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ON THE NUMBER OF CONJUGACY CLASSES OF THE SYLOW p-SUBGROUPS OF $GL(n,q)$

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If G is a finite p-group of order p^n, P. Hall determined the number of conjugacy classes of G, $r(G)$, modulo $(p^2-1)(p-1)$. Namely, he proved the existence of a constant $k \geqslant 0$ such that $r(G) = n(p^2-1) + p^e + k(p^2-1)(p-1)$. In this paper, we denote by $\mathcal{G}_n$ the group of the upper unitriangular matrices over $\mathbb{F}_q$, the finite field with $q = p^t$ elements, and we determine the number of classes of $\mathcal{G}_n$ modulo $(q-1)^5$.

In this paper we maintain the notation given in [**1**, **2**]. We count the canonical matrices by firstly fixing the configurations of entries with non-zero values which correspond to them.

LEMMA 1. *All the matrices in $\mathcal{G}_n$ having exactly one non-zero value off the main diagonal are canonical. The number of them is $\mu_1 \cdot (q-1)$, where $\mu_1 = n(n-1)/2$.*

PROOF: Let $A = I_n + a_{ij}E_{ij}$, with $a_{ij} \neq 0$. We must prove that A has the value zero at all of its inert points. The only entry with a non-zero value is (i,j) and we have $L_{ij} = a_{i,i+1}x_{i+1,j} + \cdots + a_{i,j-1}x_{j-1,j} - (a_{i+1,j}x_{i,i+1} + \cdots + a_{j-1,j}x_{i,j-1}) = 0$. Hence (i,j) is a ramification point. Finally, all these matrices are canonical. The number of them is deduced taking into account that we can have $|\mathbb{F}_q - \{0\}| = q-1$ different values at the non-zero entry and that the total number of such entries that we can consider is $\mu_1 = n(n-1)/2$. ∎

LEMMA 2. *A matrix A of $\mathcal{G}_n$ which has exactly two non-zero values off the main diagonal is canonical if and only if these values are situated in entries which are not in the same row or column. The number of such matrices is $\mu_2 \cdot (q-1)^2$, where $\mu_2 = n(n-1)(n-2)(3n-5)/24$.*

PROOF: Let $A = I_n + a_{ij}E_{ij} + a_{rs}E_{rs}$, with $a_{ij} \neq 0 \neq a_{rs}$. We show that A is a canonical matrix if and only if the entries (r,s) and (i,j) are not in the same row or column. If that condition is satisfied, (i,j) and (r,s) are ramification points, since

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$L_{ij} = 0, L_{rs} = 0$. Consequently A takes the value zero at the inert points and it is a canonical matrix. Conversely, suppose that A is a canonical matrix. According to Lemma (3.7) of [**1**], above a pivot and in the same column we cannot have another pivot. Hence the entries (i,j) and (r,s) cannot be in the same column. Suppose they are in the same row: $(i,j) \prec (r,s) = (i,s)$. Since the s-th row of $A_0 = A - I$ is zero, if there is a non-zero value in the s-th column, it must be a pivot of A, according to Lemma (3.8) of [**1**]. Thus (i,s) is a pivot, in contradiction to $a_{ij} \neq 0$. The number of possible configurations of entries $(i,j),(r,s)$ which are not in the same row or column is $\mu_2 = n(n-1)(n-2)(3n-5)/24$, as we determined in [**1**]. So the total number of canonical matrices with exactly two non-zero values is $\mu_2 \cdot (q-1)^2$. ∎

LEMMA 3. *Let* $A = I_n + a_{i_1,j_1}E_{i_1,j_1} + a_{i_2,j_2}E_{i_2,j_2} + a_{i_3,j_3}E_{i_3,j_3}$*, with* $a_{i_t,j_t} \neq 0$, $t = 1,2,3$*, and* $(i_1,j_1) \prec (i_2,j_2) \prec (i_3,j_3)$*. Then* A *is canonical if and only if one of the following conditions holds:*

