ON SOME RECENT DEVELOPMENTS IN THE THEORY OF SERIES

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In a number of recent papers, especially by Wilansky (**4**; **6**), Zeller (**8**), and Peyerimhoff (**3**), the sequence-to-sequence transformation

$$A: \qquad y_n = \sum_{k=0}^{\infty} a_{nk} x_k \qquad (n = 0, 1, \ldots)$$

has been studied under certain conditions, designated by FAK, PMI, etc. (see §3). The purpose of this note is to point out some relations among these conditions, and to show that some theorems previously obtained hold under weaker assumptions.

We begin with a remark concerning Mazur's well-known consistency theorem. This has been proved by several authors, (Mazur (**2**, Theorem 7), Banach (**1**, p. 95, Theorem 12), Wilansky (**4**, Theorem 3.3.1), Peyerimhoff (**3**, Theorem 4.2)) under various conditions and by various methods of proof. We give here a simple direct proof of the theorem, using only the assumptions that A is co-regular and reversible, as defined below.

Notation. We shall assume throughout the paper that A satisfies the "row-norm" condition: there is a constant M such that $\Sigma_k |a_{nk}| < M$ $(n = 0, 1, \ldots)$. We denote the column limits of A by

$$a_k = \lim_{n\to\infty} a_{nk} \qquad (k = 0, 1, \ldots),$$

the row sums by

$$\alpha_n = \sum_k a_{nk} \qquad (n = 0, 1, \ldots),$$

and we put

$$\alpha = \lim_n \alpha_n, \quad \rho_A = \alpha - \sum_k a_k.$$

If A limits every convergent sequence, or equivalently, if a_k, α exist, A is called *conservative*; if $\rho_A \neq 0$, A is *co-regular*, while if $\alpha = 1$, $a_k = 0$ $(k = 0, 1, \ldots)$ so that $\lim_n y_n = \lim_k x_k$ for each convergent sequence $\{x_k\}$, A is *regular*. Similarly we denote the column limits of a matrix B by b_k, and so on. We define

$$\delta^k = \{\delta_n^k\} = \{0, 0, \ldots, 0, 1, 0, \ldots\},$$

and $a^* = \{a, a, \ldots\}$ for any real number a. We denote the set of all sequences δ^k by Δ, and the same with 1^* adjoined, by Φ. The set of all sequences $\{x_k\}$ for which $\{y_n\}$ converges is denoted by (A), and the set of all $\{x_k\}$ such that

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$y_n \to 0$ by $(A)_0$. A summability method is *reversible* if to each convergent sequence $\{y_n\}$ there corresponds a unique sequence $\{x_k\}$. It is well known that if A is reversible, (A) and $(A)_0$ are Banach spaces under the norm

$$||x|| = \sup_n |\sum_k a_{nk} x_k|,$$

and that the general continuous linear functional (c.l.f.) on (A) is given by

$$f(x) = tA(x) + \sum_n t_n A_n(x), \tag{1}$$

where $A_n(x) = \Sigma_k a_{nk} x_k$, $A(x) = \lim_n A_n(x) = A\text{-}\lim x_k$, and $\Sigma |t_n| < \infty$. If two methods A, B agree on $(A) \cap (B)$, they are called *consistent*. Evidently if A is conservative and $(B) \supseteq (A)$, then B is also conservative.

THEOREM 1. *Let A be reversible and co-regular. Then in order that A be consistent with every method B such that*

$$(B) \supseteq (A) \text{ and } B(x) = A(x) \text{ on } \Phi,$$

it is necessary and sufficient that for any sequence $\{t_n\}$,

$$\text{M} \qquad \left.\begin{array}{l} \sum |t_n| < \infty \\ \sum_n t_n a_{nk} = 0 \quad (k = 0, 1, \ldots) \end{array}\right\} \text{imply } t_n = 0 \qquad (n = 0, 1, \ldots).$$

Proof. Every method B with $(B) \supseteq (A)$ represents a c.l.f. on (A), since each x_k is a c.l.f. (**1**, p. 47) and therefore so is $\lim_n \Sigma_k b_{nk} x_k$ (**1**, p. 23, Theorem 4). Conversely every c.l.f. on (A) can be represented by a matrix method B with $(B) \supseteq (A)$, for example with $f(x)$ as in (1) we may let $b_{nk} = t_0 a_{0k} + t_1 a_{1k} + \ldots + t_{n-1} a_{n-1,k} + t a_{nk}$. Hence, for A to have the property stated it is necessary and sufficient that every c.l.f. which vanishes on Φ should vanish throughout (A), that is,

$$\text{M}_1 \quad \left.\begin{array}{l} \sum |t_n| < \infty \\ t a_k + \sum_n t_n a_{nk} = 0 \quad (k = 0, 1, \ldots) \\ t\alpha + \sum_n t_n \alpha_n = 0 \end{array}\right\} \text{imply} \left\{\begin{array}{ll} t_n = 0 & (n = 0, 1, \ldots) \\ t = 0. \end{array}\right.$$

