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Nonexistence of non-Hopf Ricci-semisymmetric real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$

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Abstract. In this paper, we solved an open problem raised by Cecil and Ryan (2015, *Geometry of Hypersurfaces*, Springer Monographs in Mathematics, Springer, New York, p. 531) by proving the nonexistence of non-Hopf Ricci-semisymmetric real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$.

1 Introduction

Let $\bar{M}^n(c)$, for an integer $n \geq 2$ and a real number $c \neq 0$, be a nonflat complex space form of complex dimension n with constant holomorphic sectional curvature c. A complete and simply connected complex space form is complex analytically isometric to the complex projective space $\mathbb{C}P^n$ if $c > 0$, or the complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$.

There are a great many of studies on real hypersurfaces of $\bar{M}^n(c)$ (see, e.g., [3, 16] and the references therein). In particular, it is known that there do not exist real hypersurfaces with parallel Ricci tensor in nonflat complex space forms $\bar{M}^n(c)$ for $n \geq 2$ (see [8, 10, 19]). Moreover, it was further shown in [9, 12, 17] (cf. also Theorem 6.29 in [16]) that there do not exist Ricci-semisymmetric real hypersurfaces in $\bar{M}^n(c)$ for $n \geq 3$, and there do not exist Hopf Ricci-semisymmetric hypersurfaces in $\bar{M}^2(c)$. Based on these results, in their wonderful book [3], Cecil and Ryan stated in Remark 8.70 of [3] that "*The existence of non-Hopf Ricci-semisymmetric hypersurfaces in* $\mathbb{C}P^2$ and $\mathbb{C}H^2$ *is an open question*," and then they further raised the following interesting problem.

Problem 1.1 (cf. page 531 of [3]) *Do there exist non-Hopf Ricci-semisymmetric hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$?*

In this paper, we shall solve the above problem by proving the following theorem.

Theorem 1.1 *There do not exist non-Hopf Ricci-semisymmetric hypersurfaces in both $\mathbb{C}P^2$ and $\mathbb{C}H^2$.*

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To introduce the related notions, we recall that every connected orientable real hypersurface M of an almost Hermitian manifold $(\bar{M}, J)$ is associated with a *Reeb vector field*, called also the *structure vector field*, defined by $\xi := -JN$, where N is the unit normal vector field of $M \hookrightarrow \bar{M}$. M is called a *Hopf hypersurface* if the integral curves of ξ are geodesics. In particular, by Berndt, Bolton, and Woodward [2], a real hypersurface of any nearly Kähler manifold is Hopf if and only if its Reeb vector field is a principal vector field. As is well known, Takagi [20] investigated homogeneous real hypersurfaces in $\mathbb{C}P^n$ and gave a well-known list, which is often referred to as Takagi's list: (A_1), (A_2), (B), (C), (D), (E). Then, Montiel [15] provided a similar list of the real hypersurfaces in $\mathbb{C}H^n$, which is often referred to as Montiel's list: (A_0), (A_1), (A_2), (B). Kimura [11] proved that a Hopf hypersurface of $\mathbb{C}P^n$ with constant principal curvatures is locally congruent to one of the Takagi's list, whereas Berndt [1] proved that a Hopf hypersurface of $\mathbb{C}H^n$ with constant principal curvatures is locally congruent to one of the Montiel's list. These hypersurfaces can be seen as the tubes over a submanifold of the ambient spaces.

To study the Ricci-semisymmetric real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, we recall that a Riemannian manifold M is called Ricci-semisymmetric (or Ricci-semiparallel) if

$$g(R(X,Y)S(Z) - S(R(X,Y)Z), W) = 0, \quad \forall\, X, Y, Z, W \in TM, \tag{1.1}$$

or simply write $R \cdot S = 0$. Here, R and S are the curvature tensor and the Ricci operator of M, respectively.

The study of Ricci-semisymmetric real hypersurfaces in $\bar{M}^n(c)$ has a long history. First, Kimura and Maeda [12] proved that the Hopf hypersurfaces of $\mathbb{C}P^n, n \geq 2$, cannot be Ricci-semisymmetric. Ki, Nakagawa, and Suh [9] classified cyclic Ricci-semisymmetric (or cyclic Ryan) hypersurfaces of the nonflat complex space forms $\bar{M}^n(c), n \geq 3$. Moreover, we know from the results of [9] that all the cyclic Ricci-semisymmetric hypersurfaces are not Ricci-semisymmetric. Hence, there do not exist Ricci-semisymmetric real hypersurfaces in $\bar{M}^n(c)$ for $n \geq 3$. Niebergall and Ryan [17] proved that the Hopf hypersurfaces in $\bar{M}^2(c)$ cannot be semisymmetric, i.e., $R \cdot R = 0$. As a Riemannian 3-manifold is semisymmetric if and only if $R \cdot S = 0$, we see that Hopf hypersurfaces of $\bar{M}^2(c)$ cannot be Ricci-semisymmetric. In summary, we have the following.