(1) $(i_1,j_1),(i_2,j_2),(i_3,j_3)$ *are in different rows and columns.*

(2) $(i_1,j_1) \prec (i_2,j_2)$ *are in different rows and columns,* $i_2 = i_3$ *and* $i_1 = j_3$*.*

PROOF: Suppose that A is a canonical matrix. Then it is clear that $B = I_n + a_{i_1,j_1}E_{i_1,j_1} + a_{i_2,j_2}E_{i_2,j_2}$ is also a canonical matrix. According to the previous lemma, the points $(i_1,j_1),(i_2,j_2)$ are in different rows and columns and are pivots. Suppose that (1) does not hold and let us prove (2). The third point is then lined up with one of the other two points. Moreover, it cannot be in the same column as (i_1,j_1), since (i_1,j_1) is a pivot, and neither in the same row as (i_1,j_1), since $(i_3,j_3) \succ (i_1,j_1)$. Hence (i_3,j_3) is lined up with (i_2,j_2). Moreover, they cannot be in the same column, since (i_2,j_2) is a pivot and (i_3,j_3) would otherwise be in the same column and above a pivot. Consequently, (i_3,j_3) and (i_2,j_2) are in the same row, that is, $i_2 = i_3$ and $j_2 < j_3$. Suppose the j_3-th row of A_0 is zero. Then, according to Theorem (4.3) of [**2**], (i_3,j_3) is an inert point in a canonical matrix with a non-zero value, which is impossible. Hence, there must be a non-zero element in that row and $j_3 = i_1$.

Let us prove the converse. We must show that, if A satisfies either of conditions (1) or (2), then A is a canonical matrix. If (1) holds, we have $L_{i_1,j_1} = L_{i_2,j_2} = L_{i_3,j_3} = 0$ and the points (i_t,j_t), $t = 1,2,3$ are ramification points. Since the remaining entries of A above the main diagonal are zero, it follows that A takes the value zero at its inert points and, consequently, A is canonical. Suppose now (2) holds and let us show that the three points with non-zero values are again ramification points. This is obvious for the first two, since $L_{i_1,j_1} = 0 = L_{i_2,j_2}$. Finally, (i_3,j_3) is also a ramification point, since the form $L_{i_3,j_3} = a_{i_2,j_2}x_{j_2,j_3}$ is linearly dependent on the preceding one $L_{j_2,j_1} = -a_{i_1,j_1}x_{j_2,i_1} = -a_{i_1,j_1}x_{j_2,j_3}$. So the matrix A is canonical. ∎

LEMMA 4. *The number of canonical matrices in* $\mathcal{G}_n$ *with exactly three non-*

zero values off the main diagonal is $\mu_3 \cdot (q-1)^3$, *where* $\mu_3 = n(n-1)(n-2)(n-3)$ $(n^2-5n+8)/48$.

PROOF: The number of possible configurations of canonical matrices with three non-zero values off the main diagonal is $\mu_3 = n(n-1)(n-2)(n-3)(n^2-5n+8)/48$, as determined in [1]. Once the three possible points are fixed according to the previous lemma, in each of them we can place $q-1$ non-zero elements.

Now we give another way of calculating the number of such configurations, different from that in [1]. Our aim in doing so is twofold: on one hand, to simplify the proof in [1] and, on the other hand, to calculate some functions which are going to be used in the calculation of subsequent configurations. According to the previous lemma,

$$\mu_3(n) = \mu_3'(n) + \mu_3''(n),$$

where μ_3' is the number of 3-tuples of pairs $(i_1,j_1) \prec (i_2,j_2) \prec (i_3,j_3)$ which are in different rows and columns, and $\mu_3''(n)$ is the number of pairs $(i_1,j_1) \prec (i_2,j_2) \prec (i_3,j_3)$ such that $(i_1,j_1),(i_2,j_2)$ are in different rows and columns and $i_2 = i_3$ and $i_1 = j_3$. We calculate a recursive formula for $\mu_3'(n)$. If all the values in the first row of A_0 are zero, we have $\mu_3'(n-1)$ configurations. If there is some non-zero value in the first row, then there is exactly one. The number of configurations for the remaining two non-zero elements is $\mu_2(n-1)$. Once such a configuration is fixed, the non-zero value in the first row satisfies $(1,j_3) \in \{(1,2),\ldots,(1,n)\} - \{(1,j_1),(1,j_2)\}$. Hence we have