But M_1 is equivalent to M. For we have by absolute convergence,

$$\sum_k \sum_n t_n a_{nk} = \sum_n \sum_k t_n a_{nk} = \sum_n t_n \alpha_n.$$

Hence the left-hand side of M_1 is equivalent to

$$\begin{array}{ll} \sum |t_n| < \infty, & \\ t a_k + \sum_n t_n a_{nk} = 0 & (k = 0, 1, \ldots), \\ t(\alpha - \sum_k a_k) = 0. & \end{array}$$

Since $\alpha - \Sigma_k a_k \neq 0$, the assertion now follows. This proves the theorem.

Several theorems previously stated for normal matrices (that is, triangular with non-zero diagonal terms) can easily be extended to reversible matrices. We give one example (compare Peyerimhoff, (**3**, Theorem 4.4)). It is known (**1**, p. 50) that if $y_p = \Sigma_k a_{pk} x_k$ with A reversible, there exist constants c_k, c_{kp} with $\Sigma_p |c_{kp}| < \infty$ for each k, such that for each convergent sequence $\{y_p\}$ we have

$$x_k = c_k \lim_p y_p + \sum_p c_{kp} y_p. \tag{2}$$

THEOREM 2. *Let A be reversible and let x_k be represented as in* (2). *If the matrix* (c_{kp}) *has bounded columns, then* M *holds.*

Proof. Assume $\Sigma |t_n| < \infty$, $\Sigma_n t_n a_{nk} = 0$ $(k = 0, 1, \ldots)$. Then $\Sigma_k c_{kp}\, \Sigma_n t_n a_{nk} = 0$. By absolute convergence we have $\Sigma_n t_n\, \Sigma_k a_{nk} c_{kp} = 0$. Now by a lemma of Wilansky (**5**, Lemma 3), we have $\Sigma_k a_{nk} c_{kp} = \delta_p{}^n$. Hence $t_p = 0$ $(p = 0, 1, \ldots)$ and so M holds.

We now state the conditions referred to in the introduction. If $\Sigma a_k x_k$ converges for each $x \in (A)$, the matrix A is said to have *maximal inset* (**6**, p. 648). If every matrix B with $(B) = (A)$ has maximal inset, A has the property of *propagation of maximal inset* (briefly, A has PMI).

The conditions of Zeller which will next be stated were defined for elements of any FK-space E (**7**; **8**), but in the present paper E will be one or other of the Banach spaces (A), $(A)_0$. For a given $x = \{x_k\} \in E$, the rth *segment* (Abschnitt) is the sequence

$$x^{(r)} = \{x_0, x_1, \ldots, x_r, 0, 0, \ldots\}.$$

The property AK (Abschnittskonvergenz) is that for a given x we have $x^{(r)} \to x$ or equivalently $\Sigma x_k\, \delta^k = x$. If this holds for each $x \in E$, then E is said to have AK, which is equivalent to Δ being a basis for E (**1**, p.110). It is known (**8**, Beispiel 4.2) that if C_1 is the Cesàro method of order 1, $(C_1)_0$ has AK. Similarly SAK (schwache Abschnittskonvergenz) means $f(x^{(r)}) \to f(x)$ for each f defined on E, or equivalently $\Sigma x_k f(\delta^k) = f(x)$, FAK (funktionale Abschnittskonvergenz) that $\Sigma x_k f(\delta^k)$ converges for each f, not necessarily to $f(x)$, and AD (Abschnittsdichte) that x is a limit point of the set of all segments. For E to have AD it is necessary and sufficient that Δ be fundamental in E (**1**, p. 58).

As for the relations among these conditions, we have obviously the logical implications AK $\to$ SAK $\to$ FAK, and by a standard theorem on weak convergence (**1**, p. 134), SAK $\to$ AD. It has been proved by Wilansky (**6**, Lemma 16) that if A is reversible, co-regular and has PMI, then (A) has FAK. We shall show that by modifying the proof we may reduce the assumption that A is co-regular and arrive at the following result.

THEOREM 3. *Let A be a reversible, conservative matrix. Then A has* PMI *if and only if* (A) *has* FAK.

Proof. (a) Since every matrix B with $(B) = (A)$ represents a c.l.f. $B(x)$ on (A), with $B(\delta^k) = b_k$, it is obvious that FAK $\to$ PMI.

