SOME REMARKS ON GROUPS ADMITTING A FIXED-POINT-FREE AUTOMORPHISM

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1. Introduction. A finite group G is said to be a fixed-point-free-group (an FPF-group) if there exists an automorphism σ which fixes only the identity element of G. The principal open question in connection with these groups is whether non-solvable FPF-groups exist. One of the results of the present paper is that if a Sylow p-group of the FPF-group G is the direct product of any number of mutually non-isomorphic cyclic groups, then G has a normal p-complement. As a consequence of this, the conjecture that all FPF-groups are solvable would be true if it were true that every finite simple group has a non-trivial Sylow subgroup of the kind just described. Here it should be noted that all the known simple groups satisfy this property.

In §§ 4 and 5, conditions for abelian groups and regular p-groups to be FPF-groups are considered. Typical of the results obtained are the following. (1) A finite abelian group G is not an FPF-group if, and only if, there are fully invariant subgroups H and K in G such that $H > K$ and $|H/K| = 2$. (2) If P is a finite group of exponent p, where p is a prime >3, and of class 2, then P is an FPF-group.

If the order, N, of σ is specified, various necessary conditions for G to be an FPF-group are known. A well-known result of Thompson (**7**) states that G must be nilpotent if N is prime. For more general N and under the added hypothesis that G is solvable, various conditions that must be satisfied by the nilpotent length and p-length of G are derived in (**5**), (**6**), and (**2**). (The results in (**6**) hold for any N, while in the other two papers it is assumed that N is a power of a prime.)

2. Preliminaries. The notation is the same as in (**1**) with the addition that $A(G)$ and $O(G)$ denote the automorphism group and outer automorphism group, respectively, of the group G. All groups are assumed to be finite. The following propositions are all well known and will be assumed without proof.

2.1. *If G is abelian of odd order, then G is an FPF-group.*

2.2 *If G is an elementary abelian 2-group, then G is an FPF-group if, and only if, $|G| \geqq 4$.*

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2.3 *If* $\sigma \in A(G)$, $N \lhd G$, *and* N *admits* σ, *then* σ *is fixed-point-free on* G *if, and only if, the automorphisms of* N *and* G/N *induced by* σ *are both fixed-point-free.*

2.4. *If* H *and* K *are both FPF-groups, then* $H \times K$ *is an FPF-group. Conversely, if* $H \times K$ *is an FPF-group and either* H *or* K *is a characteristic subgroup of* $H \times K$, *then* H *and* K *are both FPF-groups.*

2.5. *If* σ *is a fixed-point-free automorphism of* G *and* $p \mid |G|$, *then there is a Sylow* p*-subgroup of* G *which admits* σ.

2.6. *If* G *is a* p*-group,* $|G| > 1$, *and* σ *is in a Sylow* p*-subgroup of* $A(G)$, *then* σ *is not fixed-point-free on* G.

An immediate consequence of 2.6 is the following.

2.7. *If* G *is an FPF* p*-group, then there exists a fixed-point-free automorphism* σ *of* G *such that* p *does not divide the order of* σ.

3. Normal p-complements of FPF-groups.

3.1. LEMMA. *Let* G *be a nilpotent group,* H *a non-trivial subgroup of* G, *and* g *an element of* G *which normalizes* H. *Then the automorphism of* H *induced by conjugation by* g *is not fixed-point-free.*

Proof. G is the direct product of its Sylow subgroups, S_i, $i = 1, 2, \ldots, n$. Let $g = \Pi_{i=1}^{n} g_i$, where $g_i \in S_i$ and let $H_i = S_i \cap H$. For some i, $i = 1$ say, $|H_i| > 1$. But $[g_j, H_1] = 1$ if $j \neq 1$ since G is nilpotent. Thus, the automorphism of H_1 induced by g is just conjugation by g_1. Since $g_1 \in S_1$, it follows from 2.6 that conjugation by g_1 is not fixed-point-free on H_1.

