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A NOTE ON THE HU–HWANG–WANG CONJECTURE FOR GROUP TESTING

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Abstract

Hu *et al*. ["A boundary problem for group testing", *SIAM J. Algebraic Discrete Meth.* **2** (1981), 81–87] conjectured that the minimax test number to find d defectives in $3d$ items is $3d-1$, a surprisingly difficult combinatorial problem about which very little is known. In this article we state three more conjectures and prove that they are all equivalent to the conjecture of Hu *et al*. Notably, as a byproduct, we also obtain an interesting upper bound for $M(d, n)$.

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1. Introduction

Consider a population of n items consisting of d defectives and $n-d$ good items. In this paper we assume that the number d of defectives is known. The problem is to identify these d defectives by means of a sequence of group tests. Each test is on a subset of items with two possible outcomes: a *pure* outcome indicates that all items in the subset are good, and a *contaminated* outcome indicates that at least one item in the subset is defective. Group testing has applications in, for example, high-speed computer networks [4], medical examination [1, 2], quantity searching [3], statistics [1] and related graph theory problems [7, 13]. Let $M_T(d, n)$ denote the maximum number of tests required by the algorithm T to identify the d defectives in n items, where the maximum is taken over all possible combinations of the d defectives among the n items. Define

$$M(d, n) = \min_T M_T(d, n).$$

Then $M(d, n)$ is the minimax test number for given d and n. We know that $M(n, n) = M(0, n) = 0$. An algorithm which achieves $M(d, n)$ is called a minimax algorithm for the (d, n) problem.

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In Section 3, we derive three statements which are equivalent to the conjecture of Hu *et al.* [11], perhaps the most major open problem in this area. Notably, as a byproduct, we also obtain an interesting upper bound for $M(d, n)$ (see Proposition 3.8).

The question studied by Hu *et al.* [11] was for what values of n and d is it the case that

$$M(d, n) = n - 1,$$

achieved by testing the first $n - 1$ items one by one. They [11] conjectured that

$$M(d, n) = n - 1 \quad \text{for} \quad 3d \geq n > d > 0.$$

It was proved in [8] that

$$M(d, n) = n - 1 \quad \text{for} \quad \left\lfloor \frac{42d}{16} \right\rfloor \geq n > d > 0,$$

where $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the largest (smallest) integer not greater (less) than x. Following the method of Du and Hwang [8], Leu *et al.* [14] improved Du and Hwang's result a little further by proving that

$$M(d, n) = n - 1 \quad \text{for} \quad \left\lfloor \frac{43d}{16} \right\rfloor \geq n > d \geq 193.$$

Also, Riccio and Colbourn [15] proved that if $\alpha < \log_{3/2} 3 \approx 2.7095$, then for sufficiently large d, $M(d, n) = n - 1$ if $n \leq \alpha d$. Note that Fischer *et al.* [10] also studied the conjecture of Hu *et al.* from a different point of view.

2. Some preliminary remarks and results

In this paper the terminology and notation which we adopt are used in [9].

A *binary tree* is a rooted tree where each node except the root has one inlink (the root has none), and each node has either zero or two outlinks. Nodes with zero outlinks are called *terminal nodes* and nodes with two outlinks are called *internal nodes*. The *path* for a node v is the alternate sequence of nodes and links which connect the root to v, excluding v itself. The *length* of a path is the number of nodes on it. Node u is the *parent* of node v and v is a child of u, if u has an outlink to v. Two children of the same parent are *siblings*.

A group testing algorithm can be represented by a binary tree where each internal node is associated with a test and its two outlinks are associated with the two possible outcomes. The *test history* $H(v)$ at node v is the set of tests and outcomes associated with the nodes and links on the path for v.

For a (d, n) problem, with d defectives among the n items, the sample space $S(d, n)$ consists of all d-subsets (called samples) of the n items. Associated with each node v is the set of samples which are consistent with the test history of v. We refer to this set

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as the sample space at v and denote it by $S(v)$. Note that if v is a terminal point, then the cardinality of $S(v)$ is necessarily unity.

