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Traceless Maps as the Singular Minimizers in the Multi-dimensional Calculus of Variations

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider the energy functional

$$\mathcal{F}[\mathbf{u}, \Omega] := \int_\Omega \mathrm{F}(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x},$$

over the space of $W^{1,2}(\Omega, \mathbb{R}^m)$ where the integrand $\mathrm{F}: \mathbb{M}_{m\times n} \to \mathbb{R}$ is a smooth uniformly convex function with bounded second derivatives. In this paper we address the question of regularity for solutions of the corresponding system of Euler–Lagrange equations. In particular, we introduce a class of singular maps referred to as *traceless* and examine them as a new counterexample to the regularity of minimizers of the energy functional $\mathcal{F}[\,\cdot\,, \Omega]$ using a method based on null Lagrangians.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. In this short note we consider variational integrals of the form

$$\mathcal{F}[\mathbf{u}, \Omega] := \int_\Omega \mathrm{F}(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.1}$$

where $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^m)$ and $\mathrm{F}: \mathbb{M}_{m\times n} \to \mathbb{R}$ is a smooth uniformly convex function with uniformly bounded second derivatives. In this paper we will consider the problem of regularity of minimizers of $\mathcal{F}$ belonging to $W^{1,2}(\Omega, \mathbb{R}^m)$. We recall that, a differentiable function f is said to be *strongly* (or *uniformly*) *convex* if there exists a constant $c > 0$ such that for all ξ and ζ in $\mathbb{M}_{m\times n}$, $\langle \nabla f(\xi) - \nabla f(\zeta), \xi - \zeta \rangle \geq c|\xi - \zeta|^2$. In the case where F is uniformly convex with uniformly bounded second derivatives, it is not difficult to see that $\mathbf{u}$ is a minimizer of $\mathcal{F}$ if and only if $\mathbf{u}$ is a weak solution of the Euler–Lagrange equation of $\mathcal{F}[\,\cdot\,, \Omega]$; $\mathbf{u}$ is a weak solution of

$$\operatorname{div}(\nabla \mathrm{F}(\xi))\big|_{\xi = \nabla \mathbf{u}(\mathbf{x})} = \mathbf{0}. \tag{1.2}$$

A partial regularity result due to Morrey [10] shows that every weak solution of (1.2) is smooth whenever $n = 2$ and $m \geq 1$. Moreover, the same result is established in the scaler case, $n \geq 2$ and $m = 1$, by fundamental work of De Giorgi [3] and Nash [11]. In this case they proceed to address the question of regularity by differentiating the Euler–Lagrange equation and consider instead a linear equation with bounded

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measurable coefficients. In contrast, however, De Giorgi gives an example in [4] and shows that these techniques fails in the vectorial case and can not be extended to the borderline case $m \geq 2$.

At the other extreme, the question of partial regularity of global minimizers of the energy functional $\mathcal{F}[\,\cdot\,,\Omega]$ when the integrand F being strong quasiconvex was settled by Evans [5]. He shows that global minimizers are smooth on the complement of a closed subset with null Lebesgue measure. This result was extended to the case of strong local minimizers by Kristensen and Taheri [9]. A question that arise naturally at this stage is the existence of such minimizer with non-empty singular set. Indeed the primary aim of this paper is to give a counterexample to the regularity of minimizers of the energy functional (1.1) where the integrand F is uniformly convex with bounded second derivatives.

Nečas [12] constructed the first example of a singular minimizer for a smooth strongly convex functional of type (1.1). He proceeded with a homogeneous degree-one map $\mathbf{u}\colon \mathbb{R}^n \to \mathbb{R}^{n^2}$ defined by $\mathbf{u}(\mathbf{x}) = \frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|}$ and constructed explicitly, for large n, a smooth uniformly convex function F with bounded second derivatives defined on $\mathbb{M}_{n\times n^2}$, for which $\mathbf{u}$ minimizes the corresponding energy functional $\mathcal{F}[\,\cdot\,,\Omega]$. Following this example, Nečas, Hao, and Leonardi [8] improved and modified this construction and make it work for dimensions $n \geq 5$ by using the new map in the following way:

$$\mathbf{u}(\mathbf{x}) = \frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|} - \frac{|\mathbf{x}|}{n}\mathbf{I}_n. \tag{1.3}$$

Šverák and Yan [16, 17] present some of the recent developments in this direction by focusing on the map (1.3). They have shown that the map $\mathbf{u}$ as given above is a counterexample for non-smooth minimizer of a smooth uniformly convex functional of type (1.1), for $n \geq 3$, $m = (n(n+1))/2 - 1$. The argument of their construction was trickier and were based on the symmetrisation of this map and the use of a quadratic null Lagrangian.

