Glasgow Mathematical Journal (2024), **66**, pp. 471–478
doi:10.1017/S0017089524000090

Glasgow Mathematical Journal

RESEARCH ARTICLE

Galois points and Cremona transformations

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Received: 11 August 2023; **Revised:** 26 February 2024; **Accepted:** 13 March 2024; **First published online:** 11 April 2024

Keywords: Galois points; Cremona transformations; Galois groups; Jonquières maps

2020 Mathematics Subject Classification: *Primary* - 14E07; *Secondary* - 14H05, 14H50

Abstract
In this article, we study Galois points of plane curves and the extension of the corresponding Galois group to $\mathrm{Bir}(\mathbb{P}^2)$. We prove that if the Galois group has order at most 3, it always extends to a subgroup of the Jonquières group associated with the point P. Conversely, with a degree of at least 4, we prove that it is false. We provide an example of a Galois extension whose Galois group is extendable to Cremona transformations but not to a group of de Jonquières maps with respect to P. In addition, we also give an example of a Galois extension whose Galois group cannot be extended to Cremona transformations.

1. Introduction

Let k be an algebraically closed field. Let C be an irreducible plane curve in $\mathbb{P}^2$. Giving a point $P \in \mathbb{P}^2$, we consider the projection $\pi_P|_C : C \dashrightarrow \mathbb{P}^1$, which is the restriction of the projection $\pi_P : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with centre P. Let $K_P = \pi_P^*(k(\mathbb{P}^1))$. If $P \in C$ [resp. $P \notin C$], we say that P is an inner [resp. outer] Galois point for C if $k(C)/K_P$ is Galois. In this case, we write $G_P = \mathrm{Gal}(k(C)/K_P)$ and call it the Galois group at P. A de Jonquières map is a birational map φ for which there exist $P \in \mathbb{P}^2$ such that φ preserves the pencil of lines passing through P. The group of a de Jonquières transformations preserving the pencil of lines passing through a given point $P \in \mathbb{P}^2$ is denoted by $\mathrm{Jonq}_P \subset \mathrm{Bir}(\mathbb{P}^2)$, this corresponding to ask that ϕ preserves a pencil of lines through the point P. As in ref. [4], we are interested in the extension of elements of G_P to $\mathrm{Bir}(\mathbb{P}^2)$. There are two interesting questions:

Question 1.1. *If P is Galois, does G_P extends to* $\mathrm{Bir}(\mathbb{P}^2)$*?*

Question 1.2. [8] *If an element extends to* $\mathrm{Bir}(\mathbb{P}^2)$*, does it extend to a de Jonquières map? i.e. to an element $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ with $\pi_P \circ \varphi = \pi_P$?*

Consider a point $P \in \mathbb{P}^2$ with multiplicity m_P on an irreducible plane curve C in $\mathbb{P}^2$ of degree d, we will show later that the extension $[k(C) : K_P]$ has degree $d - m_P$. Our first main result is the following theorem, that considers the case of degree 3.

Theorem A. *Let $P \in \mathbb{P}^2$, let $C \subset \mathbb{P}^2$ be an irreducible curve. If the extension $k(C)/K_P$ is Galois of degree at most 3, then G_P always extends to a subgroup of $\mathrm{Jonq}_P \subseteq \mathrm{Bir}(\mathbb{P}^2)$.*

Theorem A resulted from Theorem 3.2, which provides more information on the Galois extensions of degree at most 3 and the related Galois Groups at a point P. This encourages us to study higher-degree Galois extensions and determine if their Galois groups G_P can always be extended to $\mathrm{Bir}(\mathbb{P}^2)$ as well as to the group of de Jonquières map with respect to P. The following theorem gives a negative answer to Question 1.2.

Theorem B. *Let k be a field of characteristic $char(k) \neq 2$ containing a primitive fourth root of unity, and let C be the irreducible curve defined by the equation $X^4 - 4ZYX^2 - ZY^3 + 2Z^2Y^2 - YZ^3 = 0$, then the point $P = [1:0:0]$ is an outer Galois point of C and the extension induced by the projection $\pi_P : C \dashrightarrow \mathbb{P}^1$ is Galois of degree 4. The group G_P extends to $\mathrm{Bir}(\mathbb{P}^2)$ but not to Jonq_P.*

The following result gives a negative answer to Question 1.1 (see also [8, Example 5]); it follows from Lemma 5.2.

Theorem C. *Let k be a field with $char(k) \neq 5$ that contains a primitive 5th root of unity, and let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ given by $\phi : [u:v] \mapsto [uv^6 - u^7 : u^5(u^2+v^2) : v^5(u^2+v^2)]$. We define $C := \overline{\phi(\mathbb{P}^1)}$ which is an irreducible curve of $\mathbb{P}^2$, then the point $P = [1:0:0]$ is an inner Galois point of C and the extension induced by the projection $\pi_P : C \dashrightarrow \mathbb{P}^1$ is Galois of degree 5. Moreover, the identity is the only element of the Galois group that extends to $\mathrm{Bir}(\mathbb{P}^2)$.*

Remark 1.3. *After this article was uploaded to the ArXiv, [5] was uploaded. Theorem 1 in [5] corresponds to the case $k = \mathbb{C}$ of Theorem A.*

2. Preliminaries

The concept of Galois points for irreducible plane curve $C \subset \mathbb{P}^2$ was introduced by [6], [8], [1]. In order to study the extension of an element in G_P to $\mathrm{Bir}(\mathbb{P}^2)$, we need the following lemma.