When an algorithm T is in its binary tree representation, $M_T(d, n)$ is simply the maximum path length of the tree. Let $M_T(S)$ denote the maximum number of tests for the algorithm T to identify the d defectives from the subset S of $S(d, n)$, and define

$$M(S) = \min_T M_T(S).$$

In particular, let $M(m; d, n)$ denote the minimax test number necessary to identify the d defectives among n items when, among n items, a particular subset of m items is known to be contaminated.

Now we state some basic lemmas which will be used in the following section. Their proofs can be found in [9, Chapters 1 and 3].

LEMMA 2.1. *$M(S) \geq \lceil \log_2 |S| \rceil$, where $|S|$ denotes the number of samples contained in the sample space S.*

LEMMA 2.2. *Suppose that sample space S_1 is a subset of sample space S_2. Then $M(S_1) \leq M(S_2)$.*

LEMMA 2.3. *Suppose that $n - d > 1$. Then $M(d, n) = n - 1$ implies $M(d, n - 1) = n - 2$.*

LEMMA 2.4. *$M(d, n) < n - 1$ for $n > 3d > 0$.*

LEMMA 2.5. *$M(d, n) \leq M(d + 1, n)$ for $n > d + 1 > 0$.*

LEMMA 2.6. *$M(d, n) \leq n - 1$ for $n > d > 0$.*

LEMMA 2.7. *$M(m; d, n) \geq 1 + M(d - 1, n - 1)$ for $m \geq 2$ and $n > d > 0$.*

LEMMA 2.8. *$M(2; d, n) = 1 + M(d - 1, n - 1)$ for $n > d > 0$.*

PROOF. By Lemma 2.7, we know that $M(2; d, n) \geq 1 + M(d - 1, n - 1)$. Now let T be the algorithm for the $(2; d, n)$ problem which first tests one item from the given contaminated group of 2 items and then uses a minimax algorithm for the remaining problem. Then

$$\begin{aligned} M_T(2; d, n) &= 1 + \max\{M(d - 1, n - 2), M(d - 1, n - 1)\} \\ &= 1 + M(d - 1, n - 1) \quad \text{(by Lemma 2.2)}. \end{aligned}$$

Hence we conclude that $M(2; d, n) = 1 + M(d - 1, n - 1)$. □

LEMMA 2.9. *$M(1, n) = \lceil \log_2 n \rceil$.*

To state the next lemma and Conjecture 4 of Section 3, we introduce a new notation. Let $M((d_1, n) \times (d_2, m))$ denote the minimax test number necessary to identify the $d_1 + d_2$ defectives from a set X of $n + m$ items when the following extra information about set X is given: the set X is divided into two disjoint subsets $A = \{a_1, \ldots, a_n\}$

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and $B = \{b_1, \ldots, b_m\}$, where set A contains d_1 defectives and set B contains d_2 defectives. Chang and Hwang [5, 6] studied the special case $M((1, n) \times (1, m))$. In [6] they proved the following result.

LEMMA 2.10. *$M((1, n) \times (1, m)) = \lceil \log_2 mn \rceil$ for all m and n.*

REMARK 2.11. In practice, one hardly knows d exactly. Thus d is often either an estimate or an upper bound. When d is known to be an upper bound of the number of defectives, the (d, n) problem will be denoted by $(\bar{d}, n)$. Hwang *et al.* [12] proved that $M(d, n) + 1 \geq M(\bar{d}, n) \geq M(d, n + 1)$ for $n > d > 0$.

3. Equivalent statements

In this section we state four conjectures including the conjecture of Hu *et al.* [11]. At a first glance Conjectures 1, 2 and 4 seem to belong to different types of problem. However, in what follows, we will prove that they and Conjecture 3 are all equivalent to each other. On the way, as a byproduct, we also obtain an interesting upper bound for $M(d, n)$ (see Proposition 3.8).