$$\mu_3'(n) = \mu_3'(n-1) + (n-3)\mu_2(n-1) = \mu_3'(n-1) + (n-1)(n-2)(n-3)^2(3n-8)/24,$$

and consequently

$$\mu_3'(n) = n(n-1)(n-2)^2(n-3)^2/48.$$

Let us now calculate $\mu_3''(n)$. We have $1 \leqslant i_2 = i_3 < j_2 < j_3 = i_1 < j_1 \leqslant n$. Hence there are $\binom{n}{4}$ ways of choosing these indices, that is, $\mu_3''(n) = n(n-1)(n-2)(n-3)/24$, and finally

$$\begin{aligned}\mu_3(n) &= n(n-1)(n-2)^2(n-3)^2/48 + n(n-1)(n-2)(n-3)/24 \\ &= n(n-1)(n-2)(n-3)(n^2-5n+8)/48.\end{aligned}$$ ∎

REMARK. If x_{rs} appears in some form L_{ij} with a non-zero coefficient, then that coefficient is either $a_{kr} \neq 0$ and $(i,j) = (k,s)$ for some $1 \leqslant k < r$, or $a_{sl} \neq 0$ and $(i,j) = (r,l)$ for some $l > s$.

LEMMA 6. *Let* $A = I_n + a_{i_1,j_1}E_{i_1,j_1} + a_{i_2,j_2}E_{i_2,j_2} + a_{i_3,j_3}E_{i_3,j_3} + a_{i_4,j_4}E_{i_4,j_4}$, *with* $a_{i_t,j_t} \neq 0$, $t = 1,2,3,4$ *and* $(i_1,j_1) \prec (i_2,j_2) \prec (i_3,j_3) \prec (i_4,j_4)$. *Then* A *is a canonical matrix if and only if one of the following conditions holds:*

(1a) $(i_1,j_1),(i_2,j_2),(i_3,j_3),(i_4,j_4)$ *are in different rows and columns.*

(1b) $(i_1,j_1) \prec (i_2,j_2) \prec (i_3,j_3)$ *are in different rows and columns,* $i_4 = i_3$, $j_4 > j_3$, $j_4 \in \{i_1,i_2\} - \{j_1,j_2,j_3\}$ *and, if* $i_2 = j_3$, *then* $j_2 > j_1$.

(2) $1 \leqslant i_2 = i_3 < j_2 < j_3 = i_1 < j_1 \leqslant n$, $i_4 < i_3$ *and* $j_4 \notin \{j_1,j_2\}$.

PROOF: Suppose A is a canonical matrix. It is obvious that the matrix $B = I_n + a_{i_1,j_1}E_{i_1,j_1} + a_{i_2,j_2}E_{i_2,j_2} + a_{i_3,j_3}E_{i_3,j_3}$ is also canonical and the points $(i_1,j_1) \prec (i_2,j_2) \prec (i_3,j_3)$ satisfy (1) or (2) of Lemma 3. If (1) holds then $i_1 > i_2 > i_3$ and $|\{j_1,j_2,j_3\}| = 3$, and these points are pivots. So $|\{j_1,j_2,j_3,j_4\}| = 4$. If the four points are in different rows and columns, we have (1a). Otherwise, the fourth point necessarily has a common row or column with one of the preceding points. It cannot be a column, since all the points above a pivot are inert (with zero value). Hence it must be a row and, since (i_4,j_4) is the last point, necessarily $i_3 = i_4$ and $j_3 < j_4$. Proceeding as in part (2) of the previous lemma, we conclude that the j_4-th row of A_0 is non-zero and so $j_4 \in \{i_1,i_2\}$. Let us prove that if $i_2 = j_3$ (and consequently $i_1 = j_4$) then $j_2 > j_1$. Indeed, we observe that the unknown x_{j_3,j_4} appears in $L_{i,j}$ with $(i,j) \prec (i_4,j_4)$ if and only if $(i,j) = (j_3,j_1)$ and

