CYCLICALLY ORDERED SETS

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Introduction. The classical separation theorems regarding the relative position of the zeros of two consecutive orthogonal polynomials, or of the positive zeros of two consecutive Bessel Functions, are well known. Less well known is a theorem of Bôcher and Porter (**1**, p. 207), which states that the positive zeros of three consecutive Bessel Functions occur in cyclic order. This paper extends the result to symmetric orthogonal polynomials and more generally to sequences of functions satisfying a certain type of difference equation. The investigation arose in an attempt to prove a conjecture of Schoenberg (**3**) for which it was believed that knowledge of the zeros of polynomials generated by exp $(x + tx^k)$ would be helpful. This conjecture was proved by Edrei (**2**).

1. Notation. By $Z(A)$ we mean the number of elements in the set A. If A is not finite, we do not distinguish among different cardinals, and we write $Z(A) = \infty$. The elements of the various sets discussed are real numbers. Any term involving order such as "consecutive," "least," etc. refers to the natural order of the real numbers. We denote the empty set by ϕ.

If I is an interval, then we denote $Z(A \cap I)$ by $Z_I(A)$. The symbols I, J, I', I_n, J^* always refer to intervals. The term interval is used in the wide sense and includes those which are open or closed at either end and which may be finite, semi-infinite, or infinite. The symbol k always refers to some fixed but unspecified integer greater than one.

2. Cyclically ordered sets. Let $a_1 < a_2 < \ldots < a_N$ be N real numbers. We consider the sets E_μ $\mu = 1, 2 \ldots k$, such that $a_m \in E_\mu$ if and only if $m \equiv \mu$ (mod k). The finite sets E_μ are examples of cyclically ordered sets. If $0 \leqslant N < k$, some or all of the sets will be empty. We state without proof some of the properties of the sets E_μ. Let E denote $\bigcup_\mu E_\mu$:

(i) $E_\mu \cap E_\lambda = \phi, \qquad \lambda \neq \mu$

(ii) Any k consecutive elements of E include one from each of the sets E_μ.

(iii) If $Z(E_\lambda) = 0$ for some λ, $Z(E_\mu) \leqslant 1$ for all μ.

(iv) $|Z(E_\lambda) - Z(E_\mu)| \leqslant 1$ for all λ and μ.

(v) Between any two elements of E_λ there is an element of E_μ if $\lambda \neq \mu$.

(vi) If $E^*_\mu = E_\mu \cap I$, where I is a fixed interval and $E^* = E \cap I = \bigcup_\mu E_\mu{}^*$, then the sets E^*_μ and E^* satisfy (i)–(v).

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These properties are not independent. For example, (iii) is a consequence of either (iv) or (v). On the other hand (i), (ii), and (iii) imply (iv), (v), and (vi). We now formulate the following definition:

Definition. We say that the k sets $E_1, E_2, \ldots E_k$ are *cyclically ordered* (*C-O*) if and only if they satisfy the following conditions

C1: $E_\lambda \cap E_\mu = \phi, \qquad \lambda \neq \mu.$

C2: $Z_J(E_\lambda) \leqslant Z_J(E_\mu) + 1$ for all λ, μ and an arbitrary interval J.

THEOREM 1. *Let $E_1, E_2, \ldots, E_k$ be C-O. Let $E^*_\mu = E_\mu \cap J$ where J is a fixed interval. Then $E^*_1, E^*_2, \ldots, E^*_k$ are C-O.*

Proof. C1 is satisfied since $E^*_\lambda \cap E^*_\mu = E_\lambda \cap E_\mu \cap J$.

C2 is satisfied since $Z_I(E^*_\lambda) = Z_{I^*}(E_\lambda)$ where for an arbitrary interval, I, $I^* = I \cap J$.

THEOREM 2. *A necessary and sufficient condition that the sets $E_1, E_2, \ldots, E_k$ be C-O, is that each pair of sets E_λ, E_μ ($\lambda \neq \mu$) be C-O.*

Proof. The proof is immediate since C1 and C2 are required to hold for all pairs of sets E_λ, E_μ.