3.2. LEMMA. *Suppose that* σ *is a fixed-point-free automorphism of* G *and that* H *is a normal subgroup of* G *which admits* σ. *Assume further that* $O(H)$ *is nilpotent. Then* $G = HC_G(H)$. *If, in addition,* $A(H)$ *is nilpotent, then* $H \leqq Z(G)$.

Proof. Let $\bar{G}$ be the normal product of G by $\langle\sigma\rangle$ and let $C = C_{\bar{G}}(H)$. Clearly, $\bar{G}/C$ is isomorphic to a subgroup of $A(H)$. HC/C is a normal subgroup of $\bar{G}/C$ and $\bar{G}/HC$ is isomorphic to a subgroup of $O(H)$. Since GC/C is certainly normal in $\bar{G}/C$, it follows from the lemma that the automorphism of $G/C_G(H)H$ induced by σ cannot be fixed-point-free unless $|G/C_G(H)H| = 1$. This proves the first part of the theorem, and if $A(H)$ is nilpotent, the same reasoning yields that the automorphism of $G/C_G(H)$ induced by σ is not fixed-point-free unless $|G/C_G(H)| = 1$.

3.3. COROLLARY. *Suppose that* σ *is a fixed-point-free automorphism of* G, *and* H *is a normal cyclic subgroup of* G *which admits* σ. *Then* $H \leqq Z(G)$.

Proof. If H is cyclic, then $A(H)$ is abelian.

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3.4. Corollary. *Let G be an FPF-group and suppose that P, a Sylow p-subgroup of G, has a chain*

$$1 = H_0 < H_1 < H_2 < \ldots < H_m = P$$

such that H_i char P *and* H_i/H_{i-1} *is cyclic for* $i = 1, 2, \ldots, m$. *Then*

$$N_G(P) = PC_G(P).$$

Proof. Let σ be a fixed-point-free automorphism of G. Without loss of generality we may assume that P admits σ. Then $N_G(P)$ certainly admits σ. From 3.3, it follows that $H_i/H_{i-1} \leqq Z(N_G(P)/H_{i-1})$ for $i = 1, \ldots, m$. Thus, if g is an element of $N_G(P)$ whose order is not divisible by p, then $[g, H_i] \leqq H_{i-1}$ for all i. Since g is a p'-element, this implies that $[g, P] = 1$. Thus, $N_G(P)/C_G(P)$ must be a p-group, which proves the corollary.

3.5. Theorem. *Let G be an FPF-group and suppose that P, a Sylow p-subgroup of G, is of the form*

$$P = P_1 \times P_2 \times \ldots \times P_m,$$

where P_i *is cyclic of order* p^{n_i}, $i = 1, 2, \ldots, m$, *and* $n_1 < n_2 < \ldots < n_m$. *Then G has a normal p-complement.*

Proof. We shall show that the hypothesis of 3.4 is satisfied. Since P is abelian, this will imply that $P \leqq Z(N_G(P))$. As is well known, this implies that G has a normal p-complement.

Now, $P/\Omega_1(P)$ is isomorphic to

$$P_1/\Omega_1(P) \times P_2/\Omega_1(P_2) \times \ldots \times P_m/\Omega_1(P_m)$$

and $P_i/\Omega_i(P)$ is cyclic of order p^{n_i-1}. Thus, using induction on $|P|$, we may assume that there is a series

$$\Omega_1(P) = H_0 < H_1 < \ldots < H_r = P$$

such that H_j char P and H_j/H_{j-1} is cyclic for $j = 1, 2, \ldots, r$. Now let $K_i = \mho^{n_i-1}(P) \cap \Omega_1(P)$ for $i = 1, 2, \ldots, m$ and let $K_{m+1} = 1$. Clearly, K_i char P, and it is easy to verify that

$$1 = K_{m+1} < K_m < K_{m-1} < \ldots < K_1 = \Omega_1(P)$$

and K_i/K_{i+1} is cyclic of order p for $i = 1, 2, \ldots, m$. Thus, the hypothesis of 3.4 is satisfied and therefore the theorem is proved.