We slightly modify the map $\mathbf{u}$ defined by (1.3) and consider a new class of homogeneous degree-two maps $\mathbf{u}\colon \mathbb{R}^n \to \mathbb{R}^{n^3}$ defined by

$$u_{ijk}(\mathbf{x}) := \frac{x_i x_j x_k}{|\mathbf{x}|} - \frac{|\mathbf{x}|}{(n+2)}[\delta_{jk}x_i + \delta_{ik}x_j + \delta_{ij}x_k],$$

where we have used the notation $\mathbf{u} = [u_{ijk}]$. We referred these maps as *traceless* [1] and proceed with them to construct a smooth strongly convex function F with bounded second derivatives in the case $n \geq 3$ and $m = \frac{1}{6}n(n-1)(n+4)$ such that $\mathbf{u}$ will satisfy in the Euler–Lagrange equations associated with $\mathcal{F}[\,\cdot\,,\Omega]$. This, therefore, leads us to the existence of a new class of singular minimizers, which is the main result of this paper (see Theorem 3.2).

Our approach here is based upon suitably modifying a well-known technique from [16]. Indeed we use the symmetrise of the map $\mathbf{u}$ to find a quadratic null Lagrangian $\mathscr{L}$ such that $\nabla\mathscr{L} = \nabla$F holds on the set of gradients of $\mathbf{u}$, namely

[1] It is easy to see that for all $\mathbf{x}$, $\mathbf{u}(\mathbf{x})$, as a tensor product in $(\mathbb{R}^n)^{\otimes 3}$, lies in the traceless part of this space.

$\mathbb{H} := \nabla\mathbf{u}(\Omega)$, for a smooth uniformly convex function F. Then the Euler–Lagrange equation $\operatorname{div}(\nabla F(\nabla\mathbf{u}) = \operatorname{div}(\nabla\mathscr{L}(\nabla\mathbf{u}) = \mathbf{0}$ holds automatically which leads us to the existence of singular minimizer for smooth strongly convex functional.

2 Traceless Maps and Null Lagrangians

We will encounter null Lagrangians in various places in this paper. For the sake of readers not familiar with the notion here we recall the definitions and give a quick overview of some of the main properties and features. For a more detailed coverage we refer the reader to some of the classical monographs on the subject, *e.g.*, [1,13], as well as the more recent treatise by Dacorogna [2].

Definition 2.1 (Null Lagrangians) A continuous function $\mathscr{L}: \mathbb{M}_{m\times n} \to \mathbb{R}$ is referred to as a *null Lagrangian* if the identity

$$\int_\Omega \mathscr{L}(\nabla\mathbf{u} + \nabla\boldsymbol{\varphi})\, d\mathbf{x} = \int_\Omega \mathscr{L}(\nabla\mathbf{u})\, d\mathbf{x} \tag{2.1}$$

holds for every bounded open set $\Omega \subset \mathbb{R}^n$ and for all $\mathbf{u} \in C^1(\bar{\Omega}, \mathbb{R}^m)$, $\boldsymbol{\varphi} \in C_0^\infty(\Omega, \mathbb{R}^m)$.

Remark 2.2 Using (2.1) and subject to $\mathscr{L}$ being of class $C^1(\mathbb{M}_{m\times n})$, we can write

$$\begin{aligned} 0 = \frac{d}{d\varepsilon}\Big[\int_\Omega \mathscr{L}(\nabla\mathbf{u})\, d\mathbf{x}\Big] &= \frac{d}{d\varepsilon}\Big[\int_\Omega \mathscr{L}(\nabla\mathbf{u} + \varepsilon\nabla\boldsymbol{\varphi})\, d\mathbf{x}\Big] \\ &= \int_\Omega \mathscr{L}_{\xi^i_j}(\nabla\mathbf{u} + \varepsilon\nabla\boldsymbol{\varphi})\boldsymbol{\varphi}^i_{,j}\, d\mathbf{x} \\ &= \int_\Omega \langle \mathscr{L}_\xi(\nabla\mathbf{u} + \varepsilon\nabla\boldsymbol{\varphi}), \nabla\boldsymbol{\varphi}\rangle\, d\mathbf{x}. \end{aligned}$$

It follows that $\mathscr{L}$ is null Lagrangian if and only if $\operatorname{div}\nabla\mathscr{L}(\nabla\mathbf{u}(\mathbf{x})) = 0$ for all $\mathbf{u} \in C^1(\bar{\Omega}, \mathbb{R}^m)$.

To this end we recall a classical theorem due to Ball, Currie, and Olver [1], which gives a number of necessary and sufficient conditions for a function $\mathscr{L}: \mathbb{M}_{m\times n} \to \mathbb{R}$ to be a null Lagrangian.