CONJECTURE 1 (Hu *et al.* [11]). If $3d \geq n > d > 0$, then $M(d, n) = n - 1$.

CONJECTURE 2. If integers $3(d_1 + d_2) \geq n_1 + n_2 > d_1 + d_2 > 0$, $n_1 > d_1 \geq 0$ and $n_2 > d_2 \geq 0$, then

$$M(d_1, n_1) + M(d_2, n_2) < M(d_1 + d_2, n_1 + n_2).$$

CONJECTURE 3. $M(d, 3d + 2) = 3d$ for $d > 0$.

CONJECTURE 4. $M((1, 3) \times (d - 1, 3d - 1)) < M(3; d, 3d + 2)$ for $d > 1$.

REMARK 3.1. About Conjecture 2, the problem behind $M(d_1 + d_2, n_1 + n_2)$ is to identify $d_1 + d_2$ defectives from a set X of $n_1 + n_2$ items. On the other hand, the problem behind $M(d_1, n_1) + M(d_2, n_2)$ is to identify $d_1 + d_2$ defectives from the set X with extra crucial information about set X being given: that is, set X contains a known subset A of n_1 items which is known to contain d_1 defectives.

As for Conjecture 4, the problem behind $M(3; d, 3d + 2)$ is to identify d defectives from a set Y of $3d + 2$ items with extra information about set Y being given: that is, set Y contains a known subset B of 3 items which is known to be contaminated. On the other hand, the problem behind $M((1, 3) \times (d - 1, 3d - 1))$ is to identify d defectives from the set Y with more precise information on the set B: that is, set B contains only one defective.

Therefore, in appearance, the statements of Conjectures 2 and 4 seem more friendly than the original statement of Hu *et al.*. We hope that our equivalent statements will focus more attention on the Hu–Hwang–Wang conjecture.

THEOREM 3.2. *Conjecture 1 is true if and only if Conjecture 2 is true.*

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PROOF. We first prove that Conjecture 1 implies Conjecture 2. Write $n = n_1 + n_2$ and $d = d_1 + d_2$ with $n_1 > d_1 \geq 0$ and $n_2 > d_2 \geq 0$, where $3d \geq n > d > 0$. By Lemma 2.6, we find that

$$M(d_1, n_1) + M(d_2, n_2) \leq n_1 - 1 + n_2 - 1 = n - 2 < n - 1 = M(d, n).$$

Conversely, we want to prove that Conjecture 2 implies Conjecture 1. For $d = 1, 2$ and 3, it is easy to verify that $M(d, n) = n - 1$ for $3d \geq n > d$. Assuming that the statement $M(d, n) = n - 1$ is true for $d = a - 1 \geq 3$ and $3d \geq n > d$. We want to prove that $M(a, n) = n - 1$ is also true for $3a \geq n > a$. By Lemma 2.3, it is enough to prove that $M(a, n) = n - 1$ for $n = 3a$. Now, assuming Conjecture 2 is true, we find that $M(a, 3a) > M(1, 3) + M(a - 1, 3a - 3) = 2 + 3a - 4 = 3a - 2$. This forces the result $M(a, 3a) = 3a - 1$. The theorem is proved. □

The reason to introduce Conjecture 3 is to make a connection between Conjectures 1 and 4. Therefore, the next job is to prove that Conjectures 1 and 3 are equivalent to each other. In doing that Theorem 3.4 plays a key role. To prove Theorem 3.4 we need the following proposition.

PROPOSITION 3.3. *If $n > d$, $M(d, n) = n - 1$, and $n - 2 \leq M(d - 1, n) \leq n - 1$, then $M(d, n + 1) = n$.*

PROOF. Let T be an algorithm such that $M(d, n + 1) = M_T(d, n + 1)$. Then

$$M(d, n + 1) = 1 + \max\{M(m; d, n + 1), M(d, n + 1 - m)\},$$

where $m (1 \leq m \leq n + 1 - d)$ is the number of items being tested in the first test.