Theorem 1.2 (Theorem 6.29 of [16]) *In a complex space form of constant holomorphic sectional curvature $4c$, $c \neq 0$, there exists no real hypersurface $M^{2n-1}, n \geq 3$, satisfying $R \cdot S = 0$. For $n = 2$, there are no Hopf hypersurfaces satisfying $R \cdot S = 0$.*

Combining Theorems 1.1 and 1.2, we have the following.

Corollary 1.1 *There do not exist Ricci-semisymmetric hypersurfaces in nonflat complex space forms $\bar{M}^n(c), n \geq 2$. In particular, there do not exist non-Hopf semisymmetric hypersurfaces in both $\mathbb{C}P^2$ and $\mathbb{C}H^2$.*

Remark 1.1 In contrast to Corollary 1.1, we know by Theorem 6.30 of [16] and Theorem 8 of [7] that there exist real hypersurfaces in the nonflat complex space forms

$\bar{M}^n(c)$ $(n \geq 2)$ satisfying

$$g(R(X,Y)S(Z) - S(R(X,Y)Z), W) = 0, \quad \forall\, X, Y, Z, W \in \{\xi\}^{\perp}. \tag{1.2}$$

A real hypersurface of $\bar{M}^n(c)$ satisfying (1.2) is said to be pseudo-Ryan, whereas Hopf pseudo-Ryan hypersurfaces in $\bar{M}^n(c)$ coincide with the pseudo-Einstein ones for all $n \geq 2$. See Remark 8.70 of [3] for details.

Remark 1.2 As related results, we would mention some nice researches about real hypersurfaces of an almost Hermitian manifold with recurrent Ricci tensor (see [4–6, 13, 14]). Recall that a Riemannian manifold M is called having recurrent Ricci tensor S if there exists a one-form ω on M such that $(\nabla_X S)Y = \omega(X)S(Y)$ for all $X, Y \in TM$. Loo [13] and Hamada [5] proved independently that there are no real hypersurfaces with recurrent Ricci tensor in $\bar{M}^n(c)$ for $n \geq 3$, and then in Theorem 20 of [14], Loo further proved that the Ricci operator S being recurrent implies that S is semiparallel. Thus, combining with Theorem 1.1, we have shown that there do not exist non-Hopf hypersurfaces in $\bar{M}^2(c)$ with recurrent Ricci tensor. Finally, we noticed that very recently Wang [21] studied Ricci η-recurrent real hypersurfaces in $\bar{M}^2(c)$.

This paper is organized as follows. In Section 2, we review the necessary materials about real hypersurfaces of the nonflat complex space forms. In Section 3, we establish three basic lemmas about the non-Hopf Ricci-semisymmetric hypersurfaces in the nonflat complex planes $\bar{M}^2(c)$. In Section 4, we complete the proof of Theorem 1.1.

2 Preliminaries

2.1 Geometry of hypersurfaces in nonflat complex space forms

Let $\bar{M}^n(c)$ be the nonflat complex space form with the constant holomorphic sectional curvature c, the complex structure J, and the Kähler metric $\bar{g}$, respectively. Let M be a connected real hypersurface of $\bar{M}^n(c)$ with unit normal vector field N. Denote by $\bar{\nabla}$ the Levi–Civita connection of the metric $\bar{g}$, and g the induced metric on M. Put

$$JN = -\xi, \quad JX = \phi X + \eta(X)N, \quad \forall\, X \in TM, \tag{2.1}$$

where ϕX and $\eta(X)N$ are the tangential and normal parts of JX, respectively. ξ is called the Reeb vector field or the structure vector field, ϕ is a tensor field of type $(1,1)$, and η is a 1-form on M. By definition, for any $X, Y \in TM$, the following relations hold:

$$\begin{cases} \eta(X) = g(X,\xi), \ \phi^2(X) = -X + \eta(X)\xi, \ \eta(\phi X) = 0, \\ g(\phi X, Y) = -g(X, \phi Y), \ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \end{cases} \tag{2.2}$$