3.6. *Conjecture.* If G is a simple group, then there is a prime p dividing $|G|$ such that a Sylow p-subgroup of G has the structure described in the hypothesis of 3.5.

All of the known simple groups satisfy this conjecture. For example, if $G = A_n$, $n \geqq 5$, then let p be a prime such that $n/2 < p \leqq n$. It follows immediately that the Sylow p-subgroups of A_n are of order p and thus cyclic.

The verification of the conjecture for the other known simple groups is straightforward but somewhat long, and therefore is omitted.

3.7. THEOREM. *Let G be an FPF-group such that every factor in a composition series of G satisfies* 3.6. *Then G is solvable.*

Proof. Let G be a minimal counter-example and let σ be a fixed-point-free automorphism of G. Suppose that there is a non-trivial normal subgroup N in G which admits σ. Then both N and G/N are FPF-groups. By induction on $|G|$, this implies that N and G/N are solvable, and thus G is solvable.

Now, suppose that G and 1 are the only normal subgroups which admit σ. Then G must be the direct product

$$G = H_1 \times H_2 \times \ldots \times H_n$$

of isomorphic simple groups $H_1, \ldots, H_n$. If the H_i are abelian, then the proof is complete. If the H_i are not abelian, then σ must permute the H_i transitively. It follows from this that H_1 admits σ^n and σ^n must be fixed-point-free on H_1. Since $H_1 \lhd G$, H_1 satisfies 3.6. But then 3.5 would imply that either H_1 is a p-group or H_1 is not simple. Thus the theorem is proved.

4. Abelian FPF-groups.

Because of 2.4, a nilpotent group is an FPF-group if, and only if, the Sylow subgroups are FPF-groups. Then, using 2.1, we see that the problem of characterizing abelian FPF-groups is equivalent to characterizing abelian FPF 2-groups.

4.1. LEMMA. *Let P be an abelian p-group whose invariants are*

$$(\overbrace{m, m, \ldots, m}^{n}).$$

If $\tau \in A(P/D(P))$, then there exists $\sigma \in A(P)$ such that the automorphism of $P/D(P)$ induced by σ is identical with τ. Furthermore, σ is fixed-point-free on P if, and only if, τ is fixed-point-free on $P/D(P)$.

The proof of this is easy and is left to the reader.

4.2. THEOREM. *Let P be an abelian 2-group whose invariants are*

$$(\overbrace{m_1, \ldots, m_1}^{n_1}, \overbrace{m_2, \ldots, m_2}^{n_2}, \ldots, \overbrace{m_r, \ldots, m_r}^{n_r}),$$

where $0 < m_1 < m_2 < \ldots > m_r$ and $n_i > 0$ for $i = 1, 2, \ldots, r$. Then P is an FPF-group if, and only if, $n_i > 1$ for all i.

Proof. The "if" part follows from 2.2, 2.4, and 4.1. Now let $H_i = \Omega_{m_i}(P)D(P)$ for $i = 1, 2, \ldots, r$, and let $H_0 = D(P)$. Now $D(P) = \mho^1(P)$. Thus, H_i is generated by $D(P)$ together with those elements of a basis whose orders are at most 2^{m_i} (here $m_0 = 0$). It follows from this that H_i/H_{i-1} (obviously $H_i \geqq H_{i-1}$) is elementary abelian of order 2^{n_i} for $i = 1, 2, \ldots, r$. Since a group of order 2 cannot be an FPF-group, the "only if" part is proved.

Since $\mho^k(P)$ and $\Omega_k(P)$ are fully invariant subgroups of P for all k, we have also proved the following result.

4.3. COROLLARY. *Let G be an abelian group. Then G is not an FPF-group if, and only if, there exist fully invariant subgroups H and K in G such that $H > K$ and $|H/K| = 2$.*

5. Regular FPF p-groups. We now wish to consider non-abelian p-groups, but we shall restrict ourselves to regular p-groups in the sense of (**4**). Since a regular 2-group must be abelian, we shall assume that p is odd. In particular, if p is odd, then any p-group of class 2 is regular. A simple result for such groups is the following theorem.