Case 1. $m = 1$. Since $M(1; d, n + 1) = M(d - 1, n)$, so, by Lemma 2.5 and the assumption $M(d, n) = n - 1$, we find that

$$M(d, n + 1) = 1 + \max\{M(d - 1, n), M(d, n)\} = 1 + M(d, n) = n.$$

Case 2. $n + 1 - d \geq m \geq 2$. By Lemma 2.6, we know that

$$1 + M(d, n + 1 - m) \leq 1 + n + 1 - m - 1 \leq 1 + n - 2 = n - 1.$$

For the other sum $1 + M(m; d, n + 1)$, by Lemma 2.7 and the assumption $M(d - 1, n) \geq n - 2$, we have that

$$1 + M(m; d, n + 1) \geq 1 + 1 + M(d - 1, n) \geq n.$$

Thus $M(d, n + 1) \geq n$.

By Lemma 2.6, we know that $M(d, n + 1) = n$. □

THEOREM 3.4. *If $M(d, 3d) = 3d - 1$, then $M(d, 3d + 2) = 3d$ if and only if*

$$M(d + 1, 3(d + 1)) = 3(d + 1) - 1.$$

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PROOF. We first prove that if $M(d, 3d+2) = 3d$, then $M(d+1, 3d+3) = 3d+2$. Since, by Lemma 2.5, $3d-1 = M(d, 3d) \le M(d+1, 3d)$, we find that $M(d+1, 3d) = 3d-1$. So, by Proposition 3.3, $M(d+1, 3d+1) = 3d$. Next, by applying Lemmas 2.2 and 2.4, we have $M(d, 3d+1) = 3d-1$. Now, applying Proposition 3.3 on $n = 3d+1$, we obtain that $M(d+1, 3d+2) = 3d+1$. Finally, using $M(d+1, 3d+2) = 3d+1$, the assumption $M(d, 3d+2) = 3d$ and applying Proposition 3.3 on $n = 3d+2$, we have that $M(d+1, 3d+3) = 3d+2$.

Conversely, if $M(d, 3d+2) \ne 3d$, then, by Lemmas 2.2 and 2.4, $M(d, 3d+2) = 3d-1$. Let T be an algorithm for the $(d+1, 3d+3)$ problem which first tests a set K of 2 items. If the outcome is *pure*, then T uses a minimax algorithm for the remaining problem. If the outcome is *contaminated*, then T tests a single item from the set K and then uses a minimax algorithm for the remaining problem. Then

$$M_T(d+1, 3d+3) = \max\{1 + M(d+1, 3d+1), 1+1+M(d, 3d+1), 1+1+M(d, 3d+2)\}.$$

Since, by Lemmas 2.2–2.4, $1 + M(d+1, 3d+1) = 3d+1 = 1+1+M(d, 3d+1) = 1+1+M(d, 3d+2)$, we find that $M_T(d+1, 3d+3) = 3d+1$. This implies that

$$3d+2 = M(d+1, 3d+3) \le M_T(d+1, 3d+3) = 3d+1, \quad \text{a contradiction.}$$

Hence we derive that $M(d, 3d+2) = 3d$. □

Now using the fact that $M(1, 3) = 2$ and assuming either conjecture (Conjecture 1 or Conjecture 3) is true, we can prove that the other conjecture is true by induction with the help of Theorem 3.4 easily. Thus we have the following desired relation.

COROLLARY 3.5. *Conjecture 1 is true if and only if Conjecture 3 is true.*

If one is interested in testing the conjecture of Hu *et al.* for large defectives d by computer, then Theorem 3.4 might help.

The rest of this section is devoted to proving that Conjecture 4 is equivalent to Conjecture 3. In doing this Theorem 3.9 plays a key role. First we need some preliminary results.