Let ∇ be the induced Levi–Civita connection on M, and let R be its Riemannian curvature tensor. The formulas of Gauss and Weingarten are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \bar{\nabla}_X N = -AX, \ \forall X, Y \in TM, \tag{2.3}$$

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where h is the second fundamental form and A is the shape operator of M, which are related by $h(X,Y) = g(AX,Y)N$. From (2.1) and $\bar{\nabla}J = 0$, we obtain

$$\nabla_X\xi = \phi AX,\ (\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi. \tag{2.4}$$

The Gauss and Codazzi equations of M are given, for any $X,Y,Z \in TM$, by

$$\begin{aligned} R(X,Y)Z =& \tfrac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ & - 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.5}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \tfrac{c}{4}\{g(X,\xi)\phi Y - g(Y,\xi)\phi X - 2g(\phi X,Y)\xi\}. \tag{2.6}$$

By the Gauss equation (2.5), we have

$$SX = \tfrac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + mAX - A^2X,\quad \forall\, X \in TM, \tag{2.7}$$

where $m = \mathrm{trace}A$ is the mean curvature, and S is the Ricci operator.

2.2 The standard non-Hopf frame and their connections

From this subsection on, we shall restrict to the nonflat complex planes $\bar{M}^2(c)$. For a non-Hopf hypersurface M of $\bar{M}^2(c)$, there exists nonempty open subset

$$\Omega = \{p \in M \mid \xi \text{ is not a principal vector field at } p\} \subset M.$$

As our study is of local in nature, we shall assume $\Omega = M$ in the sequel. Then, as $\phi\nabla_\xi\xi = \phi^2 A\xi = -A\xi + \eta(A\xi)\xi$, we get

$$A\xi = \eta(A\xi)\xi - \phi\nabla_\xi\xi =: \alpha\xi + \beta U, \tag{2.8}$$

where $\alpha = \eta(A\xi)$, $\beta = |\phi\nabla_\xi\xi| \neq 0$, and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi$. It is clear that $\{\xi, U, \phi U\}$ is an orthonormal frame field of M which, following that in [3], is called the "*standard non-Hopf frame*" of M. Then, there are smooth functions γ, δ, and μ on M such that

$$AU = \beta\xi + \gamma U + \delta\phi U,\ A\phi U = \delta U + \mu\phi U. \tag{2.9}$$

By using (2.4), (2.8), and (2.9), direct calculations give the following.

Lemma 2.1 (cf. [18]) *With respect to the standard non-Hopf frame, there are smooth functions κ_i $(1 \le i \le 3)$ such that the following relations hold:*

$$\begin{aligned} &\nabla_U\xi = -\delta U + \gamma\phi U,\ \nabla_{\phi U}\xi = -\mu U + \delta\phi U,\ \nabla_\xi\xi = \beta\phi U, \\ &\nabla_U U = \kappa_1\phi U + \delta\xi,\ \nabla_{\phi U}U = \kappa_2\phi U + \mu\xi,\ \nabla_\xi U = \kappa_3\phi U, \\ &\nabla_U\phi U = -\kappa_1 U - \gamma\xi,\ \nabla_{\phi U}\phi U = -\kappa_2 U - \delta\xi,\ \nabla_\xi\phi U = -\kappa_3 U - \beta\xi. \end{aligned} \tag{2.10}$$

Moreover, taking $X = \xi, U, \phi U$ in (2.7), respectively, together with the use of (2.8) and (2.9), we obtain

$$\begin{aligned} S\xi &= (\tfrac{c}{2} + \alpha\gamma + \alpha\mu - \beta^2)\xi + \beta\mu U - \beta\delta\phi U, \\ SU &= \beta\mu\xi + (\tfrac{5}{4}c + \alpha\gamma + \gamma\mu - \beta^2 - \delta^2)U + \alpha\delta\phi U, \\ S\phi U &= -\beta\delta\xi + \alpha\delta U + (\tfrac{5}{4}c + \alpha\mu + \gamma\mu - \delta^2)\phi U. \end{aligned} \tag{2.11}$$

3 Basic lemmas on non-Hopf Ricci-semisymmetric real hypersurfaces

In this section, we establish three basic lemmas on non-Hopf Ricci-semisymmetric real hypersurfaces in $\bar{M}^2(c)$. To begin with, we have the following lemma.