$$L_{j_3,j_1} = \begin{cases} -a_{j_4,j_1}x_{j_3,j_4}, & \text{if } i_2 \neq j_3; \\ a_{i_2,j_2}x_{j_2,j_1} - a_{j_4,j_1}x_{j_3,j_4}, & \text{if } i_2 = j_3, j_2 < j_1. \end{cases}$$

Suppose $i_2 = j_3$. If $j_2 < j_1$ then the unknown x_{j_2,j_1} which appears in L_{j_3,j_1}, does not appear in any L_{rs} with $(r,s) \prec (j_3,j_1)$ (by using the remark), whence L_{i_4,j_4} is linearly independent on L_{j_3,j_1} and (i_4,j_4) is an inert point, which is impossible. Therefore, if $i_2 = j_3$ then $j_2 \geqslant j_1$ and necessarily $j_2 > j_1$. Suppose now that part (2) of Lemma 3 (that is, $1 \leqslant i_2 = i_3 < j_2 < j_3 = i_1 < j_1 \leqslant n$) holds and let us prove that (i_4,j_4) satisfies $i_4 < i_3$ and $j_4 \notin \{j_1,j_2,j_3\}$. Indeed, if $i_3 = i_4$ then the three points would be in the row $i_2 = i_3 = i_4$ and consequently $j_2 < j_3 < j_4$. Hence the j_4-th row of A_0 is non-zero and again we have an inert point with a non-zero value, which is impossible. We conclude that $i_4 < i_3$. The entries (i_1,j_1) and (i_2,j_2) are pivots, so $j_4 \notin \{j_1,j_2\}$.

We consider now the converse. In the case (1a) we have

$$L_{i_t,j_t} = 0, \; t = 1,2,3,4, \tag{1}$$

so the four points we are studying are ramification points and the matrix is canonical. In the case (1b), (1) holds for $t = 1,2,3$ and the forth point is a ramification point since we have the following linear dependence: $L_{i_4,j_4} = a_{i_3,j_3}x_{j_3,j_4}$ is linearly dependent on $L_{j_3,j_1} = -a_{i_1,j_1}x_{j_3,j_4}$ if $j_4 = i_1$ (note that if $j_3 = i_2$, we have $j_2 > j_1$), since $(j_3,j_1) \prec (i_4,j_4)$ and is linearly dependent on $L_{j_3,j_2} = -a_{i_2,j_2}x_{j_3,j_4}$ if $j_4 = i_2$, since $(j_3,j_2) \prec (i_4,j_4)$. Suppose now that (2) holds. The points $(i_1,j_1), (i_2,j_2)$, and (i_3,j_3) are ramification points in the matrix $B = I_n + a_{i_1,j_1}E_{i_1,j_1} + a_{i_2,j_2}E_{i_2,j_2} + a_{i_3,j_3}E_{i_3,j_3}$,

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according to Lemma 3. The same happens with (i_4, j_4). Indeed, if $j_4 \neq j_3$ we have $L_{i_4,j_4} = 0$ and we are done. Suppose $j_4 = j_3$. Then $j_4 = j_3 = i_1$ and $L_{i_4,j_4} = -a_{i_3,j_3} x_{i_4,i_3}$ which is linearly dependent on $L_{i_4,j_2} = -a_{i_2,j_2} x_{i_4,i_2} = -a_{i_2,j_2} x_{i_4,i_3}$, with $(i_4, j_2) \prec (i_4, j_4)$. This completes the proof. $\square$

LEMMA 7. *The number of configurations of canonical matrices with exactly four non-zero entries is*

$$\mu_4 = n(n-1)(n-2)(n-3)(n-4)\big(3n^3 - 30n^2 + 121n - 182\big)/1152,$$

and the number of canonical matrices with exactly four non-zero entries is $\mu_4 \cdot (q-1)^4$.