PROPOSITION 3.6. $M(d, n) < M((1, 2) \times (d, n))$ *for* $n > d > 0$.

PROOF. Let $A = \{x, y\}$ be the set containing one defective and B be the set of n items containing d defectives. By notation $(1, 2) \times (d, n)$, we mean that $A \cap B = \emptyset$. Let T be a minimax algorithm such that $M((1, 2) \times (d, n)) = M_T((1, 2) \times (d, n))$, and v be a leaf of T with the longest path length, say l. We will study the test history $H(v)$ of leaf v closely.

Case 1. Along the test history $H(v)$ of leaf v, if the tests applied at nodes starting from the first node (the root of T) up to the $(l-1)$th node do not involve items from

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the set A, then the sample space at the lth node is $\{I \cup \{x\}, I \cup \{y\}\}$, where I is a sample from the sample space $S(d, n)$.

Case 2. Along the test history $H(v)$ of leaf v, let u be the first node (before the lth node) with the test involving an item from set A. At this moment before testing, the sample space at node u is $S(u) = \{I \cup \{i\};\ I \in S_1 \text{ and } i \in A\}$, where S_1 is a subset of the sample space $S(d, n)$.

Subcase 2.1. The test $t(u)$ involves one item, say x, from set A and a nonempty subset W of set B. After executing the test $t(u)$, $S(u)$ is split into two disjoint nonempty subsets, which are $S_c(u) = \{I \cup \{x\};\ I \in S_1\} \cup \{I \cup \{y\};\ I \in S_1 \text{ and } I \cap W \neq \emptyset\}$ and $S_p(u) = \{I \cup \{y\};\ I \in S_1 \text{ and } I \cap W = \emptyset\}$. Note that $\{I \cup \{x\};\ I \in S_1\}$ says the remaining problem is to find the defectives in S_1.

Subcase 2.2. The test $t(u)$ involves only one item which is from set A, say x. Then $S_c(u) = \{I \cup \{x\};\ I \in S_1\}$ and $S_p(u) = \{I \cup \{y\};\ I \in S_1\}$.

Note that if set A does not exist, then the sample space at node u, instead of $S(u)$, is S_1. Therefore the test $t(u)$ of Case 2 is not needed.

Thus, either in Case 1 or Case 2, if we throw away the set A, then the algorithm T induces an algorithm T' on the (d, n) problem with $M_{T'}(d, n) \leq M_T((1, 2) \times (d, n)) - 1$. Hence we obtain that $M(d, n) < M((1, 2) \times (d, n))$. □

By the inequality $M((1, 2) \times (d, n)) \leq M(1, 2) + M(d, n)$ and Lemma 2.2, we have the following corollary.

COROLLARY 3.7.

(1) $M((1, 2) \times (d, n)) = 1 + M(d, n)$ *for* $n > d > 0$.
(2) $M((1, a) \times (d, n)) \leq M((1, b) \times (d, n))$ *for* $n > d > 0$ *and* $b > a > 1$.

PROPOSITION 3.8. $M(d, n) \leq 3d - 1 + \lceil (n - 3d - 1)/2 \rceil$ *for* $n \geq 3d + 1 \geq 4$.

PROOF. We prove this inequality by induction. For case $d = 1$, it is clear that the statement follows by the equality $M(1, n) = \lceil \log_2 n \rceil$. Now assume the statement is true for $d = a - 1 > 0$. By Lemma 2.4, we obtain the inequalities

$$M(a, 3a + 1) \leq 3a - 1 = 3a - 1 + \lceil (3a + 1 - 3a - 1)/2 \rceil \quad \text{and}$$
$$M(a, 3a + 2) \leq 3a = 3a - 1 + \lceil (3a + 2 - 3a - 1)/2 \rceil.$$

To proceed we assume that

$$M(a, n - 1) \leq 3a - 1 + \lceil (n - 1 - 3a - 1)/2 \rceil \quad \text{for} \quad n - 1 \geq 3a + 2.$$