Lemma 3.1 *Let M be a non-Hopf Ricci-semisymmetric hypersurface in the nonflat complex planes $\bar{M}^2(c)$, and $\{\xi, U, \phi U\}$ is the standard non-Hopf frame such that (2.8)–(2.10) hold. Then we have the following equations:*

$$\delta = 0, \tag{3.1}$$

$$\alpha\gamma - \beta^2 = -\tfrac{c}{4}, \tag{3.2}$$

$$c + 4\alpha\mu + \gamma\mu - \mu^2 = 0. \tag{3.3}$$

Proof First of all, taking $(X, Y, Z, W) = (\xi, U, \phi U, U)$ in (1.1), then by direct calculations with the use of (2.11) and Gauss equation (2.5), we can obtain

$$\tfrac{c}{4}\beta\delta = 0.$$

As $\beta \neq 0$ and $c \neq 0$, we have $\delta = 0$ as claimed.

Next, taking in (1.1),

$$(X, Y, Z, W) = (\xi, \phi U, U, \phi U),\ (\xi, \phi U, \xi, \phi U),\ (U, \phi U, U, \phi U),$$

respectively, with the use of (3.1), (2.11), and (2.5), we obtain

$$\mu(\alpha\gamma - \beta^2 + \tfrac{c}{4}) = 0, \tag{3.4}$$

$$(\tfrac{3c}{4} + \gamma\mu - \alpha\gamma + \beta^2)(\alpha\mu + \tfrac{c}{4}) - \beta^2\mu^2 = 0, \tag{3.5}$$

$$(c + \gamma\mu)(\alpha\mu - \alpha\gamma + \beta^2) - \beta^2\mu^2 = 0. \tag{3.6}$$

Now, we claim that $\mu \neq 0$. Indeed, if $\mu = 0$, from (3.6), we shall get $\alpha\gamma - \beta^2 = 0$. Then, from (3.5), we obtain $c = 0$. This is a contradiction to the assumption.

From (3.4) and $\mu \neq 0$, we obtain (3.2).

Finally, substituting (3.2) into (3.5) or (3.6), we get (3.3). ■

Next, applying Lemma 3.1, the Gauss–Codazzi equations and some techniques, we have the following basic lemma.

Lemma 3.2 *Let M be a non-Hopf Ricci-semisymmetric hypersurface in the nonflat complex planes $\bar{M}^2(c)$, and $\{\xi, U, \phi U\}$ is the standard non-Hopf frame such that (2.8)–(2.10) hold. Then we have the following equations:*

$$U(\beta) = \xi(\gamma) = \tfrac{\beta\kappa_2}{\mu^2+c}(4\alpha\gamma - 8\alpha\mu + \gamma^2), \tag{3.7}$$

$$\beta\kappa_1 + \mu\kappa_3 - \gamma\mu - \gamma\kappa_3 = 0, \tag{3.8}$$

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$$U(\alpha) = \xi(\beta) = \frac{\kappa_2(\gamma-\mu)}{\mu^2+c}\left(\alpha\gamma + 4\alpha^2 - \frac{\gamma\mu}{2}\right), \tag{3.9}$$

$$\xi(\mu) = \beta\kappa_2, \tag{3.10}$$

$$\phi U(\alpha) = \beta(\alpha + \kappa_3 - 3\mu), \tag{3.11}$$

$$\phi U(\beta) = \alpha\mu - 2\gamma\mu + \frac{c}{2} + \alpha\gamma + \beta\kappa_1, \tag{3.12}$$

$$\phi U(\gamma) = -\mu\kappa_1 + \gamma\kappa_1 + \beta\gamma + 2\beta\mu, \tag{3.13}$$

$$U(\mu) = \gamma\kappa_2 - \mu\kappa_2, \tag{3.14}$$

$$\xi(\kappa_2) - \phi U(\kappa_3) = -2\beta\mu - \beta\kappa_3 - \kappa_3\kappa_1 + \mu\kappa_1, \tag{3.15}$$

$$U(\kappa_2) - \phi U(\kappa_1) = -2\gamma\mu - \kappa_1^2 - \gamma\kappa_3 - \kappa_2^2 - \mu\kappa_3 - c, \tag{3.16}$$

$$\xi(\kappa_1) - U(\kappa_3) = \kappa_2\kappa_3 - \kappa_2\gamma, \tag{3.17}$$

$$\xi(\alpha) = \frac{\beta\kappa_2}{\mu^2+c}(c + \alpha\gamma + 4\alpha^2 + 2\alpha\mu), \tag{3.18}$$

$$U(\gamma) = \frac{\kappa_2(\gamma-\mu)}{\mu^2+c}(4c + \gamma^2 + 4\alpha\gamma + 2\gamma\mu). \tag{3.19}$$

Proof Taking $(X, Y) = (\xi, U)$ in Codazzi equation (2.6) and using (3.1), we obtain

$$U(\beta) = \xi(\gamma), \tag{3.20}$$

$$U(\alpha) = \xi(\beta), \tag{3.21}$$

$$\gamma\kappa_3 + \gamma\mu - \mu\kappa_3 - \alpha\gamma + \beta^2 - \beta\kappa_1 = \frac{c}{4}. \tag{3.22}$$

Then, from (3.2) and (3.22), we obtain (3.8).

