a_0 to a_n, it is clear that the sequence

$$P^* = (a_n^*, A_n^*, \ldots, A_2^*, a_1^*, A_1^*, a_0^*)$$

is a path in D from a_n^* to a_0^*. We call it the *conjugate path* to P.

Let M be a fixed self-conjugate subset of $E(D)$. A path P in D is *skew mod M* if there is no $A \in E(D) - M$ such that P traverses both A and A^*. When M is null, we say simply that such a path is *skew*.

1.2. *If a path P in D is simple or skew mod M, then P^* is simple or skew mod M respectively. If P is skew mod M, then $E(P) \cap E(P^*) \subseteq M$.*

2. Capacities. Let D be an antisymmetrical digraph. With each edge A of D let there be associated a non-negative integer $c(A)$ called the *capacity* of A. We impose the condition

$$c(A^*) = c(A). \qquad (2)$$

If X and Y are subsets of $V(D)$, we write $J(X, Y)$ for the sum of the capacities of the edges joining X to Y. We abbreviate $J(\{x\}, Y)$, $J(X, \{y\})$, and $J(\{x\}, \{y\})$ as $J(x, Y)$, $J(X, y)$, and $J(x, y)$ respectively.

Let U be any subset of $V(D)$. Write $V = V(D) - U$. If $u \in U$ and $v \in V$, we say that U is a *cut* between u and v in D. We define the *capacity* $c(U)$ of U by

$$c(U) = J(U, V). \tag{3}$$

The *core* of U is the subdigraph $K = K(U)$ of D specified as follows:

(i) *$V(K)$ is the set of all vertices x of D such that $\{x, x^*\}$ is a subset of U.*

(ii) *$E(K)$ is the set of all edges A of D such that $c(A) > 0$ and both ends of A are in $V(K)$.*

We note that the core of U is self-conjugate. If C is a component of K, we say that the subgraph $C \cup C^*$ of K defined by the edges and vertices of C and C^* is a *nucleus* of U. Thus the nuclei of U are non-null detached self-conjugate subdigraphs of K, and no one of them has another such subdigraph of K as one of its own subdigraphs. Moreover, each edge or vertex of K belongs to exactly one nucleus of U.

For each nucleus N of U we write

$$t(N) = J(V(N), V), \tag{4}$$

$$t_0(N) = J(U - V(N), V(N)). \tag{5}$$

We say N is an *odd* or *even* nucleus of U according as $t(N)$ is odd or even. We write $\mu(U)$ for the number of odd nuclei of U, and define the *effective capacity* $k(U)$ of U by

$$k(U) = c(U) - \mu(U). \tag{6}$$

For each odd nucleus N we must have $t(N) \geqslant 1$. Hence, by (3) and (4),

$$k(U) \geqslant 0. \tag{7}$$

2.1. *Suppose $k(U) = 0$. Then each edge joining U to V has capacity 1 or 0, and each such edge of unit capacity has its negative end in an odd nucleus of U. Moreover, each odd nucleus N of U satisfies $t(N) = 1$.*

Proof. $c(U)$ is the sum Σ_1 of the capacities of the edges of D directed from vertices of odd nuclei of U to vertices of V, together with the sum Σ_2 of the capacities of the other edges joining U to V. By (4) we have $\Sigma_1 \geqslant \mu(U)$, with equality if and only if $t(N) = 1$ for each odd nucleus N.

By hypothesis, $c(U) = \mu(U)$. Hence $\Sigma_1 = \mu(U)$ and $\Sigma_2 = 0$. The theorem follows.

2.2. *Let N be any nucleus of U. Let A be an edge of D, of non-zero capacity, which is directed from a vertex x of $U - V(N)$ to a vertex y of $V(N)$. Then $x^* \in V$.*

Proof. Suppose $x^* \in U$. Then $x \in V(K)$ and $A \in E(K)$. But this is impossible since N is a detached subdigraph of K.

COROLLARY. $t_0(N) \leqslant t(N)$.

A nucleus N of U is *simple* if it is connected and satisfies $t_0(N) = t(N)$. The set U is *simple* if each of its nuclei is simple.

2.3. *Choose $x \in V$ and write $W = U \cup \{x\}$. Let n be the number of odd nuclei N of U such that*

$$J(V(N), x) + J(x, V(N)) > 0.$$

Let p and q be the numbers of odd and even nuclei N of U respectively such that

$$J(V(N), x) \equiv 1 \mod 2.$$

Let β be 0 *or* 1 *according as*

$$n + p + q + J(x, V - \{x\}) + J(x^*, V - \{x, x^*\})$$

is even or odd. Then the following propositions hold:

$$c(W) = c(U) - J(U, x) + J(x, V - \{x\}), \tag{8}$$

$$\mu(W) = \mu(U) - p + q \qquad \text{if } x^* \in V, \tag{9}$$

$$\mu(W) = \mu(U) - n + \beta \qquad \text{if } x^* \in U. \tag{10}$$

Proof. Equation (8) follows at once from the definitions.

If $x^* \in V$, it is clear that $K(W) = K(U)$. Hence the nuclei of W are those of U. But p of them change parity from odd to even and q of them from even to odd when U is replaced by W. Hence (9) holds.