THEOREM 3. *Let $E_1, E_2, \ldots, E_k$ be C-O and let $E = \bigcup_\mu E_\mu$. Then*

$$Z_I(E) \leqslant k \Rightarrow Z_I(E_\mu) \leqslant 1,$$

for all μ. In particular $Z_I(E) = k, \Rightarrow Z_I(E_\mu) = 1$ for all μ.

Proof. By C1,

$$Z_I(E) = \sum_\mu Z_I(E_\mu)\,. \quad Z_I(E_\lambda) > 1 \Rightarrow Z_I(E_\mu) \geqslant 1$$

for all μ by C2. Hence $Z_I(E_\lambda) > 1 \Rightarrow Z_I(E) > k$ and thus $Z_I(E) \leqslant k \Rightarrow Z_I(E_\mu) \leqslant 1$, all μ.

Further, if $Z_I(E_\lambda) = 0$ for some λ, $Z_I(E) < k$. Hence $Z_I(E) = k \Rightarrow Z_I(E_\mu) = 1$ for all μ.

3. The principal theorem. Historically, cyclically ordered sets have been studied only in the case, $k = 2$ (except for the theorem of Bôcher-Porter), in which case the sets involved are said to be separated. In many cases the sets are zeros of appropriate continuous functions which are related by a linear difference equation of the second order. The functional equation is used to determine the sign of the functions at critical points and the Darboux property of continuous functions is then invoked to show that one function vanishes in some interval. It is rather curious that in such cases the method is used only to show that pairs of sets are *C-O*, since it can usually be used to show that three sets are *C-O*. This can be generalized to the case of certain kth order linear difference equations in which case $k + 1$ sets will be *C-O*. More precisely we have Theorem 4.

Definition. Let $f(x)$ be a continuous function in $a < x < b$. We say that ξ is an odd zero of $f(x)$ in (a,b) if $a < \xi < b$ and

$$\lim_{h\to 0} \operatorname{sgn} f(\xi + h) \operatorname{sgn} f(\xi - h) = -1.$$

THEOREM 4. *Let $\{f_n(x)\}$ $(n = 0, 1, 2, \ldots)$ be a sequence of functions defined in an open interval (a,b). Let E_n, F_n be the sets of odd zeros and zeros of $f_n(x)$, respectively. The functions $f_n(x)$, and the sets E_n, F_n satisfy for $n = 0, 1, 2, \ldots$:*

(1) $f_{n+k}(x) = a_n(x) f_{n+k-1}(x) - b_n(x) f_n(x)$.

(2) $f_n(x)$ *is continuous in* (a,b).

(3) $Z(F_n) \leqslant [n/k] \equiv \mu(n)$.

(4) $\lim_{x\to b^-} \operatorname{sgn} f_n(x) = +1$.

(5) $\lim_{x\to a^+} \operatorname{sgn} f_n(x) = (-1)^{\mu(n)}$.

(6) $a_n(x)$, $b_n(x)$ *are continuous and positive in* (a,b).

Then:

(a) $n > m \geqslant k \Rightarrow \max\{E_n\} > \max\{E_m\}$.

(b) $Z(E_n) = \mu(n)$.

(c) $E_n, E_{n+1}, \ldots, E_{n+k}$ *are C-O for* $n = 0, 1, 2, \ldots$.

Proof. For $0 \leqslant N < k$, $Z(E_n) \leqslant Z(F_n) = 0$. Since F_N is empty, $f_N(x) > 0$ in (a,b). By (2), (4), and (5), $Z(E_k) \geqslant 1$ and hence by (3) $Z(E_k) = 1$. Therefore (a), (b), and (c) are satisfied by the sets $E_0, E_1, \ldots, E_k$. We complete the proof by induction.

We adopt the notation $x(n,j)$ for the elements of E_n (whenever E_n is non-empty) with $x(n,1) > x(n,2) > \ldots$.

Let us assume that for some $N \geqslant k$,

(a*) $x(n,1) > x(m,1)$ for $k \leqslant m < n \leqslant N$,

(b*) $Z(E_n) = \mu(n)$ for $0 \leqslant n \leqslant N$,

(c*) $E_n, E_{n+1}, \ldots, E_{n+k}$ are *C-O*, for $0 \leqslant n \leqslant N - k$.