Similarly, taking in (2.6), $(X, Y) = (\xi, \phi U), (U, \phi U)$, respectively, with the use of (3.1), we can obtain (3.10)–(3.14).

Next, taking in Gauss equation (2.5),

$$(X, Y, Z) = (\xi, \phi U, U), (U, \phi U, \phi U), (\xi, U, U),$$

respectively, and using (3.1), we can calculate to obtain (3.15)–(3.17).

To prove the remaining equations (3.7), (3.9), (3.18), and (3.19), more computations are needed.

• Taking the derivative of (3.2) with respect to ϕU and using (3.11)–(3.13), we obtain

$$\begin{aligned} 0 &= \phi U(\alpha)\gamma + \phi U(\gamma)\alpha - 2\beta\phi U(\beta) \\ &= \beta\gamma\kappa_3 + \beta\gamma\mu - \alpha\mu\kappa_1 + \alpha\gamma\kappa_1 - \beta c - 2\beta^2\kappa_1. \end{aligned} \tag{3.23}$$

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From (3.8), we have $\gamma\mu + \gamma\kappa_3 = \beta\kappa_1 + \mu\kappa_3$. Substituting it into (3.23), we have

$$\begin{aligned}0 &= \beta(\beta\kappa_1 + \mu\kappa_3) - \alpha\mu\kappa_1 + \alpha\gamma\kappa_1 - \beta c - 2\beta^2\kappa_1\\ &= \kappa_1(\alpha\gamma - \beta^2) + \beta\mu\kappa_3 - \alpha\mu\kappa_1 - \beta c.\end{aligned}$$

By using (3.2), the above equation can be rewritten as

$$0 = -\kappa_1(\alpha\mu + \tfrac{c}{4}) + \beta(\mu\kappa_3 - c). \tag{3.24}$$

Substituting (3.2) into (3.5), we have

$$0 = (c + \gamma\mu)(\alpha\mu + \tfrac{c}{4}) - \beta^2\mu^2. \tag{3.25}$$

As $\beta\mu \neq 0$, we get from (3.25) that $\alpha\mu + \frac{c}{4} \neq 0$. It follows from (3.24) and (3.25) that

$$\begin{vmatrix} -\kappa_1 & \mu\kappa_3 - c \\ c + \gamma\mu & -\beta\mu^2 \end{vmatrix} = 0.$$

That is,

$$0 = \mu^2(\beta\kappa_1 - \gamma\kappa_3) - c(\mu\kappa_3 - \gamma\mu) + c^2.$$

By (3.8), we have $\beta\kappa_1 - \gamma\kappa_3 = \gamma\mu - \kappa_3\mu$. Substituting it into the above equation, we have

$$0 = (\mu^2 + c)(\gamma - \kappa_3)\mu + c^2, \tag{3.26}$$

which implies that

$$\kappa_3 \neq \gamma, \quad \mu^2 + c \neq 0. \tag{3.27}$$

- Taking the derivative of (3.3) along ξ and using (3.10), we have

$$0 = \mu(4\xi(\alpha) + \xi(\gamma)) + \beta\kappa_2(4\alpha + \gamma - 2\mu).$$

It follows that

$$\xi(\alpha) = -\tfrac{\beta\kappa_2(4\alpha+\gamma-2\mu)}{4\mu} - \tfrac{1}{4}\xi(\gamma). \tag{3.28}$$

- Taking the derivative of (3.3) along U and using (3.14) and (3.21), we obtain

$$\begin{aligned}0 &= \mu(4U(\alpha) + U(\gamma) - U(\mu)) + (4\alpha + \gamma - \mu)U(\mu)\\ &= \mu(4\xi(\beta) + U(\gamma)) + \kappa_2(\gamma - \mu)(4\alpha + \gamma - 2\mu).\end{aligned}$$

It follows that

$$U(\gamma) = -\tfrac{\kappa_2(\gamma-\mu)(4\alpha+\gamma-2\mu)}{\mu} - 4\xi(\beta). \tag{3.29}$$