Lemma 2.1. *The field extension $k(\mathbb{P}^1) \hookrightarrow k(C)$ induced by π_P has degree $d - m_P$, where m_P is the multiplicity of C at P, and d is the degree of C.*

Proof. Let $F(X,Y,Z) = 0$ be the defining equation of C of degree d. We may fix $P = [1:0:0] \in \mathbb{P}^2$ and choose that C is not the line $Z = 0$. Since P has multiplicity m_P, then the equation of C is $F(X,Y,Z) = F_{m_p}(Y,Z)X^{d-m_P} + \ldots\ldots + F_{d-1}(Y,Z)X + F_d(Y,Z)$, where $F_i(Y,Z)$ is a homogeneous polynomial of Y and Z of degree i ($m_P \leq i \leq d$) and $F_{m_P}(Y,Z) \neq 0$. Since $F(X,Y,Z)$ is irreducible in $k[X,Y,Z]$ and not a multiple of Z, then $f = F(X,Y,1) \in k[X,Y]$ is also irreducible in $k[X,Y]$. We can see f as an irreducible polynomial in $\tilde{k}[X]$ with $deg_X(f) = d - m_P$ where $\tilde{k} = k(Y)$. Hence, the extension $k(C)/K_P$ is isomorphic to $(k(Y)[X]/(f))/(k(Y))$, and thus, it has the same degree equal to the degree of the irreducible polynomial $f \in \tilde{k}[X]$, so $[k(C):\tilde{k}] = deg_X(f) = d - m_P$. □

It is well known [2, Ch. 1, Theorem 4.4] that for any two varieties X and Y, there is a bijection between the set of dominant rational maps $\varphi : X \dashrightarrow Y$, and the set of field homomorphisms $\varphi^\star : k(Y) \to k(X)$. In particular, we obtain:

Lemma 2.2. *For each variety X, we have a group isomorphism $\mathrm{Bir}(X) \xrightarrow{\simeq} \mathrm{Aut}_k(k(X))$ which sends φ to φ^*.*

Lemma 2.3. *For any field k, we have $\mathrm{Aut}_k(k(x)) = \{x \mapsto (ax+b)/(cx+d);\, a,b,c,d \in k,\ ad - bc \neq 0\}$ and $\mathrm{Aut}_k(k(x)) \cong \mathrm{Bir}(\mathbb{A}^1) \cong \mathrm{Bir}(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1)$.*

Definition 2.4. *Let $P \in \mathbb{P}^2$, we write $\mathrm{Jonq}_P = \{\varphi \in \mathrm{Bir}(\mathbb{P}^2) | \exists\, \alpha \in \mathrm{Aut}(\mathbb{P}^1);\, \pi_P \circ \varphi = \alpha \circ \pi_P\}$ and call it the Jonquières group of P.*

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Lemma 2.5. *Let $P = [1:0:0]$, by taking an affine chart, a de Jonquières map with respect to P is a special case of a Cremona transformation, of the form*

$$\iota^{-1} \circ \mathrm{Jonq}_P \circ \iota = \{(x, y) \mapsto (\frac{ax+b}{cx+d}, \frac{r_1(x)y + r_2(x)}{r_3(x)y + r_4(x)})\}$$

and $\iota : \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$, $(x, y) \longmapsto [x : y : 1]$, *where* $a, b, c, d \in k$ *with* $ad - bc \neq 0$ *and* $r_1(x), r_4(x), r_2(x), r_3(x) \in k(x)$ *with* $r_1(x)r_4(x) - r_2(x)r_3(x) \neq 0$.

Proof. We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{A}^2 & \overset{\iota}{\hookrightarrow} & \mathbb{P}^2 \\ {\scriptstyle \pi_x} \downarrow & & \downarrow {\scriptstyle \pi_P} \\ \mathbb{A}^1 & \overset{\psi}{\hookrightarrow} & \mathbb{P}^1, \end{array}$$

where $\iota : (x, y) \mapsto [x : y : 1]$ and $\psi : x \mapsto [x : 1]$, which gives the equality

$$\iota^{-1} \circ \mathrm{Jonq}_P \circ \iota = \{f \in \mathrm{Bir}(\mathbb{A}^2) | \exists \alpha \in \mathrm{Bir}(\mathbb{A}^1); \alpha \circ \pi_x = \pi_x \circ f\}.$$

