J. Aust. Math. Soc. **108** (2020), 402–411
doi:10.1017/S1446788719000053

ON SUB-CLASS SIZES OF FINITE GROUPS

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(Received 7 May 2018; accepted 30 January 2019; first published online 8 April 2019)

Communicated by M. Giudici

Abstract

For every element x of a finite group G, there always exists a unique minimal subnormal subgroup, say, G_x of G such that $x \in G_x$. The sub-class of G in which x lies is defined by $\{x^g \mid g \in G_x\}$. The aim of this paper is to investigate the influence of the sub-class sizes on the structure of finite groups.

2010 *Mathematics subject classification*: primary 20E45.

Keywords and phrases: finite group, class size.

1. Introduction

The study of how the arithmetical conditions on conjugacy class sizes affect group structure has a long history, and this paper is a contribution to this study. Throughout the following, G always denotes a finite group. For every element x of G, we denote by $x^G = \{x^g \mid g \in G\}$ the conjugacy class of G in which x lies.

The connection between conjugacy class sizes and the structures of finite groups has been extensively studied. For a nice survey on this subject, see a recent paper by Camina and Camina [2].

Among the many results on conjugacy class sizes, we are particularly interested in several results linked to Camina [2] and Itô [7].

The following results study the structure of finite groups when all the conjugacy class sizes (or a special subset thereof) are not divisible by a single prime.

THEOREM 1.1. *Let G be a finite group and p be a prime.*

(1) *The prime p does not divide $|x^G|$ for every $x \in G$ if and only if G has a central Sylow p-subgroup.*

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This project was supported by the NSF of China (nos. 11471054, 11671063, and 11871011), the NSF of Jiangsu Province (no. BK20161265), the Natural Science Foundation of Chongqing (cstc2016jcyjA0065, cstc2018jcyjAX0060), and a grant from the Simons Foundation (no. 499532).

(2) (*Camina* [1, Lemma 1]) *The prime p does not divide* $|x^G|$ *for every* p'*-element* $x \in G$ *if and only if* $G = O_p(G) \times O_{p'}(G)$.

(3) [10, Theorem 5] *The prime p does not divide* $|x^G|$ *for every* p'*-element* $x \in G$ *of prime power order if and only if* $G = O_p(G) \times O_{p'}(G)$.

Remark 1.2. In the previous theorem, (1) is well known, (2) strengthens (1), and (3) further strengthens (2).

The following results study the structure of finite groups when all the conjugacy class sizes are not divisible by a product of two primes.

Theorem 1.3 (Itô [7, Proposition 5.1]). *Let G be a finite group and* p, q *be different primes. If pq does not divide* $|x^G|$ *for any* $x \in G$*, then G is either p-nilpotent (that is, G has a normal p-complement) or q-nilpotent.*

In this paper, we generalize or perhaps in some sense weaken the concepts of 'conjugacy class' and 'conjugacy class size' as follows.

For every element x of G, it is easy to see that G has a unique minimal subnormal subgroup in which x lies, and this subnormal subgroup is denoted by G_x. We shall consider the sub-class

$$x^{G^\star} := \{x^g \mid g \in G_x\} = x^{G_x}$$

and the sub-class size $|x^{G^\star}|$ instead of x^G and $|x^G|$, respectively.

The main object of this paper is to investigate Theorems 1.1 and 1.3 for the case when the class size is replaced by the sub-class size.

The analog for Theorem 1.1 is as follows.

Theorem 1.4. *Let G be a finite group and p a prime. Then the following statements are equivalent.*

(1) *The prime p does not divide* $|x^{G^\star}|$ *for every* $x \in G$.

(2) *The prime p does not divide* $|x^{G^\star}|$ *for every* p'*-element* $x \in G$ *of prime power order.*

(3) *G is p-nilpotent.*

Observe that the inverse of Theorem 1.3 is clearly not true. To see this, one may consider the following example. Let $G = H \ltimes V$; here H acts faithfully on V where $H = Z_2 \wr Z_2$ and $V = \mathbb{F}_3^2$. G is 2-nilpotent and has a conjugacy class of size divisible by 6.

However, if the class size is replaced by the sub-class size, then we have the following theorem.