Let T be the algorithm for the (a, n) problem which first tests a set K of two items and then uses a minimax algorithm for the remaining problem. Then

$$\begin{aligned} M(a, n) &\leq M_T(a, n) \\ &= 1 + \max\{M(a, n-2), M(2; a, n)\} \\ &= 1 + \max\{M(a, n-2), 1 + M(a-1, n-1)\} \quad \text{(by Lemma 2.8)} \\ &\leq 3a - 1 + \left\lceil \frac{n - 3a - 1}{2} \right\rceil \quad \text{(by induction).} \end{aligned}$$

The proposition is proved. □

Now we can prove the following key step.

THEOREM 3.9. *$M((1, 3) \times (d-1, 3d-1)) \leq 3d - 2$ for $d \geq 2$.*

PROOF. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, \ldots, b_{3d-1}\}$ be the disjoint sets, where A contains exactly one defective and B contains $d - 1$ defectives. For $d = 2$, by Lemma 2.10, we know that $M((1, 3) \times (1, 5)) = \lceil \log_2 15 \rceil = 4$. For the case $d \geq 3$, let T be the algorithm defined by the following procedure.

Step 1. Set $i := 1$ and $b_0 = a_1$.

Step 2. Test the group $\{a_1, b_i\}$. If the outcome is *pure*, then use a minimax algorithm for the remaining problem which is the $(1, 2) \times (d-i, 3d-i-1)$ problem. By Propositions 3.6 and 3.8, we know that

$$\begin{aligned} M((1, 2) \times (d-i, 3d-i-1)) &= 1 + M(d-i, 3d-i-1) \\ &\leq 1 + 3d - 2i - 2 = 3d - 2i - 1. \end{aligned}$$

If this is the direction where the problem goes, then the total number of tests needed to identify all defectives is at most $2(i-1) + 1 + 3d - 2i - 1 = 3d - 2$.

If the group $\{a_1, b_i\}$ is *contaminated*, then go to Step 3.

Step 3. Test the group $\{a_2, a_3, b_i\}$. If the outcome is *pure*, which implies item a_1 is defective, then use a minimax algorithm for the remaining problem which is the $(d-i, 3d-i-1)$ problem. Note that, up until now, the identified defectives are the set $\{b_0, b_1, \ldots, b_{i-1}\}$ and the identified good items are the set $\{a_2, a_3, b_i\}$. If this is the direction where the problem goes, then the total number of tests needed to identify all defectives is at most $2i + M(d-i, 3d-i-1)$ which, by Proposition 3.8, is less than or equal to $3d - 2$.

If the group $\{a_2, a_3, b_i\}$ is *contaminated*, then, by combining the contaminated result on group $\{a_1, b_i\}$, we conclude that b_i is defective. At this stage, we check the number i. If $i < d - 1$, then set $i := i + 1$ and then go to Step 2. If $i = d - 1$, then all defectives in the set B have been identified by using $2i (= 2d - 2)$ tests and the remaining problem is the $(1, 3)$ problem which needs two more tests to complete.

By inspecting the algorithm T, we know that the maximum path length of T is at most $3d - 2$. □

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By Corollary 3.7 and Theorem 3.9, we have $1 + M(d-1, 3d-1) = M((1, 2) \times (d-1, 3d-1)) \leq M((1, 3) \times (d-1, 3d-1)) \leq 3d-2$ for $d \geq 2$. Hence the following corollary clearly holds.