PROOF: Bearing in mind the last lemma it suffices to calculate μ_4. Each configuration will be one of the types (1a), (1b) or (2) of the previous lemma. The number μ_4 will be the sum of the three corresponding numbers. Let $\mu_4'^1(n)$ be the number of configurations of type (1a), that is, with the four non-zero entries in different rows and columns. If the first row of A_0 is zero, then we have $\mu_4'^1(n-1)$ configurations. Otherwise, the first row of A_0 has exactly one non-zero entry and, since $j_4 \in \{2, \dots, n\} - \{j_1, j_2, j_3\}$, there are $n - 1 - 3 = n - 4$ possibilities for j_4. So the following recursive formula holds:

$$\mu_4'^1(n) = \mu_4'^1(n-1) + \mu_3'(n-1)(n-4).$$

Then, we have

$$\mu_4'^1(n) = n(n-1)(n-2)(n-3)(n-4)\big(15n^3 - 150n^2 + 485n - 502\big)/5760.$$

Let $\mu_4'^2(n)$ be the number of configurations of type (1b). If the first row of A_0 is zero, we have $\mu_4'^2(n-1)$ configurations. Suppose $i_4 = i_3 = 1$. Then $j_4 \in \{i_1, i_2\}$. We reorder the points $(i_1, j_1), (i_2, j_2)$ so that $j_4 = i_1$. Then $j_3 < j_4 = i_1 < j_1$, and once these three values are fixed, the value of $i_1 = j_4$ is also fixed. We count the number of points (i_2, j_2) satisfying the following conditions: $i_2 > i_3 = 1$, (because $(i_3, j_3) \succ (i_2, j_2)$, $i_2 \neq i_1$) and if $i_2 = j_3$ then $j_2 > j_1$ and $j_2 \notin \{j_1, j_3, j_4\}$. The set of the entries which satisfy these conditions is made up of the points in the rows $2, \dots, n-1$ with the exception of exactly one point in each of the columns j_3, j_4, j_1. So the number we are calculating is

$$(n-1)(n-2)/2 - (j_3 - 2) - (j_4 - 2) - (j_1 - 2) - (n - j_4 - 1) - (j_1 - j_3 - 2).$$

So taking into account the possible values for (j_3, j_4, j_1) we get the sum

$$\begin{aligned} \sum_{j_3=2}^{n-2} \sum_{j_4=j_3+1}^{n-1} \sum_{j_1=j_4+1}^{n} \big((n-1)(n-2)/2 - (j_3 - 2) - (j_4 - 2) - (j_1 - 2) \\ -(n - j_4 - 1) - (j_1 - j_3 - 2)\big) \\ = (n-1)(n-2)(n-3)(n-4)^2/12. \end{aligned}$$

The following recursive relation holds:

$$\mu_4'^2(n) = \mu_4'^2(n-1) + (n-1)(n-2)(n-3)(n-4)^2/12,$$

whence

$$\mu_4'^2(n) = n(n-1)(n-2)(n-3)(n-4)(5n-19)/360.$$

Denote by $\mu_4''(n)$ the number of configurations of type 2). $\mu_4''(n-1)$ configurations are obtained when the first row of A_0 is zero. Otherwise, for fixed $(i_1, j_1), (i_2, j_2), (i_3, j_3)$ we have $n-3$ ways of choosing j_4 in the set $\{2, \dots, n\} - \{j_1, j_2\}$. So we obtain the formula

$$\mu_4''(n) = \mu_4''(n-1) + \mu_3''(n-1) \cdot (n-3) = \mu_4''(n-1) + (n-1)(n-2)(n-3)(n-4)/24,$$

and consequently

$$\mu_4''(n) = n(n-1)(n-2)(n-3)(n-4)(5n-13)/720,$$

whence

$$\begin{aligned} \mu_4(n) &= \mu_4'^1(n) + \mu_4'^2(n) + \mu_4''(n) \\ &= n(n-1)(n-2)(n-3)^2(n-4)\big(3n^3 - 30n^2 + 121n - 182\big)/1152. \end{aligned}$$ ∎

DEFINITION 8: We denote by $\mathcal{D}$ the set of diagonal matrices of size $n \times n$ with entries in $\mathbb{F}_q - \{0\}$.