Theorem 1.5. *Let G be a finite group and* p, q *be different primes. Then pq does not divide* $|x^{G^\star}|$ *for every* $x \in G$ *if and only if G is either p-nilpotent or q-nilpotent.*

Set $cs(G) = \{|x^G| \mid x \in G\}$ and $cs^\star(G) = \{|x^{G^\star}| \mid x \in G\}$.

Let X be a set of positive integers. Let $\rho(X)$ be the set of primes dividing some member of X. The graph $\Gamma(X)$ related to X is defined as follows: its vertex set is $\rho(X)$,

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and two vertices $p, q \in \rho(X)$ are joined by an edge if pq divides some member of X. We put

$$\rho(cs(G)) = \rho(G), \quad \rho(cs^\star(G)) = \rho^\star(G),$$
$$\Gamma(G) = \Gamma(cs(G)), \quad \Gamma^\star(G) = \Gamma(cs^\star(G)).$$

Now combining Theorem 1.4 with Theorem 1.5, we conclude the following corollary.

Corollary 1.6. *For distinct primes $p, q \in \rho^\star(G)$, there always exists some $m \in cs^\star(G)$ such that $pq \mid m$. In other words, $\Gamma^\star(G)$ is a complete subgraph of $\Gamma(G)$, where 'complete' means every pair of vertices is joined by an edge.*

For every member m of a set X of positive integers, let $\sigma(m)$ be the number of distinct prime divisors of m, and $\sigma(X) = \max_{m\in X} \sigma(m)$. Write $\sigma(G) = \sigma(cs(G))$, $\sigma^\star(G) = \sigma(cs^\star(G))$. The connection between $|\rho(G)|$ and $\sigma(G)$ is studied in [8, § 33] and elsewhere (see, for example, [3, 4, 11]). This question was first proposed by B. Huppert and was often referred to as the conjugacy class version of Huppert's ρ–σ conjecture. The best known bound $|\rho(G)| \le 4\sigma(G)$ for solvable groups was obtained by Zhang [11].

Recall that $|\rho(G)| \le 2\sigma(G)$ is not true for solvable groups [3]. But for sub-class size, we have the following result.

Theorem 1.7. *For every finite solvable but not nilpotent group G, $|\rho^\star(G)| < 2\sigma^\star(G)$.*

In this paper, we shall freely use the following facts. Let $N \lhd G$ and set $\overline{G} = G/N$. Then $|x^N| \,\Big|\, |x^G|$ for any $x \in N$, and $|\overline{x}^{\overline{G}}| \,\Big|\, |x^G|$ for every $x \in G$.

Why do we consider the size $|x^{G^\star}|$? The following remarks partially answer this question.

Remark 1.8. Let N be every normal (or subnormal) subgroup of a finite group G and x be an element of N. For standard conjugacy class and class size, we have that

$$x^N \subseteq x^G \quad \text{and} \quad |x^N| \,\Big|\, |x^G|;$$

but for our sub-class, we have by definition that

$$N_x = G_x, \quad x^{N^\star} = x^{G^\star} \quad \text{and} \quad |x^{N^\star}| = |x^{G^\star}|.$$

Observe further that for an element $x \in G$, if we define $|x^{G^{\star\star}}|$ to be the size of conjugacy class in the minimal normal subgroup in which x lies, then for a normal subgroup N with $x \in N$, we obtain that $|x^{N^{\star\star}}|$ is only a divisor of $|x^{G^{\star\star}}|$. This is one of the reasons why we define the sub-class size in the minimal subnormal subgroup but not in the minimal normal subgroup.

Remark 1.9. For every pair of elements $x, y \in G$, if x, y are conjugate in G, then G_x, G_y are conjugate. In fact, if $y = x^g$ for some $g \in G$, then $G_{x^g} = (G_x)^g$. In particular, $x \longrightarrow |x^{G^\star}|$ is a class function of G.

Remark 1.10. On the one hand, for every $x \in G$, we have that $|x^{G^\star}| \bigm| |x^G|$ because G_x is subnormal in G, and that in many cases $|x^{G^\star}|$ is much smaller than $|x^G|$. On the other hand, if we replace the standard conjugacy class size $|x^G|$ by the sub-class size $|x^{G^\star}|$, some known results are still true. For example, it is shown in [5] that G is supersolvable if all members of $cs(G)$ are square-free, and it is also true that G is supersolvable if all members of $cs^\star(G)$ are square-free (see Remark 5.1).