If $x^* \in U$, then $K(W)$ is obtained from $K(U)$ by adjoining the two vertices x and x^* and the appropriate edges. Clearly $\mu(U) - n$ of the odd nuclei of U remain as odd nuclei of W, but the others become subdigraphs of a new nucleus N_x which has x and x^* as two of its vertices. Similarly, an even nucleus of U either remains as an even nucleus of W or becomes a subdigraph of N_x. We have, moreover,

$$\begin{aligned} t(N_x) &\equiv n + J(x, V - \{x\}) + p + q + J(x^*, V - \{x\}) \\ &\equiv \beta \mod 2. \end{aligned}$$

Thus (10) holds.

Let us now write M for the set of all edges A of D such that $c(A) \geqslant 2$. A path P in D is *admissible* if it satisfies the following conditions:

(i) *P is simple.*
(ii) *If $A \in E(P)$, then $c(A) \geqslant 1$.*
(iii) *P is skew mod M.*

2.4. *Let u and v be distinct vertices of D. Let U be a cut between u and v such that*

https://doi.org/10.4153/CJM-1967-101-8 Published online by Cambridge University Press

$k(U) = 0$ and u belongs to no odd nucleus of U. Then there is no admissible path from u to v in D.

Proof. Assume such a path P to exist. There is a first edge A of P directed from some $x \in U$ to some $y \in V$. Each edge of P preceding A has both ends in U. But x belongs to an odd nucleus N of U, by 2.1. Since u is not in $V(N)$, there is an edge B of P, preceding A, which joins $U - V(N)$ to $V(N)$. Then B^* joins $V(N)$ to V, by 2.2. Hence $A = B^* \notin M$ by 2.1 and so $\{A, A^*\} \subseteq E(P)$. Thus P is not admissible, contrary to assumption.

From now on we denote the number of elements of a finite set S by $|S|$. A subset U of $V(D)$ is *regular* if it is simple and each of its nuclei is odd. It is *regular mod u*, where $u \in U$, if it is simple and each nucleus is odd, except that $t(N) = 0$ for any nucleus N satisfying $u \in V(N)$.

If $u \in V(D)$, we define the *accessible set* $Ac(D, u)$ of u in D as the set of all vertices x of D such that there is an admissible path in D from u to x. Clearly all the vertices of any such path are in $Ac(D, u)$.

2.5. *If $u \in V(D)$, then $Ac(D, u)$ is regular mod u and its effective capacity is zero.*

Proof. $|E(D)|$ is even, by the definition of an antisymmetrical digraph. If $|E(D)| = 0$, then $Ac(D, u) = u$, and the theorem is trivially true. Assume as an inductive hypothesis that the theorem holds whenever $|E(D)|$ is less than some positive even integer $2q$, and consider the case $|E(D)| = 2q$. Write $U = Ac(D, u)$.

Suppose first that $c(U) = 0$. Let N be any nucleus of U. Then $t(N) = 0 = t_0(N)$, by 2.2, Corollary. Since $V(N) \subseteq Ac(D, u)$, it follows that $u \in V(N)$. If N consists of two distinct conjugate components C and C^* of $K(U)$, we may suppose that u is in $V(C)$. But then no edge of D, of non-zero capacity, joins $U - V(C^*)$ to $V(C^*)$, and this is impossible since $V(C^*) \subseteq Ac(D, u)$. We deduce that U is regular mod u. But $k(U) = 0$, by (6) and (7). Thus the theorem holds in this case.

We may now assume that $c(U) > 0$. Choose $A \in E(D)$, of non-zero capacity, directed from a vertex x of U to a vertex y of $V = V(D) - U$. If P is an admissible path from u to x, then $P(x, A, y)$ is not admissible. Hence $c(A) = 1$ and $A^* \in E(P)$.

Let D_1 be the antisymmetrical digraph obtained from D by deleting the edges A and A^*. We retain for the other edges the same capacities as in D, and we use the reverse-mapping induced by that of D. Write $U_1 = Ac(D_1, u)$. Evidently $U_1 \subseteq U$. Write $T = U - U_1$.

We have $x \in T$, for otherwise there is an admissible path P from u to x in D not traversing A or A^*, and $P(x, A, y)$ is an admissible path from u to y in D.

Let Z be the set of all edges B, of non-zero capacity, joining U_1 to T in D_1. Let each such B be directed from x_B to y_B. By 2.1 and the inductive hypothesis, x_B belongs to an odd nucleus N_B of U_1, and B is the only edge of D_1 of non-zero capacity joining $V(N_B)$ to $V(D_1) - U_1$. By 2.2 B^* is the only edge of D_1 of non-zero capacity joining $U_1 - V(N_B)$ to $V(N_B)$. Moreover, $c(B) = c(B^*) = 1$, by 2.1.

If $x^* \in V(N_B)$, where $B \in Z$, then $x \in V(N_B) \subseteq U_1$. But this is impossible since $x \in T$. We deduce that $x^* \in T$, for otherwise each admissible path from u to x in D would traverse both B and B^* for some $B \in Z$.

Let T_0 be the set of all $t \in T$ such that some admissible path from u to t in D traverses A^* and thereafter passes only through vertices of T. Thus $x^* \in T_0$.