Theorem 2.3 *If $\mathscr{L}: \mathbb{M}_{m\times n} \to \mathbb{R}$ is a continuous function, then the following statements are equivalent.*

(i) *$\mathscr{L}$ is a null Lagrangian.*
(ii) *$\mathscr{L}$ is linear combination of subdeterminants.*
(iii) *$\mathscr{L}$ is rank-one convex.*

In the case where $\mathscr{L}$ is quadratic, any of the above conditions are satisfied if and only if $\mathscr{L}(\xi) = 0$ for each rank-one matrix $\xi \in \mathbb{M}_{m\times n}$.

Definition 2.4 (Traceless maps) Let $\Omega \subset \mathbb{R}^n$ and $k \geq 2$ be a positive integer. A map $\mathbf{u} \in C(\Omega, (\mathbb{R}^n)^{\otimes k})$ is referred to as *traceless* if and only if $\mathbf{u}(\mathbf{x})$ lies in traceless part of the space $(\mathbb{R}^n)^{\otimes k}$ for every $\mathbf{x} \in \Omega$.[2]

[2] It should be noted that the image of a traceless map $\mathbf{u}: \Omega \to (\mathbb{R}^n)^{\otimes k}$ can be embedded in $\mathbb{R}^m$ for $m = n^k - (k(k+1))/2$.

From this point onward let $\Omega := \mathbb{B}(\mathbf{0},1)$ be the unit ball in $\mathbb{R}^n$ where $n \geq 3$ and consider the function $\mathbf{u} \in C(\Omega, (\mathbb{R}^n)^{\otimes 3})$ where $\mathbf{u} = [u_{ijk}]$ and

$$u_{ijk}(\mathbf{x}) := \frac{x_i x_j x_k}{|\mathbf{x}|} - \frac{|\mathbf{x}|}{(n+2)}[\delta_{jk}x_i + \delta_{ik}x_j + \delta_{ij}x_k].$$

It can be shown, by direct verification, that the function $\mathbf{u}$ defined by (2.4) is a traceless map and $\mathbf{u}(\mathbf{x})$ is a totally symmetric tensor in the space $(\mathbb{R}^n)^{\otimes 3}$, *i.e.,* $\mathbf{u}(\mathbf{x}) = [u_{ijk}(\mathbf{x})] = [u_{\sigma(ijk)}(\mathbf{x})]$ for any permutation $\sigma \in S_3$. Before going further we introduce several properties enjoyed by the traceless map $\mathbf{u}$ that are listed precisely in the following proposition.

Proposition 2.5 *Let $n \geq 3$ and suppose that $\mathbf{u}$ is the traceless map as defined in* (2.4). *Then the following observations hold.*

(i) *The set $\mathbf{u}(\Omega) := \{\mathbf{u}(\mathbf{x}) : \mathbf{x} \in \Omega\}$ can be embedded into $\mathbb{R}^{\frac{1}{6}n(n-1)(n+4)}$ and consequently the set $\mathbb{H} := \{\nabla\mathbf{u}(\mathbf{x}) : \mathbf{x} \in \Omega\}$ embeds into $\mathbb{R}^{\frac{1}{6}(n(n-1)(n+4))n}$.*

(ii) *Let $\Gamma_{ij}(\mathbf{x}) := u_{ijt,t}(\mathbf{x})$ where $u_{ijt,t} = \frac{\partial}{\partial x_t}u_{ijt}$; then we have* [3]

$$\Gamma = [\Gamma_{ij}] = \frac{n(n+3)}{(n+2)}\Big[\frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|} - \frac{|\mathbf{x}|}{n}\mathbf{I}_n\Big].$$

Proof These are easy and follow by direct verification. Indeed for (i) it is evident that $\mathbf{u}(\mathbf{x})$, for every $\mathbf{x} \in \Omega$, belongs to the space $\mathbb{A}$ where

$$\mathbb{A} := \left\{ \mathbf{A} = [A_{ijk}] \in (\mathbb{R}^n)^{\otimes 3} : \begin{array}{l} [A_{ijk}] = [A_{\sigma(ijk)}] \textit{ for all } \sigma \in S_3, \\ A_{iik} = 0 \textit{ for all } k = 1,2,\ldots,n. \end{array} \right\}$$

Now implication (i) is an easy consequence of the fact that $\mathbb{A} \cong \mathbb{R}^{\frac{1}{6}n(n-1)(n+4)}$.

(ii) A straight-forward differentiation of (2.4) with respect to x_t gives

$$\begin{aligned}\frac{\partial u_{ijk}}{\partial x_t}(\mathbf{x}) = u_{ijk,t}(\mathbf{x}) &= -\frac{x_i x_j x_k x_t}{|\mathbf{x}|^3} + \frac{x_i x_j \delta_{kt} + x_i x_k \delta_{jt} + x_j x_k \delta_{it}}{|\mathbf{x}|} \\ &\quad - \frac{1}{(n+2)|\mathbf{x}|}[x_i x_t \delta_{jk} + x_j x_t \delta_{ik} + x_k x_t \delta_{ij}] \\ &\quad - \frac{|\mathbf{x}|}{(n+2)}[\delta_{ij}\delta_{kt} + \delta_{ik}\delta_{jt} + \delta_{it}\delta_{jk}].\end{aligned}$$