At the end of this paper, we also characterize the finite groups G in which every $m \in cs^\star(G)$ is a prime power.

We shall also mention that Isaacs has pointed out to us that the concept of sub-class has some relation with what he calls 'strong conjugacy' of subgroups (see [6, Section 9D] for details).

2. On sub-class sizes avoiding a single prime

In this section we prove some preliminary results and study the sub-class analog of Theorem 1.1.

Lemma 2.1. *Let $N \lhd G$ and set $\overline{G} = G/N$. For every element $x \in G$, we have that*

$$\overline{G}_{\overline{x}} \leq \overline{G_x}, \quad |\overline{x}^{\overline{G}^\star}| \bigm| |x^{G^\star}|, \quad \text{and} \quad |\overline{x}^{\overline{G}^\star}| \bigm| |\overline{x}^{\overline{G}}| \bigm| |x^G|.$$

Proof. Observe that $\overline{x}$ belongs to $\overline{G_x} = G_xN/N$ and that $\overline{G_x}$ is subnormal in $\overline{G}$. It follows that $\overline{G}_{\overline{x}}$ is a subnormal subgroup of $\overline{G_x}$. Also, we have

$$\begin{array}{rcl} |\overline{x}^{\overline{G}^\star}| & = & |\overline{x}^{\overline{G}_{\overline{x}}}| \\ & \text{divides} & |\overline{x}^{\overline{G_x}}| = |\overline{x}^{\overline{G_xN}}| \\ & = & |G_xN/N : C_{G_xN/N}(xN)| \\ & \text{divides} & |G_xN/N : C_{G_x}(x)N/N| \\ & \text{divides} & |G_x : C_{G_x}(x)| = |x^{G_x}| = |x^{G^\star}|. \end{array}$$

The rest is clearly true. □

Lemma 2.2. *Let $\mathbb{P}$ be a group-theoretical property that is closed under taking subgroups. Suppose that for every finite group H, a normal subgroup of H maximal with respect to having property $\mathbb{P}$ is a characteristic subgroup of H. If N is a subnormal subgroup with property $\mathbb{P}$ of G, then $\langle N^g \mid g \in G\rangle$ has property $\mathbb{P}$.*

Proof. We may assume that $N < G$. Let M be a maximal normal subgroup of G such that $N \leq M$. By induction, $\langle N^g \mid g \in M\rangle$ has property $\mathbb{P}$, and so $N \leq \langle N^g \mid g \in M\rangle \leq \mathbb{P}(M)$ for some normal subgroup $\mathbb{P}(M)$ of M maximal with respect to having property $\mathbb{P}$. Observe that $\mathbb{P}(M)$ is normal in G because $\mathbb{P}(M)$ is characteristic in M. Thus $\langle N^g \mid g \in G\rangle \leq \mathbb{P}(M)$, and since the property $\mathbb{P}$ is closed by subgroups, the result follows. □

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Proof of Theorem 1.4. (1) ⇒ (2). This is clear.

(2) ⇒ (3). Note that if G is a simple group, then $|x^G|$ coincides with $|x^{G^\star}|$ for every $x \in G$, and the result follows by [10, Theorem 5]. Suppose that G is not simple and let N be a minimal normal subgroup of G. Since the hypothesis is inherited by quotient groups and subnormal subgroups (see Remark 1.8 and Lemma 2.1), we conclude by induction that both G/N and N are p-nilpotent. By a standard argument, we may assume that N is the unique minimal normal subgroup of G, and that N is a p-group. Now it suffices to show that G is a p-group.

Suppose that G is not a p-group. Let x be every p'-element of G of prime power order. If $G_x < G$, then induction yields that G_x has a normal p-complement U, and this implies by Lemma 2.2 that $\langle U^g \mid g \in G\rangle$ is a normal p'-subgroup of G, which contradicts the assumption that N is the unique minimal normal subgroup of G. Therefore, $G_x = G$ and thus x^G has p'-size for every p'-element $x \in G$ of prime power order. As $O_{p'}(G) = 1$, [10, Theorem 5] yields that G is a p-group, as required.