Suppose z is common to U_1 and $(T_0)^*$. Then $z^* \in T_0$. Hence there is an admissible path P in D_1 such that $V(P) \subseteq T_0$, from x^* to z^*. But P^* is admissible, by 1.2. Let z_1 be the last vertex of P^* in U_1. Consider the product $P_1 P_2$, where P_1 is an admissible path in D_1 from u to z_1 and P_2 is the part of P^* extending from z_1 to x. This product is admissible since $V(P_1) \subseteq U_1$ and $V((P_2)^*) \subseteq T$. Again we have the contradiction that x is in U_1. We deduce that $(T_0)^* \cap U_1$ is null.

No edge B of D_1, of non-zero capacity, is directed from some $x_1 \in U_1$ to some $x_2 \in (T_0)^*$. For suppose B has this property and P is an admissible path in D_1 from u to x_1. Then $P(x_1, B, x_2)$ is admissible since $(x_2)^* \in T$. Hence $x_2 \in (T_0)^* \cap U_1$, contrary to the preceding result.

We can now show that Z is null. For suppose B is in Z. Then $y_B \in T$. But B^* joins $U_1 - V(N_B)$ to $V(N_B)$, and so $(y_B)^* \in U_1$. Since $(T_0)^* \cap U_1$ is null, we must have $y_B \in T - T_0$. Accordingly we can choose B and an admissible path P in D from u to a member of $T - T_0$ so that

$$P = P_1(y^*, A^*, x^*)P_2(r, C, s)P_3,$$

where $V(P_1) \subseteq U_1$, $V(P_2) \subseteq T_0$, $s \in U_1$, and B is the first edge of Z in P_3. Then $r \in T_0$ and $s \in V(N_B)$. But this implies that C^* is directed from $s^* \in U_1$ to $r^* \in (T_0)^*$, contrary to the preceding result. We deduce that Z is null and therefore $T = T_0$.

We may now assert that $T \cup T^*$ does not meet U_1, and that A^* is the only edge of D, of non-zero capacity, joining U_1 to $T \cup T^*$. Let D_2 be the anti-symmetrical subdigraph of D given by the vertices of $T \cup T^*$ and the edges having both ends in $T \cup T^*$. We use the reverse-mapping and capacity-set induced by those of D. Write $U_2 = Ac(D_2, x^*)$. The product of an admissible path in D_1 from u to y^*, the path (y^*, A^*, x^*) in D, and an admissible path in D_2 from x^* to $w \in U_2$ is an admissible path in D from u to w. Thus

$$U_2 \subseteq T_0 = T.$$

Moreover, such a path exists for each $w \in T_0$. Hence $U_2 = T$ and

$$U = U_1 \cup U_2.$$

A is the only edge of non-zero capacity joining $T \cup T^*$ to V. For suppose B is another. There is an admissible path P in D from u to the negative end of B.

This must traverse B^*. But then B^* must join U_1 to $T \cup T^*$, contrary to the above assertion.

We have seen that x and x^* are both in T. Hence there is a nucleus N_x of U_2 in D_2 that contains x and x^* as vertices.

Since A^* is the only edge of D, of non-zero capacity, joining U_1 to $T \cup T^*$, we deduce that the nuclei of U in D are the nuclei of U_1 in D_1, together with the nuclei of U_2 in D_2. But U_1 and U_2 are regular mod u and x^* in D_1 and D_2 respectively, and have zero effective capacities in those digraphs, by the inductive hypothesis. Any nucleus N of U in D must thus have the following properties:

(i) *N is connected.*

(ii) *If $N = N_x$, then $t(N) = t_0(N) = 0$ in D_2 and $t(N) = t_0(N) = 1$ in D.*

(iii) *If N is not N_x, then $t(N) = t_0(N) = 1$ in D, unless $u \in V(N)$, in which case $t(N) = t_0(N) = 0$.*

We conclude that U is regular mod u in D. Moreover, we have

$$\mu(U) = \mu(U_1) + \mu(U_2) + 1,$$
$$c(U) = c(U_1) + c(U_2) + 1,$$

and therefore

$$k(U) = k(U_1) + k(U_2) = 0$$

by the inductive hypothesis, where the functions of U, U_1, and U_2 refer to the digraphs D, D_1, and D_2 respectively.

The theorem thus holds whenever $|E(D)| = 2q$. It follows in general by induction.

2.6. *Let u and v be distinct vertices of D. In order that there shall be an admissible path from u to v in D it is necessary and sufficient that some cut U between u and v shall be regular mod u and of zero effective capacity.* (By 2.4 and 2.5.)

3. Flows. Let D be a digraph, not necessarily antisymmetrical. For each vertex x of D we write $I(x)$ for the set of all edges directed to x, and $O(x)$ for the set of all edges directed from x.

Let f be a function which associates with each edge A of D a non-negative integer $f(A)$. We write

$$\text{in}(x, f) = \textstyle\sum_{A \in I(x)} f(A), \tag{12}$$

$$\text{out}(x, f) = \textstyle\sum_{A \in O(x)} f(A), \tag{13}$$

$$d(x, f) = \text{out}(x, f) - \text{in}(x, f) \tag{14}$$

for each vertex x of D.