Let $f \in \iota^{-1} \circ \mathrm{Jonq}_P \circ \iota$ given by $(x, y) \mapsto (f_1(x, y), f_2(x, y))$, then $\pi_x \circ f : (x, y) \mapsto f_1(x, y)$. Since $\alpha \circ \pi_x = \alpha(x)$, it follows that $f_1(x, y)$ depends only on x and is of the form $f_1(x, y) = (ax + b)/(cx + d)$ where $a, b, c, d \in k$ and $ad - bc \neq 0$ by Lemma 2.3. From Lemma 2.2, f^* is subjective, so $k(x, y) = k((ax + b)/(cx + d), f_2(x, y))$. To describe the second component $f_2(x, y)$, let us define the birational map $\tau : (x, y) \mapsto ((dx - b)/(-cx + a), y)$, hence $\tau \circ f : (x, y) \mapsto (x, f_2(x, y))$ is a birational map since both f and τ are birationals. By Lemma 2.2, $k(x)(f_2(x, y)) = k(x)(y)$. Apply Lemma 2.3 over the field $k(x)$, then $f_2(x, y) = (r_1(x)y + r_2(x))/(r_3(x)y + r_4(x))$. □

Lemma 2.6. *Let* $P, Q \in \mathbb{P}^2$ *and* $C, D \subset \mathbb{P}^2$ *be two irreducible curves, if* $\phi \in \mathrm{Bir}(\mathbb{P}^2)$ *and* $\phi|_C : C \dashrightarrow D$ *is birational map, and there exists* $\theta \in \mathrm{Aut}(\mathbb{P}^1)$ *such that* $\pi_Q \circ \phi = \theta \circ \pi_P$, *then P is a Galois point of C if and only if Q is a Galois point of D. Moreover, if P is Galois, an element of G_P extends an element of* $\mathrm{Bir}(\mathbb{P}^2)$ *(respectively* Jonq_P*) if and only if its image in G_Q extends an element of* $\mathrm{Bir}(\mathbb{P}^2)$ *(respectively* Jonq_Q*)*

$$\begin{array}{ccc} C & \overset{\phi}{\dashrightarrow} & D \\ {\scriptstyle \pi_P} \downarrow & & \downarrow {\scriptstyle \pi_Q} \\ \mathbb{P}^1 & \overset{\theta}{\longrightarrow} & \mathbb{P}^1. \end{array}$$

Proof. Since $\phi|_C$ is birational map from C to D, then $\phi^*|_C : k(D) \to k(C)$ is an isomorphism. Moreover, as $\pi_Q \circ \phi = \theta \circ \pi_P$, we have a commutative diagram

$$\begin{array}{ccc} k(D) & \overset{\phi^*|_C}{\longrightarrow} & k(C) \\ {\scriptstyle \pi_Q^*} \uparrow & & \uparrow {\scriptstyle \pi_P^*} \\ k(\mathbb{P}^1) & \overset{\theta^*}{\longrightarrow} & k(\mathbb{P}^1). \end{array}$$

Therefore, $k(D)/\pi_Q^*(k(\mathbb{P}^1))$ is Galois if and only if $k(C)/\pi_P^*(k(\mathbb{P}^1))$ is Galois. In addition, ϕ conjugates Jonq_P to Jonq_Q and sends any element of $\mathrm{Bir}(\mathbb{P}^2)$ that preserves C onto element of $\mathrm{Bir}(\mathbb{P}^2)$ that preserves D. □

Example 2.7. *Let* $P = Q \in \{[1:0:0], [0:1:0], [0:0:1]\}$, $\phi : [X:Y:Z] \mapsto [YZ : XZ : XY]$ *and* $\theta : [Y:Z] \mapsto [Z:Y]$, *let* $C \subset \mathbb{P}^2$ *be an irreducible curve not equal to* $x = 0$, $y = 0$ *or* $z = 0$ *and let* $D = \phi(C)$, *so we have the following diagram*

$$\begin{array}{ccc} C & \overset{\phi}{\dashrightarrow} & D \\ {\scriptstyle \pi_P} \downarrow & & \downarrow {\scriptstyle \pi_P} \\ \mathbb{P}^1 & \overset{\theta}{\longrightarrow} & \mathbb{P}^1. \end{array}$$

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Thus, if P is a Galois point for C, then P becomes a Galois point for D, this is a particular case of Lemma 2.6 corresponding to [3, *Corollary* 3].

3. Extensions of degree at most three

Lemma 3.1. *Let k be a field and let $L = k[x]/(x^3 + a_2x^2 + a_1x + a_0)$ where $f = x^3 + a_2x^2 + a_1x + a_0$ is a separable irreducible polynomial in $k[x]$, then the field extension L/K is Galois if and only if there exists an element $\sigma \in \mathrm{Gal}(L/K)$ of order 3 such that,*

$$\sigma : x \mapsto \frac{\alpha x + \beta}{\gamma x + \delta} \text{ where } \alpha, \beta, \gamma, \delta \in k \text{ with } \alpha\delta - \beta\gamma \neq 0.$$

Proof. As f is a separable irreducible polynomial of degree 3, the extension L/K is separable of degree 3. It is then Galois if and only if there exists $\sigma \in \mathrm{Aut}(L/K)$ of order 3, so it remains to prove that we can choose σ with the right form. If $\sigma \in \mathrm{Aut}(L/K)$ where, $\sigma : x \mapsto \nu_2x^2 + \nu_1x + \nu_0$ and $\nu_i \in k$ for $i = 0, 1, 2$, so the question here is can we find $\{\alpha, \beta, \gamma, \delta\} \subset K$ with $\alpha\delta - \beta\gamma \neq 0$ such that the following equality holds?