5.1. THEOREM. *Let G be a p-group of class 2 for $p > 3$. Let N be a subgroup of G and $x_1, x_2, \ldots, x_n$ elements of G such that*

(a) $Z(G) \geqq N \geqq G'$,

(b) $\{Nx_i | i = 1, 2, \ldots, n\}$ *is a basis for the abelian group* G/N,

(c) $\langle x_i \rangle \cap N = 1$ *for* $i = 1, 2, \ldots, n$.

Then G is an FPF-group.

Proof. First we remark that without (c) this theorem would be false. As will be seen later, there are p-groups of class 2 which are not FPF-groups.

To prove the theorem, note that the hypothesis implies that any element y in G can be written uniquely in the form $y = y_1 y_2 \ldots y_n u$, where $y_i \in \langle x_i \rangle$ and $u \in N$. Now let a be any integer such that

$$0 \not\equiv a \not\equiv \pm 1 \pmod{p}$$

(for example, $a = 2$ will suffice). Then define σ on G by

$$y^\sigma = y_1^{\,a} y_2^{\,a} \ldots y_n^{\,a} u^{a^2}.$$

To prove that this is a homomorphism, suppose that $z = z_1 z_2 \ldots z_n v$, where $z_i \in \langle x_i \rangle$ and $v \in N$. Now $y_i z_j = z_j y_i [y_i, z_j]$. Thus, using the fact that $Z(G) \geqq N \geqq G'$, we obtain

$$yz = y_1 \ldots y_n z_1 \ldots z_n uv = (y_1 z_1) y_2 \ldots y_n z_2 \ldots z_n uv \prod_{j=2}^{n} [y_j, z_1] =$$

$$(y_1 z_1)(y_2 z_2) \ldots (y_n z_n) \left(uv \prod_{n \geqq j > i \geqq 1} [y_j, z_i] \right).$$

Thus,

$$(yz)^\sigma = (y_1^{\,a} z_1^{\,a})(y_2^{\,a} z_2^{\,a}) \ldots (y_n^{\,a} z_n^{\,a}) \left(u^{a^2} v^{a^2} \prod_{n \geqq j > i \geqq 1} [y_j, z_i]^{a^2} \right).$$

Now $y^\sigma z^\sigma = y_1^{\,a} \ldots y_n^{\,a} z_1^{\,a} \ldots z_n^{\,a} u^{a^2} v^{a^2}$, and a similar calculation leads to

$$y^\sigma z^\sigma = (y_1^{\,a} z_1^{\,a}) \ldots (y_n^{\,a} z_n^{\,a}) \left(u^{a^2} v^{a^2} \prod_{n \geqq j > i \geqq 1} [y_j^{\,a}, z_i^{\,a}] \right).$$

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But since G is of class 2, it is easily proved that $[y_j^a, z_i^a] = [y_j, z_i]^{a^2}$. Thus $y^\sigma z^\sigma = (yz)^\sigma$, and therefore σ is at least an endomorphism of G. But from the conditions imposed on a, it is now easy to see that σ is a fixed-point-free automorphism of G.

5.2. COROLLARY. *Let G be of class 2 and exponent p, where $p > 3$. Then G is an FPF-group.*

Proof. Simply let $N = G'$.

It is not known whether 5.1 or 5.2 are true for $p = 3$.

We now wish to prove a result that will provide some examples of regular p-groups which are not FPF-groups. First, however, we need a lemma.