Let S and T be disjoint subsets of $V(D)$. We denote the sum of the numbers $f(A)$ over all edges A joining S to T by $\text{out}(S, T, f)$. If $T = V(D) - S$, we write the sum also as $\mathrm{out}(S, f)$ or $\mathrm{in}(T, f)$. By (12), (13), and (14) we have

$$\sum_{x \in S} d(x, f) = \mathrm{out}(S, f) - \mathrm{in}(S, f). \tag{15}$$

We call f a *flow* in D from u to v, of *magnitude* $M(f) \geqslant 0$, if the following conditions are satisfied:

$$d(u, f) = M(f), \tag{16}$$

$$d(v, f) = -M(f), \tag{17}$$

$$d(x, f) = 0 \quad \text{if } x \in V(D) - \{u, v\}. \tag{18}$$

These three conditions are not independent, as we may see by applying (15) with $S = V(D)$.

We now suppose that with each edge A of D there is associated a non-negative integer $c(A)$ called the *capacity* of A. A flow f in D is said to be *admissible* if

$$f(A) \leqslant c(A) \tag{19}$$

for each edge A.

In the remainder of this section we take D to be antisymmetrical. We suppose the capacities to satisfy (2). A flow f in D is *self-conjugate* if it satisfies

$$f(A^*) = f(A) \tag{20}$$

for each edge A. Such a flow is always from a vertex u to its conjugate vertex u^*.

3.1. *Let f be a self-conjugate flow in D from u to u^*. Let S be any self-conjugate subset of $V(D)$. Then*

$$\sum_{x \in S} \max(d(x, f), 0) + \mathrm{out}(S, f) \equiv 0 \mod 2.$$

Proof. Let the expression on the left of this congruence be denoted by $g(S)$. If $|S| = 0$, we have $g(S) = 0$, and the theorem is trivially true. Assume that the theorem holds whenever $|S|$ is less than some positive even integer $2q$, and consider the case $|S| = 2q$.

Write $T = V(D) - S$. Choose $x \in S$ and write $S_1 = S - \{x, x^*\}$. Since $d(x, f) = -d(x^*, f)$, by (20), we may adjust the notation so that $d(x, f) \geqslant 0$. Then

$$\begin{aligned} g(S) - g(S_1) - d(x, f) &= \mathrm{out}(\{x\}, T, f) + \mathrm{out}(\{x^*\}, T, f) \\ &\quad - \mathrm{out}(S - \{x\}, \{x\}, f) - \mathrm{out}(S - \{x^*\}, \{x^*\}, f) \\ &\quad + \mathrm{out}(\{x^*\}, \{x\}, f) + \mathrm{out}(\{x\}, \{x^*\}, f) \\ &\equiv \mathrm{out}(\{x\}, T, f) + \mathrm{out}(T, \{x\}, f) \\ &\quad + \mathrm{out}(S - \{x\}, \{x\}, f) + \mathrm{out}(\{x\}, S - \{x\}, f) \mod 2, \end{aligned}$$

for if A is directed from y to y^*, then so is A^*. Hence

$$\begin{aligned} g(S) - g(S_1) &\equiv d(x, f) + \mathrm{out}(x, f) + \mathrm{in}(x, f) \\ &\equiv 0 \mod 2, \end{aligned}$$

by (14). But $g(S_1) \equiv 0 \bmod 2$, by the inductive hypothesis. It follows that $g(S) \equiv 0 \bmod 2$. Hence the theorem is true, by induction.

Putting $S = V(D)$, we obtain

COROLLARY. $M(f)$ is even.

3.2. *Let U be a cut between u and u^*, and let f be an admissible self-conjugate flow in D from u to u^*. Then*

$$M(f) \leqslant k(U).$$

Moreover, if N is any odd nucleus of U, then

$$\text{out}(V(N), V, f) - \text{out}(V, V(N), f) < J(V(N), V). \tag{21}$$

Proof. Let N be any odd nucleus of U. Assume that N does not satisfy (21). Since f is admissible, the following propositions must hold:

(i) $f(A) = c(A)$ *if A joins $V(N)$ to V.*
(ii) $f(A) = 0$ *if A joins V to $V(N)$.*

Hence $f(A) = c(A)$ whenever A joins $U - V(N)$ to $V(N)$, by 2.2. Since N is self-conjugate, it does not have u as a vertex. So by 3.1 and (15)

$$J(V(N), V) = \text{out}(V(N), V, f) \equiv 0 \mod 2.$$

But this is contrary to the choice of N. We deduce that (21) is valid for each odd nucleus N of U.

Let Q be the class of all odd nuclei of U, and let W be the set of all vertices of U not belonging to odd nuclei. Then

$$\begin{aligned} M(f) &= \text{out}(U, f) - \text{in}(U, f) && \text{by (15)} \\ &\leqslant \text{out}(W, V, f) + \Sigma_{N \in Q} \{\text{out}(V(N), V, f) - \text{out}(V, V(N), f)\} \\ &\leqslant J(W, V) + \Sigma_{N \in Q} J(V(N), V) - \mu(U), \end{aligned}$$

by (21) and the admissibility of f. Thus

$$M(f) \leqslant c(U) - \mu(U) = k(U).$$

3.3. *Let f be an admissible self-conjugate flow in D from u to u^*, having the greatest possible magnitude $M(f)$. Then there is a cut U between u and u^* which is regular and whose effective capacity is equal to $M(f)$.*

Proof. We derive an antisymmetrical digraph D' from D in the following way. For each edge A, directed from x to y say, we adjoin a new edge A_1 directed from y to x. We arrange that if A and B are distinct edges of D, then A_1 and B_1 are distinct. The reverse-mapping of D' agrees with that of D for the vertices and for the edges of D. It is extended to the new edges by the rule

$$(A_1)^* = (A^*)_1. \tag{22}$$

To each edge X of D' we assign a capacity $c'(X)$ defined as follows:

$$c'(A) = c(A) - f(A), \tag{23}$$

$$c'(A_1) = f(A), \tag{24}$$

for each $A \in E(D)$.