$$\nu_2x^2 + \nu_1x + \nu_0 = \frac{\alpha x + \beta}{\gamma x + \delta}. \tag{1}$$

We can find a solution $\{\alpha = a_2\nu_1\nu_2 - a_1\nu_2^2 + \nu_0\nu_2 - \nu_1^2,\ \beta = a\nu_0\nu_2 - a_0\nu_2^2 - \nu_0\nu_2,\ \delta = a_2\nu_2 - \nu_1,\ \gamma = \nu_2\}$. We observe that $\alpha\delta - \beta\gamma \neq 0$, otherwise we have $\sigma(x) \in K$ and this gives a contradiction as $x \notin K$. □

Theorem 3.2. *Let $P \in \mathbb{P}^2$, let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d: with multiplicity m_P at P. We have $[k(C) : K_P] = d - m_P$.*

1. *If $d - m_P = 1$, then $\pi_P : C \dashrightarrow \mathbb{P}^1$ is a birational.*
2. *If $d - m_P = 2$, P is Galois if and only if the extension is separable, and if this holds, then the non-trivial element $\sigma \in G_P$ of order 2 extends to a de Jonquières map with respect to P.*
3. *If $d - m_P = 3$ and P is Galois, then there is a de Jonquières map with respect to P extending the action.*

Proof. The equality $[k(C) : K_P] = d - m_P$ follows from Lemma 2.1. We may assume $P = [1 : 0 : 0]$. Let $x = X/Z$ and $y = Y/Z$ be affine coordinates. Since the field extension $k(C)/\pi_P^*(k(\mathbb{P}^1))$ is of degree $d - m_P$, then $k(C) = k(y)[x]/(f)$, where $f \in k[x, y]$ is the equation of C in these affine coordinates.

(1) If $d - m_p = 1$, then $k(C) = \pi_P^*(k(\mathbb{P}^1))$ and therefore $\pi_P^* : k(\mathbb{P}^1) \to k(C)$ is an isomorphism. Hence $\pi_P : C \dashrightarrow \mathbb{P}^1$ is birational.

(2) If $d - m_p = 2$, then the extension $k(C)/\pi_P^*(k(\mathbb{P}^1))$ is of degree 2 and it is thus Galois if and only if it is separable. $k(C)/\pi_P^*(k(\mathbb{P}^1))$ is Galois ⇔ there exists an element $\sigma \in G_P$ of order 2 that permutes the roots of f ⇔ f is separable ⇔ the extension is separable. Furthermore, the element $\sigma \in G_P$ of order 2 is given by $x \mapsto -x$ up to a suitable change of coordinates.

(3) If $d - m_P = 3$, the equation of the curve C is given by $f = F_{d-3}(y, 1)x^3 + F_{d-2}(y, 1)x^2 + F_{d-1}(y, 1)x + F_d(y, 1)$. We apply Lemma 3.1, replacing k by $k(y)$. □

We now illustrate Theorem 3.2 in two examples.

Lemma 3.3. *Let k be a field with $\mathrm{char}(k) \neq 3$ that contains a primitive third root of unity. Let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ given by $\phi : [u : v] \mapsto [uv^2 + u^2v : u^3 : v^3]$. We define $C := \overline{\phi(\mathbb{P}^1)}$ is a curve of $\mathbb{P}^2$, then the point $P =$*

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$[1:0:0]$ *is a Galois point of* C *and the extension induced by the projection* $\pi_P : C \dashrightarrow \mathbb{P}^1$ *is Galois of degree* 3. *The element of order* 3 *extends to an element of* Jonq_P.

Proof. The curve C is birational to $\mathbb{P}^1$ via ϕ, with inverse $[X:Y:Z] \mapsto [X+Y:X+Z]$. Define the projection by $\pi_P : [X:Y:Z] \mapsto [Y:Z]$. Let $\psi = \pi_P \circ \phi$, then $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ maps $[u:v]$ to $[u^3:v^3]$, so the extension is Galois of degree 3 with Galois group G_P generated by $\sigma : x \mapsto \omega \cdot x$, where ω is a primitive cubic root of unity. By Theorem 3.2, every element of order 3 extends to an element of Jonq_P. Explicitly, σ extends to the map that is given by

$$[X:Y:Z] \mapsto [\frac{(Y-\omega Z)\,X + YZ(1-\omega)}{(\omega-1)X + Y\omega - Z} : Y : Z].$$

□

Lemma 3.4. *Let* k *be a field with* $char(k) = 3$ *and* $C \subset \mathbb{P}^2$ *given by the polynomial* $f = X^3 - Y^2X + Z^3$, *then the point* $P = [1:0:0]$ *is Galois point of* C *and the extension induced by the projection* $\pi_P : C \dashrightarrow \mathbb{P}^1$ *is Galois of degree* 3.