- Taking the derivative of (3.2) along ξ, we have

$$0 = \xi(\alpha)\gamma + \alpha\xi(\gamma) - 2\beta\xi(\beta). \tag{3.30}$$

- Taking the derivative of (3.2) along U, and using (3.20) and (3.21), we obtain

$$0 = \xi(\beta)\gamma + \alpha U(\gamma) - 2\beta\xi(\gamma). \tag{3.31}$$

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Substituting (3.28) into (3.30), and (3.29) into (3.31), we have

$$\begin{cases} -2\beta\xi(\beta) + (-\frac{\gamma}{4} + \alpha)\xi(\gamma) = \frac{\gamma\beta\kappa_2(4\alpha+\gamma-2\mu)}{4\mu}, \\ (\gamma - 4\alpha)\xi(\beta) - 2\beta\xi(\gamma) = \frac{\alpha(\gamma-\mu)\kappa_2(4\alpha+\gamma-2\mu)}{\mu}. \end{cases} \tag{3.32}$$

Since

$$\begin{vmatrix} -2\beta & -\frac{\gamma}{4} + \alpha \\ \gamma - 4\alpha & -2\beta \end{vmatrix} = 4\beta^2 + \frac{(\gamma-4\alpha)^2}{4} > 0,$$

we can solve (3.32) to obtain

$$\begin{cases} \xi(\beta) = \frac{\kappa_2(4\alpha+\gamma-2\mu)}{\mu(4\beta^2+\frac{(\gamma-4\alpha)^2}{4})}\left[-\frac{\beta^2\gamma}{2} - \alpha(\gamma-\mu)(-\frac{\gamma}{4}+\alpha)\right], \\ \xi(\gamma) = \frac{\beta\kappa_2(4\alpha+\gamma-2\mu)}{\mu(4\beta^2+\frac{(\gamma-4\alpha)^2}{4})}(-\alpha\gamma + 2\alpha\mu - \frac{\gamma^2}{4}). \end{cases}$$

From (3.2), we have

$$4\beta^2 = 4\alpha\gamma + c. \tag{3.33}$$

From (3.3) and the fact that $\mu \neq 0$, we have

$$\gamma + 4\alpha = \mu - \frac{c}{\mu}. \tag{3.34}$$

Thus, we obtain

$$\frac{4\alpha+\gamma-2\mu}{\mu(4\beta^2+\frac{(\gamma-4\alpha)^2}{4})} = \frac{4(4\alpha+\gamma-2\mu)}{\mu(4c+(\gamma+4\alpha)^2)} = \frac{-4(c+\mu^2)}{\mu^2(4c+(\mu-\frac{c}{\mu})^2)} = -\frac{4}{\mu^2+c}. \tag{3.35}$$

Then, with the use of (3.33)–(3.35), $\xi(\beta)$ and $\xi(\gamma)$ can be rewritten as

$$\begin{aligned} \xi(\beta) &= -\frac{4\kappa_2}{\mu^2+c}\left[-\frac{1}{2}(\alpha\gamma+\frac{c}{4})\gamma - \alpha(\gamma-\mu)(-\frac{\gamma}{4}+\alpha)\right] \\ &= \frac{\kappa_2(\gamma-\mu)}{\mu^2+c}(\alpha\gamma + 4\alpha^2 - \frac{\gamma\mu}{2}) \end{aligned} \tag{3.36}$$

and

$$\xi(\gamma) = \frac{\beta\kappa_2}{\mu^2+c}(4\alpha\gamma - 8\alpha\mu + \gamma^2). \tag{3.37}$$

From (3.20) and (3.37), we obtain (3.7). From (3.21) and (3.36), we obtain (3.9). Next, substituting (3.37) into (3.28), we obtain

$$\begin{aligned} \xi(\alpha) &= -\frac{\beta\kappa_2(4\alpha+\gamma-2\mu)}{4\mu} - \frac{\beta\kappa_2}{\mu^2+c}\left(\alpha\gamma - 2\alpha\mu + \frac{\gamma^2}{4}\right) \\ &= -\frac{\beta\kappa_2}{(\mu^2+c)}\left[\frac{(4\alpha+\gamma-2\mu)}{4}\frac{\mu^2+c}{\mu} + \alpha\gamma - 2\alpha\mu + \frac{\gamma^2}{4}\right]. \end{aligned}$$

By using (3.3), we have

$$\xi(\alpha) = \frac{\beta\kappa_2}{\mu^2+c}(c + \alpha\gamma + 4\alpha^2 + 2\alpha\mu).$$

So (3.18) is obtained.