Suppose there is a path P from u to u^* in D' which is admissible with respect to the new capacities. Then P^* is another such path, and any common edge X of P and P^* satisfies $c'(X) \geqslant 2$, by 1.2. Let $p_0(A)$ and $p_1(A)$ denote the number of paths in the set $\{P, P^*\}$ traversing A and A_1 respectively. Write also

$$p(A) = p_0(A) - p_1(A). \tag{25}$$

Define a function g on $E(D)$ as follows:

$$g(A) = f(A) + p(A). \tag{26}$$

We note that $0 \leqslant g(A) \leqslant c(A)$ for each edge A, by the admissibility of P and P^*. Moreover, $g(A^*) = g(A)$. Consider an edge B directed from x to y in D. In the digraph D the replacement of $f(B)$ by $g(B)$ increases $d(x,f)$ by $p(B)$ and diminishes $d(y,f)$ by $p(B)$. By summation we find that

$$d(x, g) = d(x,f) = 0 \tag{27}$$

if x is not u or u^*, and

$$d(u, g) = d(u,f) + 2. \tag{28}$$

We deduce that g is an admissible self-conjugate flow from u to u^* in D, of magnitude $M(f) + 2$. But this is contrary to hypothesis.

It now follows by 2.6 that there is a cut U between u and u^* in D' which is regular mod u, and which has effective capacity zero. This cut is regular since it does not contain u^*.

Evidently the cores of U in D and D' are defined by the same vertex-set. Moreover, two distinct vertices of this set are the ends of an edge of non-zero capacity in D' if and only if they are the ends of such an edge in D. Hence there is a 1–1 correspondence $N \to N'$ between the set of nuclei of U in D and the set of nuclei of U in D' which has the property $V(N) = V(N')$. We note also that the nuclei of U in D are connected like those in D'.

Applying 2.1 to the cut U in D', we deduce the following properties of U considered as a cut in D:

(i) *If A joins U to $V = V(D) - U$ and $f(A) < c(A)$, then $f(A) = c(A) - 1$ and the negative end of A belongs to a nucleus of U.*

(ii) *If A joins V to U and $f(A) > 0$, then $f(A) = 1$ and the positive end of A belongs to a nucleus of U.*

(iii) *If N is a nucleus of U, there is exactly one edge A such that either $f(A) < c(A)$ and A joins $V(N)$ to V or $f(A) > 0$ and A joins V to $V(N)$. The edge A^* has one end in $U - V(N)$.*

In deriving these rules we use the regularity of U in D'. We refer to the edge A of Rule (iii) as the *singular edge of* U associated with N. We write $e(N) = 1$ if A joins V to $V(N)$, and $e(N) = -1$ if it joins $V(N)$ to V.

Let N be a nucleus of U with associated singular edge A. By (ii) there is no edge B of D, distinct from A, which is directed from V to $V(N)$ and satisfies $f(B) > 0$. Hence the conjugates of the edges of D joining $V(N)$ to V and having non-zero capacity are all directed from vertices of $U - V(N)$ to vertices of $V(N)$. (If A is such an edge we use Rule (iii).) Thus $t_0(N) = t(N)$, that is, N is simple, by 2.2. We also have

$$\text{out}(V(N), f) = J(V(N),\ V) + e(N), \tag{29}$$

for if an edge joins $V(N)$ to $U - V(N)$ and has non-zero capacity, then its conjugate edge is directed from V to $V(N)$, by 2.2. It follows that

$$\text{out}(V(N),\ V, f) - \text{out}(V,\ V(N), f) = J(V(N),\ V) - 1. \tag{30}$$

N is an odd nucleus of U, by (29) and 3.1.

We have shown that each nucleus of U in D is connected, simple, and odd. Thus U is a regular cut between u and u^* in D.

Let Q be the class of all nuclei of U in D, and let W be the set of all vertices of U not belonging to nuclei. Then, by (15),

$$\begin{aligned} M(f) &= \text{out}(W,\ V,\ f) - \text{out}(V,\ W, f) \\ &\qquad + \textstyle\sum_{N \in Q} \{\text{out}(V(N),\ V, f) - \text{out}(V,\ V(N), f)\} \\ &= J(W,\ V) + \textstyle\sum_{N \in Q} J(V(N),\ V) - \mu(U), \end{aligned}$$

by Rules (i) and (ii), together with (30),

$$= c(U) - \mu(U) = k(U).$$

Thus the cut U has the required properties.

Given a vertex u of D we write $M(u)$ for the maximum magnitude of an admissible self-conjugate flow from u to u^*, $C(u)$ for the minimum capacity of a cut between u and u^*, and $K(u)$ for the minimum effective capacity of such a cut. An admissible self-conjugate flow from u to u^* of magnitude $M(u)$ is a *maximal flow* from u to u^*. A cut between u and u^* of capacity $C(u)$, or effective capacity $K(u)$, is a *minimal*, or *effectively minimal*, cut respectively between u and u^*. By 3.2 and 3.3 we have

$$M(u) = K(u). \tag{31}$$

This result is analogous to the Max Flow Min Cut theorem **(3)** for flows in general digraphs.