We have

LEMMA 9. *Let $A \in \mathcal{G}_n$ and $D \in \mathcal{D}$. Then $A^D \in \mathcal{G}_n$ and the nature of each entry in A^D and A is the same. Moreover, A^D is canonical if and only if A is a canonical matrix.*

PROOF: If $D = \operatorname{diag}(t_1, \dots, t_n)$, then $A^D = (b_{ij})$ with $b_{ij} = t_i^{-1} t_j a_{ij}$, so $A^D \in \mathcal{G}_n$. Let $(r,s) \in \mathcal{J}$. We note that $E \in \mathcal{G}_{(r,s)}$ if and only if $E^D \in \mathcal{G}_{(r,s)}$, since $e_{ij} = 0$ for all $(i,j) \preceq (r,s)$ if and only if $t_i^{-1} t_j e_{ij} = 0$ for all $(i,j) \preceq (r,s)$. So there exists $T \in \mathcal{G}_n$ such that $A^T = AE$ with $E \in \mathcal{G}_{(r,s)}$ if and only if $\left(A^T\right)^D = (AE)^D$ if and only if $\left(A^D\right)^{T^D} = \left(A^D\right)^{E^D}$ with $E^D \in \mathcal{G}_{(r,s)}$. On the other hand, for $(k,l) \in \mathcal{J}$ the entry (k,l) is inert for A if and only if there exists $T \in \mathcal{G}_n$ such that $A^T \mathcal{G}_{(k,l)^*} = A\mathcal{G}_{(k,l)^*}$ and $A^T \mathcal{G}_{(k,l)} \neq A\mathcal{G}_{(k,l)}$. The first equality is equivalent to the existence of a matrix $E \in \mathcal{G}_{(k,l)^*}$ such that $A^T = AE$, and the second one to the condition $A^T \neq AM$, for all $M \in \mathcal{G}_{(k,l)}$. Taking into account the above remarks, we note that these conditions are equivalent to $\left(A^D\right)^{T^D} \mathcal{G}_{(k,l)^*} = A^D \mathcal{G}_{(k,l)^*}$ and $\left(A^D\right)^{T^D} \mathcal{G}_{(k,l)} \neq A^D \mathcal{G}_{(k,l)}$, which is the same as saying that (k,l) is an inert point of A^D. ∎

Lemma 10. *Let $\mathcal{X}_r$ be the set of canonical matrices $A \in \mathcal{G}_n$ with exactly r non-zero entries off the main diagonal. Then $\mathcal{D}$ acts on $\mathcal{X}_r$ by conjugation and the size of the orbit of A is*

$$|[A]| = \frac{(q-1)^n}{|C_{\mathcal{D}}(A)|}.$$

Proof: If (k,l) is inert for A^D with $D \in \mathcal{D}$ then (k,l) is inert for A and, since A is canonical, $a_{kl} = 0$ and $t_k^{-1} t_l a_{kl} = 0$. That is, the inert points of A^D are zero and so A^D is canonical. Besides, A and A^D have the same non-zero entries so $\mathcal{D}$ acts on $\mathcal{X}_r$ by conjugation. We have $[A] = \{A^D \mid D \in \mathcal{D}\}$ and $A^D = A^{D'}$ if and only if $DD'^{-1} \in C_{\mathcal{D}}(A)$. So $|[A]| = \frac{(q-1)^n}{|C_{\mathcal{D}}(A)|}$. ∎

Lemma 11. *Let $A = (a_{ij}) \in \mathcal{G}_n$ and suppose that $a_{is} = 0 = a_{sj}$, $1 \leqslant i \leqslant s-1$, $s+1 \leqslant j \leqslant n$. Then the following conditions are equivalent:*