(3) ⇒ (1). For every $x \in G$, write $x = uv = vu$, where u is a p-element and v is a p'-element. Set $N = \langle u\rangle O_{p'}(G)$. Clearly, N is subnormal in G and p does not divide $|x^N|$. This implies by Remark 1.8 that p does not divide $|x^{N^\star}| = |x^{G^\star}|$. □

Corollary 2.3. *Let G be a finite group. Then G is nilpotent if and only if $cs^\star(G) = 1$.*

Proof. This follows by Theorem 1.4. We also give a direct proof. If G is nilpotent then for every $x \in G$, $\langle x\rangle$ is subnormal by [6, Lemma 2.1], and thus $cs^\star(G) = 1$. If $cs^\star(G) = 1$ then for every $x \in G$, $\langle x\rangle$ lies in the center of G_x, and thus $\langle x\rangle$ is subnormal in G. By [6, Theorem 2.2], G is nilpotent. □

Remark 2.4. We remark that G is abelian if and only if $cs(G) = 1$.

Proposition 2.5. *Let G be a p-solvable group. Then the p-length $l_p(G) \le 1$ if and only if p does not divide $|x^{G^\star}|$ for every p-element x of G.*

Proof. The necessary part follows directly by Theorem 1.4. Suppose now that p does not divide $|x^{G^\star}|$ for every p-element x of G. We will show $l_p(G) \le 1$.

Let N be a minimal normal subgroup of G. Since the hypothesis is inherited by quotient groups and subnormal subgroups (see Remark 1.8 and Lemma 2.1), we conclude by induction that $l_p(G/N) \le 1$ and $l_p(N) \le 1$. By a standard argument, we may assume that N is the unique minimal normal subgroup of G, and that $N = O_p(G) = C_G(N)$.

Let V be the set of p-elements (of G) outside N, and suppose that V is not empty.

Assume $H := G_x < G$ for some $x \in V$. Then by induction $l_p(H) \le 1$. If $O_{p'}(H) > 1$, then by Lemma 2.2 G has a nontrivial p'-normal subgroup $\langle (O_{p'}(H))^g \mid g \in G\rangle$, a contradiction. Therefore $O_{p'}(H) = 1$, and so H has the normal Sylow p-subgroup $O_p(H)$. It follows again by Lemma 2.2 that $\langle (O_p(H)^g \mid g \in G\rangle$ is a normal p-subgroup of G, which leads to a contradiction: $x \in O_p(H) \le \langle (O_p(H)^g \mid g \in G\rangle \le O_p(G) = N$.

Assume now that $G_x = G$ for every $x \in V$. Then p does not divide $|x^G| = |x^{G^\star}|$ for every $x \in V$, and this implies that $V \le C_G(N)$, a contradiction.

Therefore V is empty, and so $l_p(G) = 1$ as desired. □

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3. On sub-class sizes avoiding a product of two primes

In this section, we study the sub-class analog of Theorem 1.3.

Let G be a p-solvable group, $x \in G$, and let $N_p(G)$ be its unique maximal normal p-nilpotent subgroup. If G_x is p-nilpotent, then by Lemma 2.2 we have that $x \in G_x \leq \langle (G_x)^g \mid g \in G \rangle \leq N_p(G)$. If $x \in N_p(G)$, then $G_x \leq N_p(G)$ is p-nilpotent. Therefore, G_x is p-nilpotent if and only if $x \in N_p(G)$.

Lemma 3.1. *Let G be a p-solvable group. Then for every fixed element x of G, p does not divide $|x^{G^\star}|$ if and only if $x \in N_p(G)$.*

Proof. If $x \in N_p(G)$, then $G_x \leq N_p(G)$ is p-nilpotent. Since $|x^{G^\star}| = |x^{N_p(G)^\star}|$, the necessary part follows from Theorem 1.4. We now show the sufficient part. Suppose that $|x^{G^\star}|$ is a p'-number and assume that $x \notin N_p(G)$. By the claim established before, G_x is not p-nilpotent. Set $N = O_{p'}(G_x)$ and $M/N = O_p(G_x/N)$ (that is, $M = N_p(G_x)$). Observe that $M < G_x$ and that x does not lie in every proper normal subgroup of G_x. It follows that $x \notin M$. Since $C_{G_x/N}(M/N) \leq M/N$ holds true for every p-solvable group G_x (see [9, 6.4.3]), we conclude that $p \mid |x^{G_x}|$, a contradiction. □

Lemma 3.2. *Let G be a $\{p, q\}$-solvable group and suppose that pq does not divide any members of $cs^\star(G)$. Then G is either p-nilpotent or q-nilpotent.*