We conclude this section by discussing some generalizations. In one of these $c(A)$ and $f(A)$ take non-negative real values but are not necessarily integers. It can be shown that then $M(u) = C(u)$, and we proceed to sketch a proof of this theorem. When the capacities happen to be even integers it follows from

https://doi.org/10.4153/CJM-1967-101-8 Published online by Cambridge University Press

(31), for then no cut can have odd nuclei and so $K(u) = C(u)$. To extend the theorem to the general case of rational capacities we multiply the capacities by a suitable integer h to make them even integers, and we consider a corresponding maximal flow f. Clearly $hM(u) = M(f) = hC(u)$. In particular, whenever the capacities are integers, there is a maximal flow f (of magnitude $C(u)$) such that $f(A)$ is an integral multiple of $1/2$ for each edge A. The theorem can be extended to the general real case by using a sequence of rational approximations.

In another generalization we assign a non-negative integral capacity $c(x)$ to each edge and vertex x of D. A flow is then called admissible only if in addition to the requirement

$$f(A) \leqslant c(A)$$

for each edge A it satisfies also

$$\max\{\text{in}(x, f), \text{out}(x, f)\} \leqslant c(x)$$

for each vertex x. When capacities are assigned only to edges, it is convenient to say that the vertex-capacities are infinite. We can make the generalization depend on the standard case of infinite vertex-capacities by using the following device.

We replace each vertex x of D by two vertices, x_1 incident with the members of $I(x)$ and x_2 incident with the members of $O(x)$, and we introduce a new edge A_x directed from x_1 to x_2. We denote the resulting digraph by D''. It is antisymmetrical, having a reverse-mapping induced in an obvious way by that of D. We assign capacities in D'' to the edges only. An edge of D retains the same capacity in D'' as in D, while an edge A_x receives the capacity $c(x)$.

It is easy to verify that there is a 1–1 correspondence $f \to f''$ between the class of admissible self-conjugate flows f from u to u^* in D and the class of admissible self-conjugate flows f'' from u_1 to $(u_1)^*$, that is $(u^*)_2$, in D'', with the property that $f''(A) = f(A)$ for each edge A of D. Hence $M(u)$ defined for D is equal to $M(u_1)$ defined for D''.

4. Analogues of Menger's theorem. Let D be an antisymmetrical digraph with capacities, satisfying (2), assigned to the edges. As in §2 we write M for the self-conjugate set of all edges A such that $c(A) \geqslant 2$, and we define admissibility of paths in terms of M.

An *admissible path-set* from u to u^* in D is a set S of admissible paths from u to u^* which satisfies the following conditions:

(i) *If P is in S, then P^* is in S.*
(ii) *The number of paths of S traversing an edge A does not exceed $c(A)$.*

We note that the number of paths in any admissible path-set is even.

For each vertex u we define $L(u)$ as the maximum number of members of an admissible path-set from u to u^*.

An admissible path-set S from u to u^* *conforms* to an admissible self-conjugate flow f from u to u^* if the number of members of S traversing an edge A does not exceed $f(A)$.

4.1. *Let S be an admissible path-set from u to u^*. Then we can construct an admissible self-conjugate flow f from u to u^* such that S conforms to f.*

Proof. We define $f(A)$ as the number of members of S traversing A. The function f clearly satisfies the definition of a flow from u to u^*. It is self-conjugate by condition (i) and admissible by condition (ii). By the definition of f, S conforms to f.

COROLLARY. *We can choose f so that $|S| = M(f)$.*

4.2. *Let f be an admissible self-conjugate flow in D from u to u^*. Then there exists an admissible path-set S from u to u^* which conforms to f and satisfies $|S| = M(f)$.*

Proof. We proceed by induction over $M(f)$. If $M(f) = 0$, the theorem is trivially true, S being null. Assume it true whenever $M(f)$ is less than some positive even integer $2q$, and consider the case $M(f) = 2q$.

Let M' denote the set of all edges A such that $f(A) \geqslant 2$. Suppose there is a simple path P from u to u^* such that P is skew mod M' and $f(A) \geqslant 1$ for each $A \in E(P)$. Then $T = \{P, P^*\}$ is an admissible path-set conforming to f. There is an admissible self-conjugate flow g from u to u^* such that $g(A)$ is the number of members of T traversing A, by the proof of 4.1. Clearly there is an admissible self-conjugate flow f_1 from u to u^* such that $f_1(A) = f(A) - g(A)$ for each edge A, and $M(f_1) = M(f) - 2 = 2q - 2$. By the inductive hypothesis there is an admissible path-set S_1 from u to u^* which conforms to f_1 and satisfies $|S_1| = M(f_1)$. But then the set $S_1 \cup \{P, P^*\}$ is an admissible path-set from u to u^* having the required properties.

In the remaining case there is no path P from u to u^* with the specified properties. We apply 2.6, taking the numbers $f(A)$ as the capacities of the corresponding edges. With this choice of capacities there is no admissible path from u to u^*. Hence there is a regular cut U between u and u^* of effective capacity zero.