5.3. LEMMA. *Let P be a regular p-group such that $x^{p^n} = 1$ for all x in P but P does contain elements of order p^n and $n > 1$. Assume that σ is a p'-element of $A(P)$ and that T is a normal cyclic subgroup of order p^{n-1} in P such that $T \geqq D(P)$. Then there is a cyclic subgroup of order p^n in P which admits σ.*

Proof. If g is of order p^n in P, then $\langle g^p \rangle = T$ since P/T is elementary abelian. But $g^p \in D(P)$. Thus $T = D(P)$, and therefore T certainly admits σ. Now, if g and h are both of order p^n in P, then we must have $\langle g^p \rangle = \langle h^p \rangle = T$. Thus, $h^p = g^{ap}$ for some a prime to p. It follows from this that $(g^a h^{-1})^p = 1$ since P is regular. Thus, $P/\Omega_{n-1}(P)$ is cyclic of order p. $\Omega_{n-1}(P)$ certainly admits σ and $\Omega_{n-1}(P)/T$ is of index p in P/T. Now, considering P/T as a vector space over GF(p) on which σ operates, we can use the theorem of complete reducibility to conclude that there is a σ-admissible complement to $\Omega_{n-1}(P)/T$ in P/T. Thus, there is a subgroup S in P such that $S\Omega_{n-1}(P) = P$, $S \cap \Omega_{n-1}(P) = T$, and S admits σ. Since $S \nleqq \Omega_{n-1}$ and $|S/T| = p$, then S must be cyclic of order p^n.

5.4. COROLLARY. *Let P be an abelian p-group with invariants $(m_1, m_2, \ldots, m_n)$ where $m_1 \leqq m_2 \leqq \ldots \leqq m_{n-1} < m_n$, and let σ be a p'-element of $A(P)$. Then there is a cyclic subgroup of order p^{m_n} in P which admits σ.*

Proof. If $m_n = 1$, then there is nothing to prove. Thus, we assume that $m_n > 1$ and use induction on m_n. Now $\mho^1(P)$ has invariants $\{m_1 - 1, m_2 - 1, \ldots, m_n - 1\}$ and $\mho^1(P)$ certainly admits σ. Thus, by induction, there is a cyclic subgroup T of order $p^{m_n - 1}$ contained in $\mho^1(P)$ such that T admits σ. Now let $S = \Omega_1$ (P mod T). S admits σ, S/T is elementary abelian, and, since $T \leqq \mho^1(P)$, S contains elements of order p^{m_n}. Applying the lemma to S completes the proof.

5.5. THEOREM. *Let P be a regular p-group, $p > 2$, such that*

(a) *$P = RS$, $R \cap S = 1$, where R and S are subgroups;*

(b) *S is cyclic of order p^n, R is of exponent p^m, and $n > m$;*

(c) *$S \lhd P$.*

Then P is an FPF-group if, and only if, $S \leqq Z(P)$ and R is an FPF-group.

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Proof. If $S \leqq Z(P)$, then $P = R \times S$ and the "if" part of the theorem follows from 2.1 and 2.4. Now suppose that σ is a fixed-point-free automorphism whose order is prime to p.

First suppose that $S \leqq Z(P)$. Then $P = R \times S$ and $Z(P) = Z(R) \times S$. From 5.4, there is a cyclic subgroup S^* of order p^n in $Z(P)$ such that S^* admits σ. Now $\mho^{n-1}(P) = \mho^{n-1}(R) \times \mho^{n-1}(S) = \mho^{n-1}(S)$ since $m \leqq n - 1$. Since $\mho^{n-1}(S^*) \neq 1$, this implies that

$$\mho^{n-1}(S^*) \cap Z(R) = \mho^{n-1}(S^*) \cap Z(R) = 1.$$

Thus $P = S^* \times R$, and therefore $P/S^* \cong R$. Since S^* admits σ, this implies that R is an FPF-group.

It now remains to prove that $S \leqq Z(P)$. If S admitted σ, this would follow from 3.3. Unfortunately, S need not admit σ. We shall prove that $S \leqq Z(P)$ by induction on $|P|$.