Proof. Let $N_p(G)$, $N_q(G)$ be the maximal normal p-nilpotent and q-nilpotent subgroups, respectively, of G. By Lemma 3.1, we have $G = N_p(G) \cup N_q(G)$. This implies that either $G = N_p(G)$ or $G = N_q(G)$, and the result follows. □

Lemma 3.3. *Let H be a proper subgroup of G. Suppose that p does not divide $|x^{G^\star}|$ for every $x \in G - H$. Then G is p-nilpotent.*

Proof. We claim first that the result is true when G is p-solvable. In fact, if G is p-solvable, then by Lemma 3.1 we have $\langle G - H \rangle \leq N_p(G)$, and therefore $G = N_p(G)$ is p-nilpotent.

Observe that $|x^{G^\star}| = |(x^g)^{G^\star}|$ for every $x, g \in G$ (see Remark 1.9). It follows that p does not divide $|v^{G^\star}|$ for every $v \in G - \bigcap_{g \in G} H^g = \bigcup_{g \in G}(G - H^g)$. Therefore we may replace H by a normal subgroup $\bigcap_{g \in G} H^g$, and so we may assume that H is a maximal normal subgroup of G. Now G/H is simple. By Lemma 2.1 and Theorem 1.1, we obtain that G/H is p-solvable.

Case 1. Suppose that G has a maximal normal subgroup M different from H.

Then $M - (M \cap H) \leq G - H$, and by Remark 1.8, p does not divide $|x^{M^\star}|$ for every $x \in M - (M \cap H)$. This implies by induction that M is p-nilpotent.

Assume that $M \cap H > 1$. Since the hypothesis is inherited by $G/M \cap H$, it follows that $G/M \cap H$ is p-nilpotent. Therefore G is p-solvable, and the result follows by the claim established before.

Assume that $M \cap H = 1$. Then $G = H \times M$. Observe that both H and M are simple. Suppose that G is not p-solvable. Then H is a nonabelian simple group with $p \mid |H|$. Let $1 \neq u \in M$, $1 \neq v \in H$, and let us consider G_{uv}. We claim that $G_{uv} = G$.

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Otherwise, $1 < G_{uv} < G$. If G_{uv} is p-solvable, then $G_{uv} = M$, which yields a contradiction: $uv \in M$. If G_{uv} is not p-solvable, then $G_{uv} = H$, which is also impossible. Thus $G_{uv} = G$ as claimed. Now p does not divide $|(uv)^{G^\star}| = |(uv)^G|$, and since $|(uv)^G| = |u^M||v^H|$, we have that $\gcd(p, |v^H|) = 1$. Then Theorem 1.1 yields a contradiction. Thus G is p-solvable, and then G is p-nilpotent.

Case 2. Suppose that H is the unique maximal normal subgroup of G.

In this case, we get that $G_x = G$ and so that p does not divide $|x^G|$ for every $x \in G - H$. Let P be a Sylow p-subgroup of G. Then $G - H$ is a proper subset of $\bigcup_{g\in G} C_G(P^g)$, and so

$$\frac{1}{2}|G| \le |G - H| < |G : N_G(P)||C_G(P)| = |G|\frac{|C_G(P)|}{|N_G(P)|}.$$

Hence $N_G(P) = C_G(P)$, and so G is p-nilpotent by a well-known result of Burnside. □

Lemma 3.4. *Let G be a p-solvable group and suppose that pq does not divide any members of $cs^\star(G)$. Then G is either p-nilpotent or q-nilpotent.*

Proof. Let $N_p(G)$ be the unique maximal normal p-nilpotent subgroup of G. Suppose that G is not p-nilpotent. Then $N_p(G) < G$, and for every $x \in G - N_p(G)$, we conclude by Lemma 3.1 that $p \mid |x^{G^\star}|$, and so q does not divide $|x^{G^\star}|$. It follows by Lemma 3.3 that G is q-nilpotent, and we are done. □

Proof of Theorem 1.5. Suppose that G is p-nilpotent or q-nilpotent. By Theorem 1.4, pq does not divide any members of $cs^\star(G)$.

Suppose conversely that pq does not divide any members of $cs^\star(G)$. Assume that G is neither p-nilpotent nor q-nilpotent, and let G be of minimal order. We will work toward a contradiction.