To complete the proof we must show that:

(a′) $x(N + 1, 1) > x(N, 1)$,

(b′) $Z(E_{N+1}) = \mu(N + 1)$,

(c′) $E_{N-k+1}, E_{N-k+2}, \ldots, E_{N+1}$ are *C-O*.

From (1) we obtain $\operatorname{sgn} f_{N+1}(x(N,1)) = -\operatorname{sgn} f_{N-k+1}(x(N,1))$. But from (2), (4), (6), and (a*) we have $\operatorname{sgn} f_{N-k+1}(x(N,1)) = +1$. Hence $\operatorname{sgn} f_{N+1}(x(N,1)) = -1$, and by (2) and (4) $x(N + 1, 1) > x(N,1)$, which proves (a′).

In the remainder of the proof, m will denote an integer for which $N - k < m \leqslant N$. The symbol $I(j)$ will denote the smallest interval (possibly empty or a single point) which contains $x(m,j)$ for all such m. $I(j)$ is closed. Let j be the smallest integer for which $I(j)$ and $I(j + 1)$ are not disjoint. Then there are two distinct values of m, say m' and m'', for which $x(m',j + 1) \geqslant x(m'',j)$. But this is impossible since the sets $E_{m'}$, $E_{m''}$ are *C-O* by (c*) and Theorem 2 and if $x(m',j + 1) = x(m'',j)$, C1 is violated and if

$x(m', j+1) > x(m'', j)$, C2 is violated in the closed interval $[x(m', j+1), x(m', 1)]$. Hence the intervals $I(j)(j = 1, 2, \ldots)$ are mutually disjoint.

For fixed j, the points $x(m, j)$ are monotonic with respect to m. Specifically $x(N, j) > x(N-1, j) > \ldots$. This is clearly so for $j = 1$, by (a*). For $j > 1$, it is a consequence of Theorem 3 applied to the C-O sets $\{E_m\}$.

Let F denote $\bigcup_m E_m$. If $kj \leqslant N$ the set $F \cap I(j)$ has exactly k elements. The set $I(\mu(N+1))$ has $(N+1) - k\mu(N+1)$ elements.

Since by (c*) E_N and E_{N+1-k} are C-O, it follows that $\operatorname{sgn} f_N(x(N-k+1, j))$ takes the values of ± 1 alternately as j assumes consecutive integer values. This is because all the zeros of $f_N(x)$ are odd zeros and there is precisely one zero of $f_N(x)$ between consecutive zeros of $f_{N-k+1}(x)$. We also know that $\operatorname{sgn} f_N(x(N-k+1, 1)) = -1$. Hence we obtain $\operatorname{sgn} f_N(x(N-k+1, j)) = (-1)^j$. Similarly, $\operatorname{sgn} f_{N-k+1}(x(N, j)) = (-1)^{j+1}$. Using (1) we then obtain $\operatorname{sgn} f_N(x(N, j) = \operatorname{sgn}(x(N-k+1, j)) = (-1)^j$. This means that at the extreme points of $I(j)$, $\operatorname{sgn} f_{N+1}(x) = (-1)^j$ for $j = 1, 2, \ldots, \mu(N)$. But the sets are disjoint. Hence between each consecutive pair of non-empty sets $I(j)$, $f_{N+1}(x)$ must have an odd zero. Also there is an odd zero between $I(1)$ and b. Thus $f_{N+1}(x)$ has at least $\mu(N)$ zeros to the right of $I(\mu(N))$. We have already seen that $I(j)$ is non-empty for $j \leqslant \mu(N)$. If $\mu(N) = \mu(N+1)$ then by (3), $Z(E_{N+1}) = \mu(N+1)$. If $\mu(N) \neq \mu(N+1)$, then the signs of $f_{N+1}(x)$ at the extreme points of $I(\mu(N))$ and at $(a+0)$ are different and hence there is another odd zero between a and $I(\mu(N))$. Hence again $Z(E_{N+1}) = \mu(N+1)$. Thus (b′) is proved and also since E_{N+1} and $I(j)$ is disjoint for every j, we have that C1 is satisfied by E_m and E_{N+1} for all m in $N - k < m \leqslant N$.

To prove (c′) it suffices because of Theorem 2 to show that E_m and E_{N+1} are C-O. We have already established C1 for these sets and it remains only to establish C2.