In what follows, motivated by the latter, for all $1 \leq i, j \leq n$, we can write

$$\begin{aligned}\Gamma_{ij} &= \sum_{t=1}^{n} u_{ijt,t} \\ &= \sum_{t=1}^{n} \Big\{ -\frac{x_i x_j x_t^2}{|\mathbf{x}|^3} + \frac{x_i x_j \delta_{tt} + x_i x_t \delta_{jt} + x_j x_t \delta_{it}}{|\mathbf{x}|} \\ &\qquad - \frac{1}{(n+2)|\mathbf{x}|}[x_i x_t \delta_{jt} + x_j x_t \delta_{it} + x_t^2 \delta_{ij}] - \frac{|\mathbf{x}|}{(n+2)}[\delta_{ij}\delta_{tt} + \delta_{it}\delta_{jt} + \delta_{it}\delta_{jt}]\Big\}.\end{aligned}$$

[3] For the sake of convenience, throughout this paper, we use Einstein's summation convention.

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Hence, upon simplifying, we arrive at

$$\Gamma_{ij} = \frac{n(n+3)}{(n+2)}\Big[\frac{x_i x_j}{|\mathbf{x}|} - \frac{|\mathbf{x}|}{n}\delta_{ij}\Big],$$

which is the required result. ∎

For the sake of convenience and reasons that will become clear shortly, we introduce the tensor space

$$\mathbb{T} := \Big\{ \mathbf{T} = [T_{ijkt}] \in (\mathbb{R}^n)^{\otimes 4} : \begin{array}{l} [T_{ijkt}] = [T_{\sigma(ijk)t}] \text{ for all } \sigma \in S_3, \\ T_{iikt} = 0 \text{ for all } k, t = 1, 2, \dots, n. \end{array} \Big\} \tag{2.2}$$

Indeed, one can identify the tensor space $\mathbb{T}$ with $\mathbb{R}^{\frac{1}{6}(n(n-1)(n-4))n}$, or matrix space $\mathbb{M}_{m\times n}$ where $m = \frac{1}{6}n(n-1)(n-4)$, in an obvious way. The primary task is to find a quadratic null Lagrangian, explicitly, over the space $\mathbb{T}$. We begin by decomposing the tensor space $\mathbb{T}$ into its irreducible subspaces. First, we decompose this space by writing $\mathbb{T} = \mathbb{T}' \oplus \mathbb{T}_3$, where $\mathbb{T}'$ and $\mathbb{T}_3$ are the traceless part of $\mathbb{T}$ and its orthogonal supplement respectively. Furthermore, by a slight abuse of notation Γ_{ij} if denoting $\Gamma_{ij} = T_{ijtt}$ a straight-forward calculation shows that the projection on $\mathbb{T}_3$ is given as

$$T_{ijkt} \longmapsto \frac{1}{n(n+4)} \left\{ \begin{array}{l} (n+2)\delta_{kt}\Gamma_{ij} - 2\delta_{ij}\Gamma_{kt} \\ +(n+2)\delta_{jt}\Gamma_{ik} - 2\delta_{ik}\Gamma_{jt} \\ +(n+2)\delta_{it}\Gamma_{jk} - 2\delta_{jk}\Gamma_{it} \end{array} \right\} =: T^3_{ijkt}. \tag{2.3}$$

Next, in order to decompose the subspace $\mathbb{T}'$ into irreducible subspaces, we use symmetrisations that correspond to the following Young tableaux: [4]

$$\begin{array}{|c|}\hline i\\ \hline j\\ \hline k\\ \hline t\\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline i & j\\ \hline k & \\ \cline{1-1} t & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline i & j\\ \hline k & t\\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline i & j & k\\ \hline t & & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline i & j & k & t\\ \hline\end{array}$$

Indeed symmetrisations related to the first, second, and fifth Young tableaux give trivial subspace. As a result, deduce that $\mathbb{T}' = \mathbb{T}_1 \oplus \mathbb{T}_2$, where spaces $\mathbb{T}_1$ and $\mathbb{T}_2$ are correspond to the third and fourth Young tableaux. Toward this end a direct calculation shows that projections on $\mathbb{T}_1$ and $\mathbb{T}_2$ are given, respectively, by

$$T_{ijkt} \longmapsto \tfrac{1}{4}[T_{ijkt} + T_{ktij} + T_{jitk} + T_{tkji}] =: T^1_{ijkt},$$
$$T_{ijkt} \longmapsto \tfrac{1}{4}[3T_{ijkt} - T_{tjki} - T_{tikj} - T_{tijk}] =: T^2_{ijkt}.$$

For further reading on the subject matter of the above discussion and topics related to Young tableaux see [6,7].