COROLLARY 3.10. *For $d \geq 2$, if $M(d-1, 3d-1) = 3d-3$, then*

$$M((1, 3) \times (d-1, 3d-1)) = 1 + M(d-1, 3d-1).$$

REMARK 3.11. By Lemma 2.10, we know that $M((1, 3) \times (1, 6)) = \lceil \log_2 18 \rceil = 5 \neq 4 = 1 + M(1, 6)$. Hence, in general, the equation $M((1, 3) \times (d, n)) = 1 + M(d, n)$ does not hold. We also note that, by Lemma 2.2,

$$M((1, 3) \times (d, n)) \leq M(3; d+1, n+3) \quad \text{for} \quad n > d > 0.$$

COROLLARY 3.12. *For $d \geq 2$, if $M(d_0, 3d_0) = 3d_0 - 1$ for $0 < d_0 \leq d$, then $M(d, 3d+2) = 3d$ if and only if $M((1, 3) \times (d-1, 3d-1)) < M(3; d, 3d+2)$.*

PROOF. First, by the assumption on $M(d_0, 3d_0)$ and applying Lemma 2.3 and Theorem 3.4, we know that $M(d, 3d-1) = 3d-2$ and $M(d-1, 3d-1) = 3d-3$, respectively. Also, by Lemmas 2.2 and 2.4, we know that $3d-1 = M(d, 3d) \leq M(d, 3d+2) \leq 3d$.

Suppose that $M(d, 3d+2) = 3d$. Then $3d = M(d, 3d+2) = \min_T M_T(d, 3d+2) \leq 1 + \max\{M(3; d, 3d+2), M(d, 3d+2-3)\} = 1 + M(3; d, 3d+2)$. Now, by applying Corollary 3.10, we obtain that $M(3; d, 3d+2) \geq 3d-1 > 3d-2 = 1 + M(d-1, 3d-1) = M((1, 3) \times (d-1, 3d-1))$.

Conversely, suppose that $M(3; d, 3d+2) > M((1, 3) \times (d-1, 3d-1)) = 3d-2$. To prove $M(d, 3d+2) = 3d$, we employ the equation

$$\begin{aligned} M(d, 3d+2) &= \min_T M_T(d, 3d+2) \\ &= 1 + \max\{M(m; d, 3d+2), M(d, 3d+2-m)\} \\ &\qquad \text{for some } m > 0. \end{aligned}$$

In the case $m = 1$ or $m = 2$, we have

$$M(d, 3d+2) \geq 1 + M(d, 3d+2-m) \geq 1 + M(d, 3d) = 3d.$$

If $m \geq 3$, then by Lemma 2.2 we still have

$$M(d, 3d+2) \geq 1 + M(m; d, 3d+2) \geq 1 + M(3; d, 3d+2) \geq 1 + 3d - 1 = 3d.$$

In conclusion, we have $M(d, 3d+2) = 3d$.

The corollary is proved. □

Now we are ready to prove that Conjecture 3 is equivalent to Conjecture 4.

THEOREM 3.13. *Conjecture 3 is true if and only if Conjecture 4 is true.*

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PROOF. Assuming that Conjecture 3 is true, then, by Corollary 3.5, we have that Conjecture 1 is also true. Now, by applying Corollary 3.12, we know that Conjecture 4 is true.

Conversely, assuming Conjecture 4 is true, we prove that Conjecture 3 is true by induction.

For $d = 1$, by Lemma 2.9, $M(1, 5) = 3$ holds. Suppose that $M(d, 3d + 2) = 3d$ holds for $d \leq k$. Then, by applying Theorem 3.4 repeatedly, we have $M(d, 3d) = 3d - 1$ for $0 < d \leq k + 1$. Now, applying Corollary 3.12, we obtain $M(k + 1, 3(k + 1) + 2) = 3(k + 1)$. Thus Conjecture 3 is true.

The theorem is proved. □

REMARK 3.14. Inspired by the statements of Conjectures 2 and 4 many problems arise. For example, we could ask what relations would exist between $M(d_1, n_1) + M(d_2, n_2)$ and $M(d_1 + d_2, n_1 + n_2)$ for $n_1 > d_1 > 0$ and $n_2 > d_2 > 0$. Similarly, we could ask what relations could be between $M(m; d, n)$ and $M((1, m) \times (d - 1, n - m))$. Of course, many more questions could be asked.

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