Proof. Define the birational map $\phi : \mathbb{P}^1 \dashrightarrow C$ by $\phi : [u:v] \mapsto [v^3 : u^3 : u^2v - v^3]$ with inverse $[X:Y:Z] \mapsto [Y : Z+X]$. Define the projection by $\pi_P : [X:Y:Z] \mapsto [Y:Z]$. Let $\psi = \pi_P \circ \phi$, then $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ maps $[u:v]$ to $[u^3 : u^2v - v^3]$, so the extension is Galois of degree 3 with Galois group G_P generated by $\sigma : [u:v] \mapsto [u : u+v]$. By Theorem 3.2, we know that every element of order 3 extends to an element of Jonq_P. Explicitly σ extends to the map that is given by σ that is given by $[X:Y:Z] \mapsto [X+Y:Y:Z]$. □

4. Curves that are Cremona equivalent to a line

Definition 4.1. *Let* X *be a smooth projective variety and* D *a divisor in* X. *Let* K_X *denote a canonical divisor of* X. *We define the Kodaira dimension of* $D \subset X$, *written* $\mathcal{K}(D,X)$ *to be the dimension of the image of* $X \mapsto P(H^0(m(D+K_X)))$ *for* $m >> 0$. *By convention we say that the Kodaira dimension is* $\mathcal{K}(D,X) = -\infty$ *if* $|m(D+K_X)| = \varnothing \; \forall \, m > 0$.

Definition 4.2. [7] *Let* $C \subset \mathbb{P}^2$ *be an irreducible curve. If* C *is a smooth curve, we define* $\bar{\mathcal{K}}(C,\mathbb{P}^2)$ *to be* $\mathcal{K}(C,\mathbb{P}^2)$. *If* C *is a singular curve, we take* $X \to \mathbb{P}^2$ *to be an embedded resolution of singularities of* C *in* $\mathbb{P}^2$ *where* $\tilde{C}$ *is the strict transform of* C, *then we define* $\bar{\mathcal{K}}(C,\mathbb{P}^2)$ *to be* $\mathcal{K}(\tilde{C},X)$. *This does not depend on the choice of the resolution.*

Definition 4.3. *Let* C *be an irreducible smooth plane curve. The curve* C *is said to be Cremona equivalent to a line if there is a birational map* $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *that sends* C *to a line.*

Theorem 4.4. (Coolidge) [7, Theorem 2.6] *Let* $C \hookrightarrow \mathbb{P}^2$ *be an irreducible rational curve. Then, there exists a Cremona transformation* σ *of* $\mathbb{P}^2$ *such that* $\sigma(C)$ *is a line if and only if* $\overline{\mathcal{K}}(C,\mathbb{P}^2) = -\infty$.

Lemma 4.5. *If* $C \hookrightarrow \mathbb{P}^2$ *is an irreducible rational curve of degree* $d < 6$, *then* C *is equivalent to a line.*

Proof. Let $\pi_1 : X_1 \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at P_1, and let $\pi_i : X_i \to X_{i-1}$ the blow-up of X_{i-1} at $P_i \in X_{i-1}$ for $i \geq 2$, $\mathcal{E}_i = \pi_i^{-1}(P_i)$ is a (-1)-curve, where $\mathcal{E}_i^2 = -1$ and $\mathcal{E}_i \cong \mathbb{P}^1$. After blowing up n points, let $\pi : Y \mapsto \mathbb{P}^2$ be the composition of the blow-ups π_i, where $Y = X_n$ we choose enough points such that the strict transform of C is smooth. By induction, we have $E_i = (\pi_{i+1} \circ \ldots\ldots \circ \pi_n)^*(\mathcal{E}_i)$,

$Pic(Y) = \pi^*(Pic(\mathbb{P}^2)) \oplus \mathbb{Z}E_1 \oplus \ldots \oplus \mathbb{Z}E_n$, and $E_i^2 = -1$ for every $i = 1, \ldots, n$, and $E_i \cdot E_j = 0$ for every $i \neq j$. Moreover,

$$K_Y = \pi_n^* \ldots . \pi_1^*(K_{\mathbb{P}^2}) + \Sigma_{i=1}^n \pi_n^* \ldots . \pi_{i+1}^*(\epsilon_i) = \pi^*(K_{\mathbb{P}^2}) + \Sigma_{i=1}^n E_i = -3\pi^*(L) + \Sigma_{i=1}^n E_i.$$

The strict transform $\tilde{C} \subset C$ is equivalent to $\tilde{C} = d \cdot \pi^*(L) - \Sigma_{i=1}^n m_{P_i}(C)E_i$. Hence we have $2K_Y + \tilde{C} = (-6 + d) \cdot \pi^*(L) + \Sigma_{i=1}^n (2 - m_{P_i}(C))E_i$, so $\pi^*(L) \cdot (2K_Y + \tilde{C}) = -6 + d$, thus $|2K_Y + \tilde{C}| = \phi$ for every curve of degree $d < 6$. [7, Corollary 2.4] shows that $|2K_Y + \tilde{C}| = \varnothing$ is equivalent to $\bar{\mathcal{K}}(C, \mathbb{P}^2) = -\infty$, so C is equivalent to a line by Theorem 4.4. □

Lemma 4.6. *If C is a Cremona equivalent to a line $L \subseteq \mathbb{P}^2$ and P is a Galois point, then every non-trivial element in G_P extends to an element in* $\mathrm{Bir}(\mathbb{P}^2)$.