Proposition 2.6 *Let the tensor space $\mathbb{T}$ and traceless map $\mathbf{u}$ be as defined in (2.2) and (2.4), respectively. Then there exists a quadratic null Lagrangian on $\mathbb{T}$ that is constant on the set $\mathbb{K} := \{\nabla\mathbf{u}(\mathbf{x}) : \mathbf{x} \in \mathbb{S}^{n-1}\}$.*

[4] Interestingly, the first and last Young tableaux give totally symmetric and totally anti-symmetric part of any tensor in $\mathbb{T}$ respectively. Moreover it is also evident that, here, the totally anti-symmetric part of any tensor in the space $\mathbb{T}$ is zero.

Proof We begin by identifying the tensor space $\mathbb{T}$ with $\mathbb{A}\otimes\mathbb{R}^n$, which is isomorphic to $\mathbb{M}_{m\times n}$ for $m = n(n-1)+1$. In what follows we consider a rank-one matrix $\mathbf{T} = [T_{ijkt}]$ in this space, *i.e.*, $T_{ijkt} = A_{ijk}\xi_t$, where $\mathbf{A} = [A_{ijk}]$ lies in $\mathbb{A}$ and $\xi \in \mathbb{R}^n$. Based on the earlier discussion, we can decompose the matrix $\mathbf{T} = [T_{ijkt}]$ as

$$\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 \quad \text{and} \quad T_{ijkt} = T^1_{ijkt} + T^2_{ijkt} + T^3_{ijkt},$$

where T^r_{ijkt} are projections of T_{ijkt} to the space $\mathbb{T}_r$. Furthermore, a straight-forward calculation shows that

$$\begin{aligned}
\left|\mathbf{T}_1\right|^2 &= \frac{1}{4}\Big\{|\mathbf{A}|^2|\xi|^2 + \frac{3(n+2)}{n+4}|\Gamma|^2\Big\},\\
\left|\mathbf{T}_2\right|^2 &= \frac{3}{4}\Big\{|\mathbf{A}|^2|\xi|^2 - \frac{(n+2)}{n}|\Gamma|^2\Big\},\\
\left|\mathbf{T}_3\right|^2 &= \frac{3(n+2)}{n(n+4)}|\Gamma|^2.
\end{aligned}$$

To this end, pick the tensor $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 \in \mathbb{T}$ and define $\mathscr{L}\colon \mathbb{T} \to \mathbb{R}$ by

$$\mathscr{L}(\mathbf{T}) = \mathscr{L}(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3) = -3|\mathbf{T}_1|^2 + |\mathbf{T}_2|^2 + (n+1)|\mathbf{T}_3|^2,$$

where $\mathbf{T}_r$ are projections of the tensor $\mathbf{T}$ on the space $\mathbb{T}_r$. Evidently the function $\mathscr{L}$ is quadratic, continuous, and vanishes over whole rank-one matrices in $\mathbb{M}_{m\times n}$. This, together with Theorem 2.1, reveals that $\mathscr{L}$ is a null Lagrangian. Thus to complete the proof it remains to show that the null Lagrangian $\mathscr{L}$ is constant on $\mathbb{K}$. Indeed with regards to the $\nabla\mathbf{u}(\mathbf{x}) = [u_{ijk,t}(\mathbf{x})]$, an element of $\mathbb{K}$, we can decompose it as

$$u_{ijk,t}(\mathbf{x}) = u^1_{ijk,t}(\mathbf{x}) + u^2_{ijk,t}(\mathbf{x}) + u^3_{ijk,t}(\mathbf{x}),$$

where $u^r_{ijk,t}$ lies in $\mathbb{T}_r$. Then using (2.3) and Proposition 2.5(ii), along with the fact that $|\mathbf{x}| = 1$, we can write

$$\begin{aligned}
u^3_{ijk,t}(\mathbf{x}) &= \frac{1}{n(n+4)}\left\{\begin{array}{l}(n+2)\delta_{kt}\Gamma_{ij}(\mathbf{x}) - 2\delta_{ij}\Gamma_{kt}(\mathbf{x})\\ +(n+2)\delta_{jt}\Gamma_{ik}(\mathbf{x}) - 2\delta_{ik}\Gamma_{jt}(\mathbf{x})\\ +(n+2)\delta_{it}\Gamma_{jk}(\mathbf{x}) - 2\delta_{jk}\Gamma_{it}(\mathbf{x})\end{array}\right\},\\
&= \frac{(n+3)}{n(n+2)(n+4)}\left\{\begin{array}{l}(n+2)\delta_{kt}(nx_ix_j - \delta_{ij}) - 2\delta_{ij}(nx_kx_t - \delta_{kt})\\ +(n+2)\delta_{jt}(nx_ix_k - \delta_{ik}) - 2\delta_{ik}(nx_jx_t - \delta_{jt})\\ +(n+2)\delta_{it}(nx_jx_k - \delta_{jk}) - 2\delta_{jk}(nx_ix_t - \delta_{it})\end{array}\right\}
\end{aligned}$$