By Lemma 3.4, we may assume that G is neither p-solvable nor q-solvable. Also G is not simple by Theorem 1.3.

We claim that G possess a unique minimal normal subgroup, say E. Suppose G has distinct minimal normal subgroups E_1 and E_2. Since the hypothesis is inherited by the groups $E_1, E_2, G/E_1, G/E_2$, these mentioned groups are either p-nilpotent or q-nilpotent. Since G is neither p-solvable nor q-solvable, by a standard argument we may assume that E_1 is not q-solvable and that E_2 is not p-solvable. Let V_i be a simple factor of E_i, $i = 1, 2$. Then both V_1 and V_2 are nonabelian simple groups, and $q \mid |V_1|$, $p \mid |V_2|$. Let $v_1 \in V_1$, $v_2 \in V_2$ be such that $q \mid |v_1^{V_1}|$ and $p \mid |v_2^{V_2}|$. Since $G_{v_1v_2} = V_1 \times V_2$, pq divides $|(v_1v_2)^{V_1\times V_2}| = |(v_1v_2)^{G^\star}|$, a contradiction.

Now E is the unique minimal normal subgroup of G. Since G is neither p-solvable nor q-solvable, by induction we may assume that G/E is p-nilpotent, that E is q-nilpotent but not p-nilpotent. In particular, E is a nonabelian q'-group with $p \mid |E|$.

We claim that G possesses a unique maximal normal subgroup, say M. Otherwise, G has distinct maximal normal subgroups M_1 and M_2. Then $E \le M_1 \cap M_2$. By induction M_i is either p-nilpotent or q-nilpotent where $i = 1, 2$, thus M_1, M_2 are both q-nilpotent because E is not p-nilpotent. This implies that $G = M_1M_2$ is q-nilpotent, a contradiction.

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We claim that E is simple. Suppose that E is not simple. Let E_i be simple factors of E, $i = 1, 2, \ldots, s$. Then $E = E_1 \times \cdots \times E_s$. Observe that G acts transitively on $\{E_1, \ldots, E_s\}$; it follows that $H := \bigcap_{i=1}^{s} N_G(E_i)$ is a proper normal subgroup of G. Let x be an element centralizing a Sylow p-subgroup of E. Then x centralizes a Sylow p-subgroup of E_i for every $i = 1, \ldots, s$. It follows that $x \in H$ because G acts transitively on $\{E_1, \ldots, E_s\}$. Since M is the unique maximal normal subgroup of G, we have $G - M \subseteq G - H$. Now for every $y \in G - M$, p divides $|y^G| = |y^{G^\star}|$ because y cannot centralize any Sylow p-subgroups of E, and thus q does not divide $|y^{G^\star}|$. Applying Lemma 3.3, we obtain that G is q-nilpotent, a contradiction.

Now E is a nonabelian simple group, and then $G/E \leq \mathrm{Out}(E)$ because E is the unique minimal normal subgroup of G. By the classification of finite simple groups, G/E is necessary solvable. Since E is q-solvable, G is q-solvable, a final contradiction. □

4. On sub-class version of the Huppert's ρ–σ conjecture

In this section, we study the sub-class analog of the Huppert's ρ–σ conjecture.

PROOF OF THEOREM 1.7. We argue using Casolo's method (see [8, Theorem 33.10]). For every $p \in \rho^\star(G)$, we put

$$\Delta_p = \{g \in G \mid p \mid |g^{G^\star}|\}.$$

Then by Lemma 3.1, we have

$$G - \Delta_p = \{g \in G \mid p \nmid |g^{G^\star}|\} = N_p(G),$$

where $N_p(G)$ is the unique maximal normal p-nilpotent subgroup of G. Write $m_p = |G : N_p(G)|$. By Theorem 1.4, we have $m_p \geq 2$ for every $p \in \rho^\star(G)$, and then

$$|\Delta_p| = \frac{m_p - 1}{m_p}|G| \geq \frac{|G|}{2}.$$

We consider in $\rho^\star(G) \times (G - 1)$ the subset

$$S = \{(p, g) \mid p \mid |g^{G^\star}|\} = \bigcup_{p \in \rho^\star(G)} (p, \Delta_p).$$

Then

$$(|G| - 1)\sigma^\star(G) \geq \sum_{1 \neq g \in G} \sigma(|g^{G^\star}|) = |S| = \sum_{p \in \rho^\star(G)} |\Delta_p| = \sum_{p \in \rho^\star(G)} \frac{m_p - 1}{m_p}|G| \geq |\rho^\star(G)||G|/2.$$

Hence $|\rho^\star(G)| < 2\sigma^\star(G)$. □

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5. Remarks

It has been shown in [5, Theorem 1] that G is supersolvable if all the members of $cs(G)$ are square-free. Furthermore, G is also supersolvable if $|x^G|$ is square-free for every element x of prime power order (see [10, Theorem 8]).