Proof. Let $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ that sends C onto a line L. For each $g \in G_P$, $\varphi|_C : C \dashrightarrow L$ conjugates g to an element of $\mathrm{Aut}(L)$, that extends to $\hat{g} \in \mathrm{Aut}(\mathbb{P}^2)$. Hence, g extends to $\varphi^{-1}\hat{g}\varphi \in \mathrm{Bir}(\mathbb{P}^2)$. □

Remark 4.7. *Let C be the smooth conic given by $C = \{[X : Y : Z] | Y^2 = XZ\} \subset \mathbb{P}^2$, then the natural embedding of* $\mathrm{Aut}(\mathbb{P}^2, C) = \{g \in \mathrm{Aut}(\mathbb{P}^2) | g(C) = C\} = \mathrm{PGL}_2$ *in* $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3$ *is the one induced from the injective group homomorphism*

$$\mathrm{GL}_2(k) \to \mathrm{GL}_3(k), \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{1}{ad - bc} \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}$$

where $\rho : [u : v] \mapsto [u^2 : uv : v^2]$, and the following diagram commutes.

$$\begin{array}{ccc} \mathbb{P}^1 & \overset{\rho}{\dashrightarrow} & \tilde{C} \subset \mathbb{P}^2 \\ \downarrow g & & \downarrow \tilde{g} \\ \mathbb{P}^1 & \overset{\rho}{\longrightarrow} & \mathbb{P}^2 \end{array}$$

Lemma 4.8. *Let k be a field of characteristic $char(k) \neq 2$ containing a primitive fourth root of unity, and let C be the irreducible curve defined by the equation*

$$X^4 - 4ZYX^2 - ZY^3 + 2Z^2Y^2 - YZ^3 = 0, \tag{2}$$

then the point $P = [1 : 0 : 0]$ is an outer Galois point of C and the extension induced by the projection $\pi_P : C \dashrightarrow \mathbb{P}^1$ is Galois of degree 4. Furthermore, the group G_P extends to $\mathrm{Bir}(\mathbb{P}^2)$ *but not to* Jonq_P.

Proof. Define the birational map $\phi : \mathbb{A}^1 \dashrightarrow C$ by $\phi : t \mapsto [t + t^3 : t^4 : 1]$ with inverse $[X : Y : Z] \mapsto (X(Y + Z)) / (X^2 - YZ + Z^2)$. Hence C is a rational irreducible curve of degree 4 and therefore, C is equivalent to a line by Lemma 4.5. Furthermore, every non-trivial element in G_P extends to an element in $\mathrm{Bir}(\mathbb{P}^2)$ by Lemma 4.6. We will also prove it explicitly below. We have $K(C) = k(t)$ and define the projection by $\pi_P : [X : Y : Z] \mapsto [Y : Z]$. Let $x = X/Z$ and $y = Y/Z$ be affine coordinates, so the affine equation $x^4 - 4yx^2 - y^3 + 2y^2 - y = 0$ is defining the extension field $k(y)[x]/k(y) = k(t)/k(t^4)$. Since k contains the 4*th* root of unity, the extension is Galois of degree 4 with basis $\{1, t, t^2, t^3\}$, and we have the following diagram

$$\begin{array}{ccc} \mathbb{A}^1 & \overset{\phi}{\dashrightarrow} & C \subseteq \mathbb{P}^2 \\ & \underset{\psi}{\searrow} & \downarrow f \\ & & \mathbb{A}^1 \end{array}$$

where ψ is given by $\psi : t \mapsto t^4$. By contradiction, we prove that there is no de Jonquières map f extending the action. Let us assume that there exists a de Jonquières map g that extends the action, *i.e* there exists $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \in k(y)$ with $\tilde{\alpha}\tilde{\delta} - \tilde{\beta}\tilde{\gamma} \neq 0$ such that $g : (x, y) \mapsto (\frac{\tilde{\alpha}x + \tilde{\beta}}{\tilde{\gamma}x + \tilde{\delta}}, y)$. Since $g \circ \phi = \phi \circ \sigma$, writing

https://doi.org/10.1017/S0017089524000090 Published online by Cambridge University Press

$\alpha = \tilde{\alpha}(t^4)$, $\beta = \tilde{\beta}(t^4)$, $\gamma = \tilde{\gamma}(t^4)$ and $\delta = \tilde{\delta}(t^4)$ where $\alpha, \beta, \gamma, \delta \in k(t^4)$. We obtain the equation

$$\mathrm{i}t - \mathrm{i}t^3 = \frac{\alpha(t+t^3)+\beta}{\gamma(t+t^3)+\delta}.$$