and consequently

$$u^3_{ijk,t}(\mathbf{x}) = \frac{(n+3)}{(n+2)(n+4)}\left\{\begin{array}{l}(n+2)\delta_{kt}x_ix_j - 2\delta_{ij}x_kx_t - \delta_{ij}\delta_{kt}\\ +(n+2)\delta_{jt}x_ix_k - 2\delta_{ik}x_jx_t - \delta_{ik}\delta_{jt}\\ +(n+2)\delta_{it}x_jx_k - 2\delta_{jk}x_ix_t - \delta_{jk}\delta_{it}\end{array}\right\}.$$

In a similar fashion, one can show that $u^2_{ijk,t}(\mathbf{x}) = 0$ and

$$u^1_{ijk,t}(\mathbf{x}) = -x_ix_jx_kx_t + \frac{1}{(n+4)}\left\{\begin{array}{l}\delta_{kt}x_ix_j + \delta_{ij}x_kx_t - \frac{1}{(n+2)}\delta_{ij}\delta_{kt}\\ +\delta_{jt}x_ix_k + \delta_{ik}x_jx_t - \frac{1}{(n+2)}\delta_{ik}\delta_{jt}\\ +\delta_{it}x_jx_k + \delta_{jk}x_ix_t - \frac{1}{(n+2)}\delta_{jk}\delta_{it}\end{array}\right\}.$$

Furthermore, in view of what seen in above, by a straight-forward calculation, we have

$$|u^1_{ijk,t}|^2 = \frac{(n-1)(n+1)}{(n+2)(n+4)} \quad \text{and} \quad |u^3_{ijk,t}|^2 = \frac{3(n-1)(n+3)^2}{(n+2)(n+4)}.$$

Thus the conclusion now follows by observing that for fixed $\nabla \mathbf{u}(\mathbf{x}) \in \mathbb{K}$

$$\mathscr{L}(\nabla \mathbf{u}(\mathbf{x})) = -3|u^1_{ijk,t}|^2 + |u^2_{ijk,t}|^2 + (n+1)|u^3_{ijk,t}|^2 = 3(n^2-1), \tag{2.4}$$

which is the required result. ■

3 The Main Result

Before presenting the main result of this paper, we take a moment to make the following useful observation on the null Lagrangian $\mathscr{L}$, which was introduced earlier in Proposition 2.6.

Proposition 3.1 *Let $\mathscr{L}$ be the null Lagrangian as in the Proposition 2.6. Then there exists a constant $c_{\mathrm{n}} > 0$ such that*

$$\left\langle \nabla \mathscr{L}(\mathbf{X}) - \nabla \mathscr{L}(\mathbf{Y}), \mathbf{X} - \mathbf{Y} \right\rangle \geq c_{\mathrm{n}} |\mathbf{X} - \mathbf{Y}|^2,$$

for all $\mathbf{X}, \mathbf{Y} \in \mathbb{K}$.

Proof In order to establish this estimate, we use the following claim, from which the assertion follows immediately.

Claim For every $\mathbf{X}, \mathbf{Y} \in \mathbb{K}$, there exists a positive constant $c_{\mathrm{n}} > 0$ such that

$$\left\langle \nabla \mathscr{L}(\mathbf{X}), \mathbf{X} - \mathbf{Y} \right\rangle \geq c_{\mathrm{n}} |\mathbf{X} - \mathbf{Y}|^2. \tag{3.1}$$

For the proof of this claim, fix $\mathbf{X} = \nabla \mathbf{u}(\mathbf{x}) = [u_{ijk,t}(\mathbf{x})]$ and $\mathbf{Y} = \nabla \mathbf{u}(\mathbf{y}) = [u_{ijk,t}(\mathbf{y})]$ in $\mathbb{K}$. In view of $\mathscr{L}$ being a real quadratic form, we can write

$$\left\langle \nabla \mathscr{L}(\mathbf{X}), \mathbf{X} - \mathbf{Y} \right\rangle = 2\left\{ \mathscr{L}(\mathbf{X}) - \mathscr{B}(\mathbf{X}, \mathbf{Y}) \right\}, \tag{3.2}$$

where $\mathscr{B}(\cdot, \cdot)$ is the associated symmetric bilinear form of $\mathscr{L}$. Moreover, a straight-forward calculation reveals that

$$\begin{aligned}
\mathscr{B}(\mathbf{X}, \mathbf{Y}) &= -3\left\langle u^1_{ijk,t}(\mathbf{x}), u^1_{ijk,t}(\mathbf{y}) \right\rangle + \left\langle u^2_{ijk,t}(\mathbf{x}), u^2_{ijk,t}(\mathbf{y}) \right\rangle \\
&\quad + (n+1)\left\langle u^3_{ijk,t}(\mathbf{x}), u^3_{ijk,t}(\mathbf{y}) \right\rangle, \\
&= -3\Big[\langle \mathbf{x}, \mathbf{y} \rangle^4 - \frac{6}{(n+4)} \langle \mathbf{x}, \mathbf{y} \rangle^2 + \frac{3}{(n+2)(n+4)} \Big] \\
&\quad + (n+1)\Big[\frac{3n(n+3)^2}{(n+2)(n+4)} \langle \mathbf{x}, \mathbf{y} \rangle^2 - \frac{3(n+3)^2}{(n+2)(n+4)} \Big].
\end{aligned}$$