(1) *A is a canonical matrix of $\mathcal{G}_n$.*

(2) *$B = (b_{uv})$ is a canonical matrix of $\mathcal{G}_{n-1}$, where*

$$b_{uv} = \begin{cases} a_{uv}, & \text{if } u < s,\ v < s; \\ a_{u,v+1}, & \text{if } u < s,\ v \geqslant s; \\ a_{u+1,v+1}, & \text{if } u \geqslant s. \end{cases}$$

Proof: Both A and B are canonical if and only if the value at any inert point is zero, that is if any non-zero value corresponds to a ramification point. The linear forms of A can be split up into families each involving disjoint sets of unknowns. If $i \neq s \neq j$, then no unknown $x_{\sigma\tau}$ appears in the form $L_{ij,A} = \sum_{k=i+1}^{j-1} (a_{ik}x_{kj} - a_{kj}x_{ik})$ when (σ,τ) is in the set

$$\mathcal{K} = \{(1,s), \ldots, (s-1,s), (s,s+1), \ldots, (s,n)\}$$

since, for the indices in this set, the corresponding sumand is $a_{is}x_{sj}$ with $a_{is} = 0$, or $-a_{sj}x_{is}$ with $a_{sj} = 0$. On the other hand, the forms $L_{sj,A} = \sum_{k=s+1}^{j-1} -a_{kj}x_{sk}$, $j = s+1, \ldots, n$, and $L_{is,A} = \sum_{k=i+1}^{s-1} a_{ik}x_{ks}$, $i = 1, \ldots, s-1$, only involve unknowns indexed by elements of $\mathcal{K}$. So, in order to determine the nature of a point $(i,j) \in \mathcal{J} - \mathcal{K}$ of A, only the forms $L_{kl,A}$, $(k,l) \prec (i,j)$, $(k,l) \in \mathcal{J} - \mathcal{K}$ are relevant. Now, the map

$$(u,v) \mapsto (u,v)^+ = \begin{cases} (u,v), & \text{if } u < s,\ v < s; \\ (u,v+1), & \text{if } u < s,\ v \geqslant s; \\ (u+1,v+1), & \text{if } u \geqslant s, \end{cases}$$

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establishes a bijection between the sets $\mathcal{J}_{n-1}$ and $\mathcal{J} - \mathcal{K}$ so that

$$L_{uv,B}(x_{\alpha\beta}) = L_{(u,v)^+,A}\left(x_{(\alpha,\beta)^+}\right).$$

Thus, (u,v) is a ramification or inert point of B if and only if $(u,v)^+$ is a ramification or inert point of A. $\square$

REMARK 12. The nature of the points of $\mathcal{K}$ in the matrix A is determined according to the following criterion: by Theorem (4.3) of [**2**], (i,s) is inert if and only if $\pi(i) < s$ and (s,j) is inert if and only if there exists $k > s$ such that $\pi(k) = s$. Also, we can give the relation between the order of the centralisers: $\left|C_{\mathcal{G}_n}(A)\right| = \left|C_{\mathcal{G}_{n-1}}(B)\right| q^{n-1-c(\pi)}$, where $c(\pi) = |\{i \mid \pi(i) < s \quad \text{or} \quad i > s \text{ and } \pi(i) \leqslant n\}|$.

LEMMA 13. *Set* $\mathcal{X} = \bigcup_{r\geqslant 5} \mathcal{X}_r$. *Then* $|\mathcal{X}| \equiv 0 \left(\mod (q-1)^5 \right)$.