This implies that $\beta = \beta(t) = -(\mathrm{i}t^6 - \mathrm{i})\gamma t^2 - (\mathrm{i}\delta + \alpha)t^3 + (\delta\mathrm{i} - \alpha)t \in k(t^4)$ and is then $(\mathrm{i}t^4 - \mathrm{i})\gamma = 0$, $\mathrm{i}\delta + \alpha = 0$ and $\delta\mathrm{i} - \alpha = 0$. This gives $\alpha = 0, \gamma = 0$ which is a contradiction. Viewing C as an irreducible curve in $\mathbb{P}^2$ of degree 4, there are three singular points on the curve $[0:1:1]$, $\left[\mathrm{i}\sqrt{2}: -1:1\right]$, $\left[\mathrm{i}\sqrt{2}:1:-1\right]$. After suitable change of coordinates, $\sigma : \mathbb{P}^2 \to \mathbb{P}^2$ is given by $[X:Y:Z] \mapsto \left[-\mathrm{i}\sqrt{2}X - \mathrm{i}\sqrt{2}Z : 2X + Y - Z : 2X - Y + Z\right]$, this map sends the curve C to $\tilde{C}$, which is given by $\tilde{f} = X^2Y^2 + 6X^2YZ + X^2Z^2 + 4Y^2Z^2 = 0$ and this new equation has $\{[1:0:0], [0:1:0], [0:0:1]\}$ as multiple points of order 2. After blowing up the three points $\{[1:0:0], [0:1:0], [0:0:1]\}$ in $\mathbb{P}^2$ and contract again, the strict transform curve is of degree $d' = 2\cdot d - m_1 - m_2 - m_3 = 2\cdot 4 - 2 - 2 - 2 = 2$, so it is a conic given by the equation $F = 4X^2 + Y^2 + 6YZ + Z^2 = 0$. We change the coordinates using the following matrix

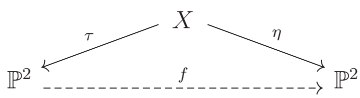
$$\begin{bmatrix} 4\mathrm{I} & 0 & -\mathrm{I} \\ 0 & 2\sqrt{2} & 0 \\ 8 & -6\sqrt{2} & 2 \end{bmatrix}$$

to send the conic to $Y^2 - XZ = 0$ and we extend G_P explicitly using Remark 4.7. □

5. Example where G_P cannot be extended to Bir($\mathbb{P}^2$)

Lemma 5.1. *Let k be an algebraically closed field, $C \subset \mathbb{P}^2$ be an irreducible curve, $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map sends the curve C to itself, and $X \to \mathbb{P}^2$ is an embedded resolution of singularities of C in $\mathbb{P}^2$ where $\tilde{C}$ is the strict transform of C. If all singular points of C have a multiplicity $m_P(C) < deg(C)/3$, then f is an automorphism of $\mathbb{P}^2$.*

Proof. Let $deg(f) = d$, and assume for contradiction that $d > 1$. We take a commutative diagram where π and η are sequences of blow-ups



and we can assume that the strict transform of C is smooth. As in Lemma 4.5 $\eta^*(L) = d\cdot\pi^*(L) - \Sigma m_i E_i$, $K_X = -3\pi^*(L) + \Sigma E_i$, and $\eta^*(C) = \tilde{C} = deg(C)\cdot\pi^*(L) - \Sigma m_{P_i}(C)E_i$. Since $deg(C) = C\cdot L = \eta^*(C)\cdot\eta^*(L) = d\cdot deg(C) - \Sigma m_i\cdot m_{P_i}(C)$, then $deg(C)(d-1) = \Sigma m_i \cdot m_{P_i}(C) < \Sigma m_i \cdot deg(C)/3$ and therefore $3(d-1) < \Sigma m_i$ Noether equality, which is a contradiction as $\Sigma m_i = 3(d-1)$: this equation follows from $\eta^*(L)^2 = L^2 = 1$ and from the adjunction formula $\eta^*(L)\cdot(\eta^*(L) + K_X) = -2$, which gives $\eta^*(L)\cdot K_X = -3$. □

Lemma 5.2. *Let k be a field with $char(k) \neq 5$ that contains a primitive 5th root of unity, and let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ given by $\phi : [u:v] \mapsto [uv^6 - u^7 : u^5(u^2+v^2) : v^5(u^2+v^2)]$. We define $C := \overline{\phi(\mathbb{P}^1)}$, then the point $P = [1:0:0]$ is an inner Galois point of C and the extension induced by the projection $\pi_P : C \dashrightarrow \mathbb{P}^1$ is Galois of degree 5. Moreover, there is no birational map f extending the action of the generator of the Galois group.*