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(b) Let A have PMI, and let $f(x) = tA(x) + \Sigma_n t_n A_n(x)$ be a c.l.f. on (A). If θ, θ^n denote the solutions of $y = Ax$ when y equals $1^*, \delta^n$ respectively, we have

$$t_n = f(\theta^n), \quad t = t(f) = f(\theta) - \sum f(\theta^n).$$

It is well known that if $t \neq 0$, the corresponding matrix B (see the proof of Theorem 1) has $(B) = (A)$. (Indeed $B = TA$, where T is the matrix whose nth row is $(t_0, t_1, \ldots, t_{n-1}, t, 0, 0, \ldots)$ and it can be shown (**6**, Lemma 1) that T sums only convergent sequences.) We then have at once from PMI that $\Sigma x_k f(\delta^k) = \Sigma b_k x_k$ converges. If however $t = 0$, we define $g(x) = A(x) + f(x)$ on (A). Then

$$g(\theta) = 1 + f(\theta), \; g(\theta^n) = f(\theta^n),$$

so $t(g) = g(\theta) - \Sigma g(\theta^n) \neq 0$, and $\Sigma x_k g(\delta^k)$ converges. But $g(\delta^k) = a_k + f(\delta^k)$ and so

$$\sum x_k f(\delta^k) = \sum x_k g(\delta^k) - \sum a_k x_k$$

converges. Hence (A) has FAK.

THEOREM 4. *Let A be reversible and co-regular. Then A has* PMI *if and only if Φ is a basis for (A).*

This is proved by Wilansky (**6**, p. 650) under the assumption that A is normal. But an examination of the proof shows that this is introduced only because at a certain point it is shown that A satisfies condition M, and one wishes to conclude that Φ is fundamental in (A). Theorem 1 shows that reversibility is sufficient for this.

THEOREM 5. *Let A be reversible and regular. Then $(A)_0$ has* AK *if and only if $(A)_0$ [or equivalently (A)] has* FAK.

Proof. Let $(A)_0$ have AK; then by a general implication already mentioned, $(A)_0$ has FAK, whence by an easy deduction (**8**, Beispiel 4.4), (A) has FAK. Conversely, let (A) have FAK. Then by Theorem 3, A has PMI, and by Theorem 4, Φ is a basis for (A). But $\Delta \subset (A)_0$ and $1^* \notin (A)_0$, hence Δ is a basis for $(A)_0$, and $(A)_0$ has AK.

Remark. It is shown by Zeller (**8**, Theorem 3.4) that for any FK-space E, FAK and AD together imply AK. Theorem 5 shows that for certain spaces AD can be dropped.

The relation between M and PMI for a regular reversible method can be summarized: M means that Δ is fundamental in $(A)_0$, or Φ in (A); PMI that Δ is a basis for $(A)_0$, or Φ for (A).

THEOREM 6. *Let A be reversible, regular, and have* PMI. *Then for any matrix B with $(B) \supseteq (A)$ we have the representation*

$$B(x) = \rho_B A(x) + \sum b_k x_k, \tag{3}$$

valid for each $x \in (A)$.

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Proof. By Theorem 5, $(A)_0$ has AK and therefore SAK. Now $B(x)$ is a c.l.f. on $(A)_0$ and so $B(x) = \Sigma x_k B(\delta^k) = \Sigma b_k x_k$ for $x \in (A)_0$. For any $x \in (A)$ with $A(x) = \sigma$, we write $x = \sigma^* + (x - \sigma^*)$, with $x - \sigma^* \in (A)_0$. Then

$$\begin{aligned} B(x) &= B(\sigma^*) + B(x - \sigma^*) \\ &= \beta\sigma + \sum b_k(x_k - \sigma) \\ &= \rho_B\sigma + \sum b_k x_k, \end{aligned}$$

which proves the theorem.

If B is co-regular a simpler argument, based on the matrix $(1/\rho_B)\ (b_{nk} - b_k)$, suffices. The condition PMI is obviously necessary, as without it (3) would not be defined even for all B with $(B) = (A)$.

References

1. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).

2. S. Mazur, *Eine Anwendung der Theorie der Operationen bei der Untersuchung der Toeplitzschen Limitierungsverfahren*, Studia Mathematica, *2* (1930), 40–50.

3. A. Peyerimhoff, *Konvergenz- und Summierbarkeitsfaktoren*, Math. Z., *55* (1951), 23–54.

4. A. Wilansky, *An application of Banach linear functionals to summability*, Trans. Amer. Math. Soc., *67* (1949), 59–68.

5. ———, *Convergence fields of row-finite and row-infinite reversible matrices*, Proc. Amer. Math. Soc., *3* (1952), 389–391.

6. ———, *Summability: The inset, replaceable matrices, the basis in summability space*, Duke Math. J., *19* (1952), 647–660.

7. K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z., *53* (1951), 463–487.

8. ———, *Abschnittskonvergenz in* FK-*Räumen*, Math. Z., *55* (1951), 55–70.

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