Remark 5.1. Suppose that $|x^{G^\star}|$ is square-free for every element $x \in G$ of prime power order. Then G is supersolvable.

Proof. If G is simple, then the result follows by [10, Theorem 8]. Suppose that G is not simple and let N be a minimal normal subgroup of G. Since the hypothesis is inherited by quotient groups and normal subgroups, we conclude by induction that G/N and N are supersolvable, and in particular that G is solvable. By standard arguments and induction, we may assume that G possesses a unique minimal normal subgroup N, that G/N is supersolvable, and also that $N = \mathrm{Fitting}(G)$. Now it suffices to show that N is of prime order.

Let L/N be a chief factor of G. Then there exists an element y of order q with $\gcd(q, |N|) = 1$ such that $L = N\langle y\rangle$. Observe that $L' = N$ since N is minimal normal in G; it follows that $C_N(y) = 1$. Suppose that $L_y < L$ and let M be a maximal normal subgroup of L with $L_y \leq M$. Then $N = L' \leq M < L$, and then $M = N$ which is clearly impossible. Thus $L_y = L$ and $|y^{G^\star}| = |y^{L^\star}| = |y^L| = |N|$. This implies by our hypothesis that N is of prime order, and so G is supersolvable. □

The finite groups in which every conjugacy class size is a prime power are studied in [5]. In what follows, we will describe the finite groups G in which every member of $cs^\star(G)$ is a prime power. Let $N_{pn}(G)$ be such that $N_{pn}(G)/O_p(G) = \mathrm{Fitting}(G/O_p(G))$.

Remark 5.2.

(1) For a fixed element x of G and a fixed prime p, $|x^{G^\star}|$ is a power of prime p if and only if $x \in N_{pn}(G)$.

(2) Every member of $cs^\star(G)$ is a prime power if and only if $G/O_p(G)$ is nilpotent for some prime p. And in this case, $\rho^\star(G) \subseteq \{p\}$.

Proof. (1) If $x \in N_{pn}(G)$, then by Theorem 1.4, $|x^{G^\star}|$ is a power of prime p. Assume conversely that $|x^{G^\star}|$ is a power of prime p. Suppose first that $G_x < G$. Then $x \in N_{pn}(G_x)$ by induction. Observe that $N_{pn}(G_x)$ is a characteristic subgroup of G_x; it is easy to see that $N_{pn}(G_x) \leq N_{pn}(G)$ (see Lemma 2.2). This implies that $x \in N_{pn}(G)$, and we are done. Suppose now that $G_x = G$. Thus x does not belong to any proper normal subgroups of G, and so $\langle x^g \mid g \in G\rangle = G$. Appealing to a theorem of Kazarin [8, Theorem 15.7], we get that G is solvable. Now by Lemma 3.2, we get that $x \in N_q(G)$ and thus $G = \langle x^g \mid g \in G\rangle = N_q(G)$ for every prime $q \neq p$. Therefore $G/O_p(G)$ is nilpotent, and we are done.

(2) If $G/O_p(G)$ is nilpotent for some prime p, then by Theorem 1.4 we get $\rho^\star(G) \subseteq \{p\}$. If every member of $cs^\star(G)$ is a prime power, then Corollary C yields that $\rho^\star(G) \subseteq \{p\}$ for some prime p, that is, every member of $cs^\star(G)$ is a power of p, and then $G/O_p(G)$ is nilpotent by statement (1). □

https://doi.org/10.1017/S1446788719000053 Published online by Cambridge University Press

Remark 5.3. An interesting question is whether a result similar to Theorem 1.7 might also hold true for arbitrary groups.

Remark 5.4. The study of sub-class sizes defined in this paper seems to be new, and we hope it will promote some future research interest in this direction.

Acknowledgement

The authors would like to thank the referee for valuable suggestions.

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