Thus, substituting this along with (2.4) into (3.2), upon simplification, we arrive at

$$\begin{aligned}
\langle \nabla \mathscr{L}(\mathbf{X}), \mathbf{X} - \mathbf{Y} \rangle &= 2\left\{ \mathscr{L}(\mathbf{X}) - \mathscr{B}(\mathbf{X}, \mathbf{Y}) \right\} \\
&=: \frac{6}{(n+2)} \left[1 - \langle \mathbf{x}, \mathbf{y} \rangle^2 \right] \left[(n+2) \langle \mathbf{x}, \mathbf{y} \rangle^2 + \alpha \right],
\end{aligned}$$

where for the sake of convenience we have introduced $\alpha = \alpha(n) := (n^3+3n^2+2n+1)$. Furthermore, with regards to the term $|\mathbf{X}-\mathbf{Y}|^2$, one can easily verify that

$$\begin{aligned}|\mathbf{X}-\mathbf{Y}|^2 &= |u_{ijk,t}(\mathbf{x})|^2 + |u_{ijk,t}(\mathbf{y})|^2 - 2\langle u_{ijk,t}(\mathbf{x}), u_{ijk,t}(\mathbf{y})\rangle \\ &= \frac{(n-1)(3n+7)}{(n+2)} + \frac{(n-1)(3n+7)}{(n+2)} \\ &\quad - 2\Big[\langle \mathbf{x},\mathbf{y}\rangle^4 + \frac{3(n^2+2n-1)}{(n+2)}\langle \mathbf{x},\mathbf{y}\rangle^2 - 3\Big] \\ &=: \frac{2}{(n+2)}\big[1-\langle \mathbf{x},\mathbf{y}\rangle^2\big]\big[(n+2)\langle \mathbf{x},\mathbf{y}\rangle^2 + \beta\big],\end{aligned}$$

where for the sake of brevity we have set $\beta = \beta(n) := (3n^2+7n-1)$. Therefore, by putting together all the above fragments, we conclude that [5]

$$\begin{aligned}\langle \nabla\mathscr{L}(\mathbf{X}), \mathbf{X}-\mathbf{Y}\rangle \ge c_{\mathrm{n}}|\mathbf{X}-\mathbf{Y}|^2 &\iff 3\big\{(n+2)\langle \mathbf{x},\mathbf{y}\rangle^2+\alpha\big\} \ge c_{\mathrm{n}}\big[(n+2)\langle \mathbf{x},\mathbf{y}\rangle^2+\beta\big] \\ &\iff 3\Big[\frac{(n+2)\langle \mathbf{x},\mathbf{y}\rangle^2+\alpha}{(n+2)\langle \mathbf{x},\mathbf{y}\rangle^2+\beta}\Big] \ge c_{\mathrm{n}} \\ &\iff 3\inf_{t\in[0,1]}\Big[\frac{(n+2)t+\alpha}{(n+2)t+\beta}\Big] \ge c_{\mathrm{n}} \\ &\iff 3\alpha\beta^{-1} \ge c_{\mathrm{n}}.\end{aligned}$$

Therefore, the inequality (3.1) holds with the choice of $c_{\mathrm{n}} \in]0, 3\alpha\beta^{-1}]$, which is the required conclusion. ■

Proposition 3.1 together with the fact that the null Lagrangian $\mathscr{L}$ is constant over $\mathbb{K}$, give a sufficient condition for the existence of a strongly convex function F, satisfying $\nabla \mathrm{F} = \nabla\mathscr{L}$. The next theorem is in charge of the construction of such a function, explicitly.