First, suppose that $(xy)^p = 1$ for $x \in R$, $y \in S$. Since P is regular, we must have $x^p = y^{-p}$. Since $R \cap S = 1$, this implies that $x^p = y^p = 1$. Thus, $\Omega_1(P) = \Omega_1(R)\Omega_1(S)$. $\Omega_1(P)$ admits σ and therefore $P/\Omega_1(P)$ is an FPF-group. Now, if $P = \Omega_1(P)$, then $n = 1$, $m = 0$, and the result is obvious. If $P \neq \Omega_1(P)$, then $P/\Omega_1(P)$, which equals $(R\Omega_1(S)/\Omega_1(R)\Omega_1(S))(S\Omega_1(R)/\Omega_1(R)\Omega_1(S))$, satisfies the hypothesis of the theorem. Thus, by induction we obtain $[P, S] \leqq \Omega_1(R)\Omega_1(S)$. But $S \lhd P$. Thus, $[P, S] \leqq \Omega_1(S)$. Hence, if $x \in P$, $g \in S$, then $1 = [x, g]^p = [x, g^p]$. Therefore $\mho^1(S) \leqq Z(P)$.

Now from $S \lhd P$ we can easily prove that $\mho^1(P) = \mho^1(R)\mho^1(S)$. From this, it follows that

$$\mho^{n-1}(P) = \mho^{n-1}(R)\mho^{n-1}(S) = \mho^{n-1}(S) = \Omega_1(S).$$

Thus, $\Omega_1(S)$ is a characteristic subgroup of P and therefore it certainly admits σ. Now let $M = Z\,(P \bmod \Omega_1(S))$. M admits σ and $S \leqq M$ since $[P, S] \leqq \Omega_1(S)$. It now follows that $M = Z(R)S$.

Suppose that there is a cyclic subgroup S^* of order p^n contained in M such that S^* admits σ. $\mho^{n-1}(P) = \Omega_1(S)$ implies that $S^* > \Omega_1(S^*) = \Omega_1(S)$. Thus, $S^* \lhd P$ since $[P, M] \leqq \Omega_1(S)$. 3.3 now implies that $S^* \leqq Z(P)$. But $S^* \cap Z(R) = 1$ since $\Omega_1(S^*) \cap Z(R) = \Omega_1(S) \cap Z(R) = 1$. Thus, $M = S^*Z(R)$ which implies that $[M, R] = 1$. This certainly implies that $S \leqq Z(P)$.

We now complete the proof by showing the existence of such an S^*. $\mho^1(M) = \mho^1(S)\mho^1(Z(R))$ is an abelian group satisfying the hypothesis of 5.4. Thus, there is a cyclic subgroup T of order p^{n-1} in $\mho^1(M)$ such that T admits σ. But $\mho^{n-1}(P) = \Omega_1(S)$ and $T \leqq \mho^1(P)$. Thus $T \geqq \mho^{n-2}(T) = \Omega_1(S)$. Now let $N = \Omega_1\,(M \bmod T)$. N admits σ and N/T is elementary abelian. N contains elements of order p^n since $T \leqq \mho^1(M)$. Thus, from 5.3, there is a cyclic subgroup S^* of order p^n contained in N which admits σ. This completes the proof of the theorem.

Example. Let p be an odd prime, $n > 1$, and let P be the group with generators x, y and relations

$$x^{p^n} = y^p = 1, \qquad y^{-1}xy = x^{1+p^{n-1}}.$$

Then P is a regular p-group since it is of class 2 but P is not an FPF-group since $\langle x \rangle \nleqq Z(P)$.

It seems difficult to formulate conditions sufficient for non-abelian 2-groups to be FPF-groups. Since a simple non-abelian group must be of even order, theorems of this type would be of interest with respect to the conjecture that all FPF-groups are solvable. There is some evidence, however, to suggest that there are not too many non-abelian FPF 2-groups. For example, of the 311 non-abelian 2-groups of order at most 64 listed in (**3**), there are only three which are FPF-groups. These three, in the notation of (**3**), are $64\Gamma_9 e$, $64\Gamma_{13}a_1$, and $64\Gamma_{13}a_5$.

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