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Substituting (3.36) into (3.29), we obtain

$$\begin{aligned} U(\gamma) &= -\frac{\kappa_2(\gamma-\mu)(4\alpha+\gamma-2\mu)}{\mu} - 4\frac{\kappa_2(\gamma-\mu)}{\mu^2+c}(\alpha\gamma + 4\alpha^2 - \frac{\gamma\mu}{2}) \\ &= -\frac{\kappa_2(\gamma-\mu)}{\mu^2+c}[(4\alpha+\gamma-2\mu)\frac{\mu^2+c}{\mu} + 4(\alpha\gamma + 4\alpha^2 - \frac{\gamma\mu}{2})]. \end{aligned}$$

By using (3.3), we have

$$U(\gamma) = \frac{\kappa_2(\gamma-\mu)}{\mu^2+c}(4c + \gamma^2 + 4\alpha\gamma + 2\gamma\mu).$$

So (3.19) is obtained. We have completed the proof of Lemma 3.2. ∎

Lemma 3.3 *Let M be a non-Hopf Ricci-semisymmetric hypersurface in $\bar{M}^2(c)$, and $\{\xi, U, \phi U\}$ is the standard non-Hopf frame such that (2.8)–(2.10) hold. Then there exists at least one point $p \in M$ such that $\kappa_2 \neq 0$ at p.*

Proof In fact, if $\kappa_2 \equiv 0$ on M, by (3.10) and (3.14), we have $\xi(\mu) = U(\mu) = 0$. Thus, we have

$$0 = \{[\xi, U] - (\nabla_\xi U - \nabla_U \xi)\}(\mu) = -(\kappa_3 - \gamma)\phi U(\mu).$$

From (3.27), we have $\phi U(\mu) = 0$. Thus, we obtain that μ is a constant. It follows from (3.26) that $\gamma - \kappa_3$ is a constant. Then, taking the derivative of $\gamma - \kappa_3$ with respect to ϕU and using (3.13), (3.15), and the fact that $\kappa_2 = 0$, we obtain

$$\begin{aligned} 0 &= -\mu\kappa_1 + \gamma\kappa_1 + \beta\gamma + 2\beta\mu + (-2\beta\mu - \beta\kappa_3 - \kappa_3\kappa_1 + \mu\kappa_1) \\ &= (\beta + \kappa_1)(\gamma - \kappa_3). \end{aligned}$$

As $\kappa_3 \neq \gamma$, we have $\beta = -\kappa_1$.

Next, by $\kappa_2 = 0$, (3.7), and (3.19), we have $\xi(\gamma) = U(\gamma) = 0$. Thus,

$$0 = \{[\xi, U] - (\nabla_\xi U - \nabla_U \xi)\}(\gamma) = -(\kappa_3 - \gamma)\phi U(\gamma). \tag{3.38}$$

As $\kappa_3 \neq \gamma$, from (3.38), $\beta = -\kappa_1$, and (3.13), we obtain

$$0 = \phi U(\gamma) = -\mu\kappa_1 + \gamma\kappa_1 + \beta\gamma + 2\beta\mu = 3\beta\mu,$$

which is a contradiction to $\beta \neq 0$ and $\mu \neq 0$. This completes the proof of Lemma 3.3. ∎

4 Proof of Theorem 1.1

Now, we are ready to complete the proof of Theorem 1.1.

Suppose on the contrary that $\bar{M}^2(c)$ admits a non-Hopf Ricci-semisymmetric hypersurface M, and $\{\xi, U, \phi U\}$ is the standard non-Hopf frame on M such that (2.8)–(2.10) hold. Then, from (3.3), we have $c = \mu^2 - \gamma\mu - 4\alpha\mu$. Substituting it into (3.24), we obtain

$$\begin{aligned} 0 &= -\kappa_1(\alpha\mu + \frac{\mu^2 - \gamma\mu - 4\alpha\mu}{4}) + \beta(\mu\kappa_3 + 4\alpha\mu + \gamma\mu - \mu^2) \\ &= \mu[\frac{\kappa_1(\gamma-\mu)}{4} + \beta(\kappa_3 + 4\alpha + \gamma - \mu)]. \end{aligned}$$

As $\mu \neq 0$, we have

$$\kappa_1(\mu-\gamma) = 4\beta(\kappa_3 + 4\alpha + \gamma - \mu). \tag{4.1}$$

From (3.26), we have

$$\kappa_3 = \gamma + \tfrac{c^2}{\mu(\mu^2+c)}. \tag{4.2}$$

Now, we claim that

$$\gamma - \mu \neq 0.$$

In fact, if $\gamma = \mu$, from (3.3), we get $4\alpha\gamma + c = 0$. This combining with (3.2) implies that $\beta^2 = 0$. This is a contradiction, which verifies the claim.