Theorem 3.2 *There exists uniformly convex energy functional of type* (1.1) *with nonsmooth minimizer.*

Proof The proof will be justified by showing that there exists smooth uniformly convex integrand F, with bounded second derivatives, such that traceless map $\mathbf{u}$ defined by (2.4) is the minimizer of energy functional (1.1). The idea behind the proof can be briefly described as follows. First we taking the convex hull of $\mathbb{K}$ and its corresponding Minkowski function. Second using homogeneity of $\mathscr{L}$ to extend it on $\mathbb{H}$. Fix $\delta > 0$ and let

$$\mathbb{G}_\delta := \mathrm{Conv}\Big\{\bigcup_{\mathbf{X}\in\mathbb{K}} \mathbb{B}\big(\mathbf{X}-\delta\nabla\mathscr{L}(\mathbf{X}), \delta|\nabla\mathscr{L}(\mathbf{X})|\big)\Big\} \subseteq \mathbb{T}_1 \oplus \mathbb{T}_3,$$

and consider the Minkowski function on $\mathbb{G}_\delta$ in the following form:

$$\begin{aligned}&\mathrm{M}\colon \mathbb{T}_1\oplus\mathbb{T}_3 \to \mathbb{R} \\ &\mathrm{M}(\mathbf{X}) := \inf\{t : t\ge 0, \mathbf{X}\in t\mathbb{G}_\delta\}.\end{aligned}$$

[5] In passing we note that $0 \le \langle \mathbf{x},\mathbf{y}\rangle^2 \le 1$ since $\mathbf{x},\mathbf{y}\in\mathbb{S}^{n-1}$.

Evidently by considering for $\delta > 0$ sufficiently small, the function M^2 is smooth and strongly convex in small neighbourhood of $\mathbb{K}$ (*cf. e.g.,* Phelps [14] or Rockafellar [15]). Moreover, by direct verification for every $\mathbf{X} \in \mathbb{K}$, we have $\nabla \mathscr{L}(\mathbf{X}) = \nabla \mathrm{M}^2(\mathbf{X})$, which is a consequence of homogeneity of Minkowski function and $\mathscr{L}$ as well as the fact that $\mathscr{L}$ is constant on $\mathbb{K}$. In doing so we need a modification of the function M^2, defined as follows:

$$\mathrm{M}_{\varepsilon,\lambda}(\mathbf{X}) := (\Phi_\varepsilon \star \mathrm{M}^2)(\mathbf{X}) + \lambda|\mathbf{X}|^2,$$

where $\lambda > 0$, Φ is a standard mollifier and $\Phi_\varepsilon(\mathbf{X}) = \varepsilon^{-n}\Phi(\mathbf{X}/\varepsilon)$. To this end, let Ψ be a cut-off function [*i.e.,* $\Psi \in \mathrm{C}_c^\infty(\Omega)$] satisfying the following additional properties:

$$\begin{cases} 0 \le \Psi \le 1 & \text{in } \mathbb{H}, \\ \Psi = 1 & \text{in } \mathbb{U}, \\ \Psi = 0 & \text{in } \mathbb{H}\backslash\mathbb{U}, \end{cases}$$

where $\mathbb{U}$ is a small open neighbourhood of $\mathbb{K}$. We now consider the function $\mathrm{F}_{\varepsilon,\lambda}\colon \mathbb{T}_1 \oplus \mathbb{T}_3 \to \mathbb{R}$ by

$$\mathrm{F}_{\varepsilon,\lambda}(\mathbf{X}) := [1 - \Psi(\mathbf{X})]\mathrm{M}_{\varepsilon,\lambda}(\mathbf{X}) + \Psi(\mathbf{X})\mathrm{M}^2(\mathbf{X}).$$

In the first place, an easy calculation shows that above function is two-homogeneous, smooth and

$$\nabla^2 \mathrm{F}_{\varepsilon,\lambda} = [1 - \Psi]\nabla^2 \mathrm{M}_{\varepsilon,\lambda} + \Psi\nabla^2\mathrm{M}^2 + 2\nabla\Psi \otimes \nabla[\mathrm{M}^2 - \mathrm{M}_{\varepsilon,\lambda}] + [\mathrm{M}^2 - \mathrm{M}_{\varepsilon,\lambda}]\nabla^2\Psi.$$

The above identity reveals that more is true, namely, that $\mathrm{F}_{\varepsilon,\lambda}$ is strongly convex on $\mathbb{T}_1 \oplus \mathbb{T}_3$ provided that ε and λ are small enough. We are now in a position to define the strongly convex integrand, which is

$$\mathrm{F}\colon \mathbb{T}_1 \oplus \mathbb{T}_2 \oplus \mathbb{T}_3 \longrightarrow \mathbb{R}, \quad \mathrm{F}(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3) := \mathrm{F}_{\varepsilon,\lambda}(\mathbf{T}_1 + \mathbf{T}_3) + |\mathbf{T}_2|^2.$$

Clearly, F is smooth, strongly convex function with bounded second derivatives everywhere and as a result of $\nabla \mathrm{F}_{\varepsilon,\lambda}(\mathbf{X}) = \nabla\mathscr{L}(\mathbf{X})$ on $\mathbb{K}$ together with the fact that $\mathrm{F}_{\varepsilon,\lambda}$ is two-homogeneous, we conclude that $\nabla \mathrm{F}(\mathbf{X}) = \nabla\mathscr{L}(\mathbf{X})$ for all $\mathbf{X} \in \mathbb{H}$. This completes the proof. ∎

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