Since $M(f) = 2q > 0$, there is an edge B joining U to $V = V(D) - U$ such that $f(B) > 0$. By 2.1 we must have $f(B) = 1$. Moreover, there is an odd nucleus N of U such that B is the only edge, satisfying $f(B) > 0$, that joins $V(N)$ to V. Since $M(f) > 0$, we can choose B and N such that there is no edge A, satisfying $f(A) > 0$, that joins V to $V(N)$, by (15). Then there is no edge A, satisfying $f(A) > 0$, that joins $V(N)$ to $U - V(N)$, as we may see by applying 2.2 to A^*. We can thus choose B and N so that $\text{out}(V(N), f) = 1$. But this is contrary to 3.1.

The preceding argument shows that the theorem is true when $M(f) = 2q$. It follows in general by induction.

https://doi.org/10.4153/CJM-1967-101-8 Published online by Cambridge University Press

Given a vertex u of D we deduce from 4.1, its corollary, and 4.2, together with (31) that

$$L(u) = M(u) = K(u). \tag{32}$$

When $c(A) = 1$ for each edge A, then $L(u)$ is the maximum number of simple skew paths from u to u^* such that no two traverse a common edge, subject to the restriction that if we choose a path P we must also choose P^*.

In a related problem we consider the maximum number $L_1(u)$ of simple skew paths from u to u^*, always counting P^* with P, such that no two pass through a common vertex other than u and u^*. For simplicity we suppose that no edge is directed from u to u^*. We assign unit capacity to each vertex of D and arbitrarily large positive capacities to the edges. Then $L_1(u)$ is equal to $L(u_1)$, the latter number being defined for the digraph D'' of §3.

The problem of finding $L_1(u)$ is related to the result known as Menger's theorem for ordinary digraphs.

5. A matching problem. Let D be an antisymmetrical digraph in which the edges have capacities satisfying (2).

Let X be a subset of $V(D)$ not meeting X^*. An *admissible path-set* from X to X^* in D is a set S of admissible paths that satisfies the following conditions:

(i) *Each $P \in S$ has its origin in X and its terminus in X^*.*
(ii) *If P is in S, then P^* is in S.*
(iii) *The number of paths of S traversing an edge A does not exceed the capacity of A.*

With each vertex x of X or X^* we associate a non-negative integer $g(x)$, imposing the restriction

$$g(x) = g(x^*). \tag{33}$$

A path-set S from X to X^* is *acceptable* if it is admissible and the number of paths of S originating or terminating at a vertex x of $X \cup X^*$ never exceeds $g(x)$. We write $L(X)$ for the maximum number of members in an acceptable path-set from X to X^*.

We can express $L(X)$ as the magnitude of a maximal flow by the following device. We adjoin two new vertices u and u^*, regarded as conjugate. For each $x \in X$ we adjoin a new edge A_x directed from u to x and a conjugate edge $(A_x)^*$ directed from x^* to u^*. The digraph D_1 thus constructed from D is antisymmetrical, with a reverse-mapping agreeing with that of D for the vertices and edges of D, and given by the specified relation of conjugacy for the new vertices and edges. In D_1 we assign the capacity $g(x)$ to A_x and (A_x^*). The other edges of D_1 are assigned the same capacities as in D.

5.1. *$L(X)$ for D is equal to $K(u)$ for D_1.*

Proof. Given an admissible path-set S from u to u^* in D_1 we can delete the first and last edges and the first and last vertices of each member of S, and so obtain an acceptable path-set S' from X to X^* in D such that $|S'| = |S|$. Considering this operation and its converse, we find that $L(X)$ for D is equal to $L(u)$ for D_1. The theorem follows, by (32).

There is a special case that seems to deserve further study. We define a *bipartition* of D as an ordered pair $\{X, X^*\}$ of conjugate complementary subsets of $V(D)$ such that each edge of D has one end in X and one in X^*. The bipartition is *restricted* if it has the following property: if $x \in X$ there are just two edges, necessarily conjugate, directed from x^* to x, and there is no edge directed from x^* to any vertex of $X - \{x\}$. We denote the two edges directed from x^* to x by $B(x)$ and $B(x)^*$, and we write their common capacity as $h(x)$.

A bipartition $\{X, X^*\}$ of D is *directed* if each edge has its negative end in X and its positive end in X^*. For our present purposes a directed bipartition is equivalent to a restricted one in which $h(x) = 0$ for each $x \in X$. This implies that $b(x) = g(x)$. Thus for a directed bipartition $L(X)$ is the maximum number of edges such that the number incident with a vertex y does not exceed $g(y)$. Theorem 5.1 thus gives rise to an antisymmetrical matching theorem.

6. A problem of Gallai. Let G be a graph. We define a *path* in G as a sequence of the form (1) in the same way as for a digraph, except that an edge A_i is required to have a_{i-1} and a_i as its two ends instead of to be directed from a_{i-1} to a_i.

A path P in G is *circular* if it is non-degenerate and re-entrant and no vertex-term, other than the last, is a repetition of a preceding one. A *track* in G is a non-degenerate path which is simple or circular.

With each vertex v of G let there be associated two non-negative integers $g(v)$ and $h(v)$. An *admissible track-set* in G is a set S of tracks in G satisfying the following conditions:

(i) *No two members of S traverse a common edge.*

(ii) *The number of tracks of S originating or terminating at a vertex v, circular tracks being counted twice, does not exceed $g(v)$.*

(iii) *The number of tracks of S passing through a vertex v, other than as origin or terminus, does not exceed $h(v)$.*

Let $L(G)$ denote the maximum number of members for an admissible track-set S in G. We discuss the problem of characterizing $L(G)$ in terms of the structure of G. This problem was posed and solved by T. Gallai **(5)**. Here we express the problem in terms of the theory of antisymmetrical digraphs.

