Bull. Aust. Math. Soc. **97** (2018), 166–167
doi:10.1017/S0004972717001010

GENERALISATIONS OF THE DOYEN–WILSON THEOREM

ROSALIND A. HOYTE

(Received 11 September 2017; first published online 4 December 2017)

2010 *Mathematics subject classification*: primary 05C70; secondary 05B30, 05C38.

Keywords and phrases: Doyen–Wilson theorem, cycle system, subsystem, graph decomposition, complete graph with a hole, cycle switching.

In 1973, Doyen and Wilson [7] famously solved the problem of when a 3-cycle system can be embedded in another 3-cycle system. There has been much interest in the literature in generalising this result for m-cycle systems when $m > 3$. Although there are several partial results, including complete solutions for some small values of m and strong partial results for even m, this still remains an open problem [4, 5, 8, 9].

The main results of this thesis concern generalisations of the Doyen–Wilson theorem for odd m-cycle systems and cycle decompositions of the complete graph with a hole. The complete graph of order v with a hole of size u, $K_v - K_u$, is constructed from the complete graph of order v by removing the edges of a complete subgraph of order u (where $v \geq u$).

For each odd $m \geq 3$ we completely solve the problem of when an m-cycle system of order u can be embedded in an m-cycle system of order v, barring a finite number of possible exceptions. The problem is completely resolved in cases where u is large compared to m, where m is a prime power, or where $m \leq 15$. In other cases, the only possible exceptions occur when $v - u$ is small compared to m. This result is proved as a consequence of a more general result which gives necessary and sufficient conditions for the existence of an m-cycle decomposition of $K_v - K_u$ in the case where $u \geq m - 2$ and $v - u \geq m + 1$ both hold.

We prove that $K_v - K_u$ can be decomposed into cycles of arbitrary specified lengths provided that the obvious necessary conditions are satisfied, $v - u \geq 10$, each cycle has length at most $\min(u, v - u)$, and the longest cycle is at most three times as long as the second longest. This complements existing results for cycle decompositions of graphs such as the complete graph [1, 3, 10], complete bipartite graph [6, 8] and complete multigraph [2].

Thesis submitted to Monash University in December 2016; degree approved on 4 April 2017; supervisor Daniel Horsley, associate supervisor Ian M. Wanless.

We obtain these cycle decomposition results by applying a cycle switching technique to modify cycle packings of $K_v - K_u$. The tools developed by cycle switching enable us to merge collections of short cycles to obtain longer cycles. The methodology therefore relies on first finding decompositions of various graphs into short cycles, then applying the merging results to obtain the required decomposition. Similar techniques have previously been successfully applied to the complete graph and the complete bipartite graph. These methods also have potential to be further developed for the complete graph with a hole as well as other graphs.

We also give a complete solution to the problem of when there exists a packing of the complete multigraph with cycles of arbitrary specified lengths. The proof of this result relies on applying cycle switching to modify cycle decompositions of the complete multigraph obtained from known results.

The results in this thesis make substantial progress towards generalising the Doyen–Wilson theorem for arbitrary odd cycle systems and towards constructing cycle decompositions of the complete graph with a hole. However, there still remain unsolved cases. Moreover, the cycle switching and base decomposition methods used to obtain these results give rise to several interesting open problems.

References

[1] B. Alspach and H. Gavlas, 'Cycle decompositions of K_n and $K_n - I$', *J. Combin. Theory Ser.* B **81**(1) (2001), 77–99.

[2] D. E. Bryant, D. Horsley, B. M. Maenhaut and B. R. Smith, Decompositions of complete multigraphs into cycles of varying lengths', Preprint, 2015, arXiv:1508.00645 [math.CO].

[3] D. E. Bryant, D. Horsley and W. Pettersson, 'Cycle decompositions V: Complete graphs into cycles of arbitrary lengths', *Proc. Lond. Math. Soc. (3)* **108**(5) (2014), 1153–1192.

[4] D. E. Bryant and C. A. Rodger, 'The Doyen–Wilson theorem extended to 5-cycles', *J. Combin. Theory Ser.* A **68**(1) (1994), 218–225.

[5] D. E. Bryant, C. A. Rodger and E. R. Spicer, 'Embeddings of m-cycle systems and incomplete m-cycle systems: $m \leq 14$', *Discrete Math.* **171**(1–3) (1997), 55–75.

[6] C.-C. Chou, C.-M. Fu and W.-C. Huang, 'Decomposition of $K_{m,n}$ into short cycles', *Discrete Math.* **197/198** (1999), 195–203.

[7] J. Doyen and R. M. Wilson, 'Embeddings of Steiner triple systems', *Discrete Math.* **5** (1973), 229–239.

[8] D. Horsley, 'Decomposing various graphs into short even-length cycles', *Ann. Comb.* **16**(3) (2012), 571–589.

[9] E. Mendelsohn and A. Rosa, 'Embedding maximal packings of triples', *Congr. Numer.* **40** (1983), 235–247.

[10] M. Šajna, 'Cycle decompositions III. Complete graphs and fixed length cycles', *J. Combin. Des.* **10**(1) (2002), 27–78.

ROSALIND A. HOYTE,
Department of Mathematics and Statistics,
Memorial University of Newfoundland,
St John's, NL, Canada A1C 5S7
e-mail: rahoyte@outlook.com