PROOF: Let $A \in \mathcal{X}_r$. We note that $D = \operatorname{diag}(t_1, \dots, t_n) \in C_{\mathcal{D}}(A)$ if and only if $t_i = t_j$ for any (i,j) with $a_{ij} \neq 0$. Let v be the rank of this system of linear equations. Then $|C_{\mathcal{D}}(A)| = (q-1)^{n-v}$ and, according to Lemma 10, $|[A]| = (q-1)^v$. In order to complete the proof it suffices to prove that $v \geqslant 5$ for $r \geqslant 5$. More precisely, we show that the rank of the five first equations

$$t_{i_1} = t_{j_1},\ t_{i_2} = t_{j_2},\ t_{i_3} = t_{j_3},\ t_{i_4} = t_{j_4},\ t_{i_5} = t_{j_5}$$

equals 5. First of all, it is immediate that, if A is a canonical matrix, then $I + a_{i_1,j_1}E_{i_1,j_1} + \dots + a_{i_5,j_5}E_{i_5,j_5}$ is also canonical. Hence we can suppose $r = 5$ without loss of generality.

Let us consider the non-oriented graph Γ defined over the set $\{i_1, j_1, \dots, i_5, j_5\}$ by the arcs $(i_1, j_1), \dots, (i_5, j_5)$. For each arc there is a relation $t_{i_k} = t_{j_k}$ among the elements of the diagonal matrix D. We shall prove that these equations are linearly independent (that is, that $v = 5$) by showing that the graph Γ has no cycles. Suppose, on the contrary, there exists a cycle involving the indices $k_1 < \dots < k_u$ and such that $a_{k_1,k_2} \cdots a_{k_{u-1},k_u} a_{k_1,k_u} \neq 0$. Since a_{k_1,k_u} is above a_{k_{u-1},k_u} and in the same column, (k_{u-1}, k_u) is not a pivot. Moreover

$$(k_{u-1}, k_u) \prec (k_{u-2}, k_{u-1}) \prec \dots \prec (k_1, k_2) \prec (k_1, k_u)$$

and, among the non-zero entries of A, there are at least $u-1$ $(u \geqslant 3)$ after (k_{u-1}, k_u). Let $A_4 = I + a_{i_1,j_1}E_{i_1,j_1} + \dots + a_{i_4,j_4}E_{i_4,j_4}$. Then A_4 is also canonical and belongs to $\mathcal{X}_4$. So one of the cases described in Lemma 6 must hold. Case (1a) is ruled out because all non-zero elements of A_4 would be pivots, which is impossible. If (1b) holds, the only non-zero element which is not a pivot is a_{i_4,j_4}, whence $(k_{u-1}, k_u) = (i_4, j_4)$. But among the non-zero entries of A only (i_5, j_5) appears after (i_4, j_4), which contradicts the relation $(i_4, j_4) = (k_{u-1}, k_u) \prec (k_1, k_2) \prec (k_1, k_u)$. Finally, if case (2) holds, (i_3, j_3) is the only non-zero element which is not a pivot and we must have

$$(i_3, j_3) = (k_{u-1}, k_u) \prec (i_4, j_4) = (k_1, k_2) \prec (i_5, j_5) = (k_1, k_u)$$

so that $\{i_1, j_1, \ldots, i_5, j_5\} = \{i_1, i_2, i_4, j_1, j_2\}$. Now, applying successively Lemma 11, the size of the matrix can be reduced to 5 and the set of indices $\{i_1, j_1, i_2, j_2, i_3, j_3, i_4, j_4, i_5, j_5\}$ transforms into $\{1, \ldots, 5\}$. An inspection of the canonical matrices in $\mathcal{G}_5$, proves that no such matrix has 5 non-zero entries off the main diagonal and this case must also be ruled out. ∎

THEOREM 14. *There exists a non-negative integer k such that*

$$r(\mathcal{G}_n) = 1 + \sum_{i=1}^{4} \mu_i \cdot (q-1)^i + k \cdot (q-1)^5,$$

where

$$\begin{aligned}
\mu_1 &= n(n-1)/2, \\
\mu_2 &= n(n-1)(n-2)(3n-5)/24, \\
\mu_3 &= n(n-1)(n-2)(n-3)\big(n^2 - 5n + 8\big)/48, \\
\mu_4 &= n(n-1)(n-2)(n-3)(n-4)\big(3n^3 - 30n^2 + 121n - 182\big)/1152.
\end{aligned}$$

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