Proof. The curve C is birational to $\mathbb{P}^1$ via ϕ, with inverse $[X:Y:Z] \mapsto [X^4Y + 4X^3Y^2 - 2X^3Z^2 + 6X^2Y^3 - 2X^2YZ^2 + 4XY^4 + XZ^4 + Y^5 : Z(X^4 + 2X^3Y + X^2Y^2 - 3X^2Z^2 - 3XYZ^2 + Z^4)]$. Define the projection by $\pi_P : [X:Y:Z] \mapsto [Y:Z]$. Let $\psi = \pi_P \circ \phi$ then $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ maps $[u:v]$ to $[u^5{:}v^5]$, hence the extension is Galois of degree 5 with Galois group G_P generated by $\sigma : x \mapsto \zeta \cdot x$, where ζ is the 5th root of unity. We now prove that the curve C does not have a point P of multiplicity $m_P(C) \geq 3$. By contradiction, we take a point $P = [P_0 : P_1 : P_2]$ of multiplicity 3, and then we take two distinct lines $a_1x + a_2y - a_3z = 0$ and $b_2y - b_3z = 0$ passing through the point P. We take the preimage in $\mathbb{P}^1$, so we get a common factor of degree at least 3. Let $f_1(u, v) = a_1 \left(-u^7 + u\, v^6\right) + a_2u^5 \left(u^2 + v^2\right) + a_3v^5 \left(u^2 + v^2\right)$ and $f_2(u, v) = \left(u^2 + v^2\right) \left(b_2u^5 + b_3v^5\right)$. We check now that it is not possible for the polynomials f_1 and f_2 to have a factor of degree 3 in common. Assume first that $u + \mathrm{i}v$ divides both polynomials, so we should have $f_1(1, \mathrm{i}) = f_2(1, \mathrm{i}) = 0$ implies to $a_1 = 0$, hence $P = [1 : 0 : 0]$ is a smooth point and this gives a contradiction. If we assume that $u - \mathrm{i}v$ divides both polynomials, then we should have $f_1(1, -\mathrm{i}) = f_2(1, -\mathrm{i}) = 0$, again we have $a_1 = 0$, so the factor of degree 3 must divide $b_2u^5 + b_3v^5$. If we assume that u divides the polynomial f_2, then $b_3 = 0$ and u^3 should divide f_1, but this is not true as $f_1 = -u(u^6 - v^6)$. If we assume that v divides the polynomial f_2, then $b_2 = 0$ and v^3 should divide f_1, but this is not true as $f_1 = uv^2(u^4 + v^4)$. So the factor of degree 3 must divide $b_2u^5 + b_3v^5$, $b_2 \neq 0$ and $b_3 \neq 0$. Hence we can assume that $a_1 = 1$ and replace f_1 by $f_1 - a_3f_2/b_3$ so we can put $a_3 = 0$ and up to multiple, we can assume that $b_3 = 1$, $a_1 = 1$ and $b_2 = -\xi^5$. So $f_2 = -u^5\xi^5 + v^5$, $f_1 = \left(-u^7 + u\, v^6\right) + a_2u^5 \left(u^2 + v^2\right)$ this implies that the roots of f_2 are $(u, v) = (1, \xi\zeta^i)$, where ζ is a 5th root of unity. Since f_1 and f_2 should have three roots in common, therefore let $\{(1, \xi), (1, \xi\zeta), (1, \xi\rho)\}$ are the three roots in common where $\rho^5 = 1$ and $\zeta^5 = 1$, and $\rho^5 \neq \zeta$ and they are not equal to 1, so f_1 should vanish on these three roots. This gives three equations $q_1 = \xi^6 - 1 + a_2 \left(\xi^2 + 1\right) = 0$, $q_2 = a_2 \left(\xi^2\zeta^2 + 1\right) + \xi^6\zeta - 1 = 0$, $q_3 = a_2 \left(\rho^2\xi^2 + 1\right) + \rho\,\xi^6 - 1 = 0$, by solving this system in a_2, ζ and ρ, we found that $\zeta = \rho = (\xi^4 + 1)/(\xi^6 - 1)$, which is a contradiction as $\zeta \neq \rho$. Finally, f_1 and f_2 cannot have a factor of degree $d \geq 3$. Since $m_P(C) < 3$ for each $P \in \mathbb{P}^2$, let us assume that there exists a birational map g that extends the generator of the Galois group, then by Lemma 5.1, g is a linear automorphism of $\mathbb{P}^2$, so it is given by a matrix let us say $A \in \mathrm{PGL}_3$, Since $g \circ \phi = \phi \circ \sigma$, so we have

$$\begin{bmatrix} UV^6 - U^7 \\ U^5(U^2 + V^2) \\ V^5(U^2 + V^2) \end{bmatrix} = A \cdot \begin{bmatrix} \zeta UV^6 - \zeta^2 U^7 \\ U^5(\zeta^2U^2 + V^2) \\ V^5(\zeta^2U^2 + V^2) \end{bmatrix}$$

where ζ is the $5th$ root of unity. Since UV^6, U^7, U^5V^2, and V^7 are linearly independent, after checking the calculation we found that A should be diagonal, but $U^5(\zeta^2U^2 + V^2)$ is not a multiple of $U^5(\zeta^2U^2 + V^2)$, then we have a contradiction. □

Acknowledgements. I would like to thank my PhD advisor Jérémy Blanc for suggesting the question and for interesting discussions during the preparation of this text. I would also like to thank the Department of Mathematics and Computer Science at Basel for the hospitality. Thanks also to the referee Gavin Brown for his remarks that helped to improve the presentation of the article.

The author acknowledges support from the Federal Commission for Scholarships for Foreign Students of Switzerland.

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