We derive an antisymmetrical digraph D from G as follows. We replace each vertex v of G by two conjugate vertices v and v^* of D. An edge A of G with ends x and y is replaced by two conjugate edges, A and A^*, one directed from x to y^* and the other from y to x^*. Finally, for each vertex v of G we adjoin two con-

https://doi.org/10.4153/CJM-1967-101-8 Published online by Cambridge University Press

jugate edges $B(v)$ and $(B(v))^*$, each directed from v^* to v. We assume no unnecessary coincidences between the edges and vertices so defined. Thus $E(G)$ and $(E(G))^*$ are disjoint subsets of $E(D)$, and neither of them meets $\{B(v), (B(v))^*\}$.

For each $v \in V(G)$ we write $g(v^*) = g(v)$. We assign the capacity $h(v)$ to $B(v)$ and $(B(v))^*$. For each $A \in E(G)$ we assign unit capacities to A and A^* in D.

We note that D has a restricted bipartition $\{X, X^*\}$, where $X = V(G)$.

Consider a path

$$P = (a_0, A_1, a_1, A_2, a_2, \ldots, a_{n-1}, A_n, a_n)$$

in G. Then there is a path

$$P' = (a_0, C_1, (a_1)^*, C(a_1), a_1, C_2, (a_2)^*, C(a_2), a_2, \ldots, a_{n-1}, C_n, (a_n)^*)$$

from a_0 to $(a_n)^*$ in D, where C_i is whichever of the edges A_i and $(A_i)^*$ of D is directed from a_{i-1} to $(a_i)^*$, and $C(a_i)$ is chosen arbitrarily from the pair $\{B(a_i), (B(a_i))^*\}$. We call P' a *derived path* of P.

6.1. *If P is a track in G from u to v, then any derived path P' of P is a simple skew path in D from u to v^*.*

Proof. If P' is not simple, it has a vertex-repetition $a_i = a_j$ or $a_i = (a_j)^*$, where $i \neq j$. Hence P has the vertex-repetition $a_i = a_j$. We may therefore suppose that $i = 0$ and $j = n$. From P we have $a_0 = a_n$, and from P' we have $a_0 = (a_n)^*$. Hence $(a_n)^* = a_n$, contrary to the definition of D.

If P' is not skew, it traverses two conjugate edges A and A^*. Evidently neither of these is of the form $B(v)$. Hence P traverses A twice, contrary to the definition of a track.

6.2. *If Q is a simple skew path in D from u to v^*, then there is a unique track P in G from u to v such that Q is a derived path of P.*

Proof. We can obtain such a P by replacement of each subsequence of Q of the form $((a_i)^*, C(a_i), a_i)$ by the corresponding single term a_i, and of the last term $(a_n)^*$ of Q by a_n. By the definition of a derived path Q determines P uniquely.

Given a track P in G we can associate with it a derived path P' in D and the conjugate path $(P')^*$ of P'. Conversely, given two conjugate simple skew paths Q and Q^* in D we can associate them with a single track P in G such that Q is a derived path of P. This association enables us to construct an acceptable path-set S from X to X^* in D from an admissible track-set T in G, and conversely, in such a way that $|S| = 2|T|$. Accordingly we have

6.3. $2L(G) = L(X)$, *where $L(X)$ is computed for D.*

There is an extensive literature dealing with problems connected with Gallai's theorem. It is concerned mainly with the special case in which $h(x) = 0$

for each vertex x, and with the related problem of finding a spanning subgraph H of G, that is one such that $V(H) = V(G)$, with a specified valency at each vertex. The works listed as references, apart from **(3)**, are samples of this literature.

References

1. H. B. Belck, *Reguläre Faktoren von Graphen*, J. Reine Angew. Math., *188* (1950), 228–252.
2. J. Edmonds, *Paths, trees and flowers*, Can. J. Math., *17* (1965), 449–467.
3. L. R. Ford, Jr., and D. R. Fulkerson, *Flows in networks* (Princeton, 1962).
4. T. Gallai, *On factorization of graphs*, Acta Math. Acad. Sci. Hungar., *1* (1950), 133–153.
5. ——— *Maximum-minimum Sätze und verallgemeinerte Faktoren von Graphen*, Acta Math. Acad. Sci. Hungar., *12* (1961), 131–173.
6. O. Ore, *Graphs and subgraphs, I–II*, Trans. Amer. Math. Soc., *84* (1957), 109–136 and *93* (1959), 185–204.
7. J. Petersen, *Die Theorie der regularen Graphs*, Acta Math., *15* (1891), 193–220.
8. A. Sainte-Lagüe, *Les réseaux*, Mém. Sci. Math. (Paris), *18* (1926).
9. W. T. Tutte, *The factorization of linear graphs*, J. London Math. Soc., *22* (1947), 107–111.
10. ——— *The factors of graphs*, Can. J. Math., *4* (1952), 314–328.

University of Waterloo,
Waterloo, Ontario

https://doi.org/10.4153/CJM-1967-101-8 Published online by Cambridge University Press