If C2 is not satisfied by E_m and E_{N+1}, there is some interval, J, in which $|Z_J(E_m) - Z_J(E_{N+1})| > 1$. Hence there must be a subinterval J' in which one of $Z_{J'}(E_m)$ and $Z_{J'}(E_{N+1})$ is 0 and the other is 2.

If $Z_{J'}(E_m) = 2$, J' must intersect $I(j)$ for two values of j. But we have seen that between consecutive intervals $I(j)$ there is a point of E_{N+1} and if $Z_{J'}(E_{N+1}) = 2$, J' must contain an interval $I(j)$, since between consecutive intervals $I(j)$ there is precisely one point of E_{N+1} by (3).

Hence C2 is satisfied by E_{N+1}, and E_m for $N - k < m \leqslant N$ and thus (c′) is proved.

4. Applications of the principal theorem. Let $\{p_n(x)\}$ denote a sequence of orthonormal polynomials in an interval symmetric about the origin, with respect to an even non-negative weight function. We call such sequences "Symmetric orthonormal polynomials." Then the well-known recurrence formula **(4)** specializes to

$$p_{n+1}(x) = A_n x p_n(x) - B_n p_{n-1}(x) \qquad (n \geqslant 1)$$

where A_n, B_n are positive constants. For $k = 2$, the polynomials $\{p_n(x)\}$ satisfy the hypotheses of Theorem 5 in the interval $(0, \infty)$. To see this we note that (1) is satisfied since they satisfy the recursion formula. Since they are polynomials, they are continuous and (2) is satisfied. Condition 3 is satisfied since the zeros are symmetric about the origin. They satisfy (4) since the normalization which asserts that $A_n > 0$, implies that the coefficient of x^n in $p_n(x)$ is positive. To show that (5) is satisfied we note that for n odd, $\operatorname{sgn} p_{n+1}(0) = -\operatorname{sgn} p_{n-1}(0) \neq 0$ and for n even

$$\operatorname{sgn} \frac{p_{n+1}(x)}{x}\bigg|_{x=0} = -\operatorname{sgn} \frac{p_{n-1}(x)}{x}\bigg|_{x=0}.$$

Hence it is only necessary to verify (5) for $n = 0, 1$. But $p_0(x)$ is a positive constant and $p_1(x) = \alpha x$ where $\alpha > 0$.

Hence Theorem 4 applies and we have the following result:

THEOREM 5. *Let $\{p_n(x)\}$ be a sequence of symmetric orthonormal polynomials. Let P_n denote the set of positive zeros of $p_n(x)$. Then P_n, P_{n+1}, P_{n+2} are C-O for $n = 0, 1, 2, \ldots$.*

Theorem 5 supplements the well-known fact that E_n, E_{n+1} are *C-O* where E_n denotes the set of zeros of $p_n(x)$. We cannot expect E_n, E_{n+1}, E_{n+2} to be *C-O*, since C2 would be violated. Similarly we cannot expect P_n, P_{n+1}, P_{n+2}, P_{n+3} to be *C-O*.

We mention here without proof that Theorem 5 could be proved independently of Theorem 4, by using only the following well-known properties of symmetric orthonormal polynomials **(4)**.

(1) E_n, E_{n+1} are *C-O*.

(2) If $n > m$, there is a point of E_n in any open interval bounded by points of E_m.

(3) E_n is symmetric about zero.

(4) $Z(E_n) = n$.

We now consider another example. Let $p_n(t, k) = p_n = p_n(t)$ be defined by the generating function:

$$F(x,t) = \exp(x + tx^k) = \sum_{n=0}^{\infty} p_n(t;k)x^n. \tag{4.1}$$

As usual, k denotes some integer > 1. Differentiating both sides of (4.1) with respect to x we obtain

$$(1 + kt\, x^{k-1}) \sum_{n=0}^{\infty} p_n x^n = \sum_{n=0}^{\infty} np_n x^{n-1}. \tag{4.2}$$

If we equate coefficients we obtain

$$mp_m = p_{m-1} + ktp_{m-k} : m \geqslant k. \tag{4.3}$$

We wish to apply Theorem 4 to the sequence $\{p_n(t)\}$ in the interval $(-\infty, 0)$.

Now $\exp(x + tx^k) = \exp(x) \exp(tx^k)$ and hence

$$p_n(t) = \sum_{m+kq=n} \frac{1}{m!} \frac{t^q}{q!} \tag{4.4}$$

where the summation is extended over all non-negative integers m, q satisfying the indicated condition.

This shows that $p_n(t)$ is a polynomial of precise degree $[n/k]$ with all coefficients positive. Hence conditions (2), (3), (4), (5) of Theorem 4 are satisfied.

We use the following consequences of (4.4) in showing conditions (4) and (5) are satisfied.

$$p_n(o) = \frac{1}{n!}$$

$$p_n(t) \sim \frac{t^{[n/k]}}{(n - k\,[n/k])! \cdot [n/k]!} \text{ as } |t| \to \infty\ .$$

Hence if E_m denotes the set of zeros of $p_m(t;k)$ which are negative, the sets E_n, E_{n+1}, . . . E_{n+k} are *C-O*.

When $k = 2$, this result can be interpreted in terms of the Hermite polynomials, $H_n(x)$ defined by

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} w^n = \exp(2xw - w^2). \tag{4.5}$$

Hence

$$H_n(z) = 2^n n! p_n\left(-\frac{1}{4z^2}, 2\right) \cdot z^n$$

and if y_1, y_2, . . . are zeros of $p_n(y)$ then

$$Z_j = 2\sqrt{1/y_j}$$

are positive zeros of $H_n(x)$. Hence the sets of positive zeros of $H_n(x)$, $H_{n+1}(x)$, $H_{n+2}(x)$ are *C-O*. This was obtained earlier as a special case of the result concerning symmetric orthogonal polynomials.

5. Further examples. We have already seen that the sets of three consecutive Legendre polynomials are *C-O*. For fixed $n \geqslant 2$, let Q, Q', Q'' denote the sets of positive zeros of $P_n(x)$, $P'_n(x)$, $P''_n(x)$ respectively. We shall show that Q, Q', Q'' are *C-O*. The Legendre polynomials satisfy two important functional equations:

$$(n + 1)P_{n+1}(x) - (2n + 1)P_n(x) + nP_{n-1}(x) = 0; \tag{5.1}$$

and

$$(1 - x^2)P''_n(x) - 2xP'_n(x) + n(n + 1)P_n(x) = 0. \tag{5.2}$$

In Theorem 4, we used (5.1) to deduce $\operatorname{sgn} P_{n+1}(x)$ at various critical points. Similarly, we may use (5.2) to deduce $\operatorname{sgn} P_n(x)$ at the zeros of $P'_n(x)$ and $P''_n(x)$.

$$\text{For } x \in Q, \quad \operatorname{sgn} P'_n(x) = \operatorname{sgn} P''_n(x).$$

$$\text{For } x \in Q', \quad \operatorname{sgn} P_n(x) = -\operatorname{sgn} P''_n(x).$$

$$\text{For } x \in Q'', \quad \operatorname{sgn} P_n(x) = \operatorname{sgn} P'_n(x).$$

Also since $Q \subset (0,1)$, we have by the standard normalization,

$$\operatorname{sgn} P_n(1) = \operatorname{sgn} P'_n(x) = \operatorname{sgn} P''_n(1) = 1.$$

From (5.2) and the simplicity of the zeros of $P_n(x)$ it is clear that Q and Q'' are disjoint. It remains only to verify C2.

Let us assume an interval J for which $Z_J(Q) > Z_J(Q'') + 1$. Then there is an interval $I \subset J$, such that

$$Z_I(Q) = 2, \qquad Z_I(Q'') = 0.$$

Since Q, Q' are *C-O*, there is precisely one point of Q' in I. Let x_1, x_2 be the two points of Q in I. Then $\operatorname{sgn} P'_n(x_1) = -\operatorname{sgn} P'_n(x_2)$, and hence $\operatorname{sgn} P''_n(x_1) = -\operatorname{sgn} P''_n(x_2)$, and thus $Z_I(Q'') \geqslant 1$. But this is a contradiction. Hence there is no interval J in which $Z_J(Q) > Z_J(Q'') + 1$. Similarly there is no interval J in which $Z_J(Q'') > Z_J(Q) + 1$. Hence C2 is satisfied and Q, Q'' are *C-O*.

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