Then, substituting (4.2) into (4.1) and applying (3.34), we obtain

$$\kappa_1 = \tfrac{4\beta}{\mu-\gamma}\left(\gamma + \tfrac{c^2}{\mu(\mu^2+c)} - \tfrac{c}{\mu}\right) = \tfrac{4\beta}{\mu-\gamma}\left(\gamma - \tfrac{c\mu}{\mu^2+c}\right). \tag{4.3}$$

• Taking the derivative of (4.2) along U and applying (3.14) and (3.19), we obtain

$$U(\kappa_3) = U\left(\gamma + \tfrac{c^2}{\mu(\mu^2+c)}\right) = \tfrac{\kappa_2(\gamma-\mu)}{\mu^2+c}\left[4c + \gamma^2 + 4\alpha\gamma + 2\gamma\mu - \tfrac{c^2(c+3\mu^2)}{\mu^2(\mu^2+c)}\right]. \tag{4.4}$$

• Taking the derivative of (4.3) along ξ, then applying (3.7), (3.9), and (3.10), we obtain

$$\begin{aligned}
\xi(\kappa_1) =& \tfrac{4\xi(\beta)(\mu-\gamma)-4\beta(\xi(\mu)-\xi(\gamma))}{(\mu-\gamma)^2}\left(\gamma - \tfrac{c\mu}{\mu^2+c}\right) \\
&+ \tfrac{4\beta}{\mu-\gamma}\left[\xi(\gamma) - \tfrac{c\xi(\mu)(\mu^2+c)-c\mu(2\mu\xi(\mu))}{(\mu^2+c)^2}\right] \\
=& \tfrac{1}{(\mu-\gamma)^2}\Big[\tfrac{-4\kappa_2(\mu-\gamma)^2}{\mu^2+c}\left(\alpha\gamma + 4\alpha^2 - \tfrac{\gamma\mu}{2}\right) \\
&\qquad - 4\beta\Big(\beta\kappa_2 - \tfrac{\beta\kappa_2}{\mu^2+c}\left(4\alpha\gamma - 8\alpha\mu + \gamma^2\right)\Big)\Big]\left(\gamma - \tfrac{c\mu}{\mu^2+c}\right) \\
&+ \tfrac{4\beta}{\mu-\gamma}\left[\tfrac{\beta\kappa_2}{\mu^2+c}(4\alpha\gamma - 8\alpha\mu + \gamma^2) - \tfrac{c(c-\mu^2)\beta\kappa_2}{(\mu^2+c)^2}\right] \\
=& \tfrac{\kappa_2}{\mu^2+c}\Big\{\tfrac{1}{(\mu-\gamma)^2}\Big[-4(\mu-\gamma)^2\left(\alpha\gamma + 4\alpha^2 - \tfrac{\gamma\mu}{2}\right) \\
&\qquad - 4\beta^2\Big(\mu^2 + c - \left(4\alpha\gamma - 8\alpha\mu + \gamma^2\right)\Big)\Big]\left(\gamma - \tfrac{c\mu}{\mu^2+c}\right) \\
&+ \tfrac{4\beta^2}{\mu-\gamma}\left[4\alpha\gamma - 8\alpha\mu + \gamma^2 - \tfrac{c(c-\mu^2)}{\mu^2+c}\right]\Big\}.
\end{aligned} \tag{4.5}$$

From (3.2) and (3.3), we have

$$4\beta^2 = 4\alpha\gamma - (4\alpha\mu + \gamma\mu - \mu^2) = (\mu-\gamma)(\mu-4\alpha). \tag{4.6}$$

Substituting (4.6) into (4.5) and applying (3.3), we obtain

$$\begin{aligned}
\xi(\kappa_1) =& \tfrac{\kappa_2}{\mu^2+c}\Big\{\Big[-4\left(\alpha\gamma + 4\alpha^2 - \tfrac{\gamma\mu}{2}\right) - \tfrac{\mu-4\alpha}{\mu-\gamma}\left(2\mu^2 - \gamma\mu - 4\alpha\gamma + 4\alpha\mu - \gamma^2\right)\Big] \\
&\qquad \times\left(\gamma - \tfrac{c\mu}{\mu^2+c}\right) + \left(\mu - 4\alpha\right)\left[4\alpha\gamma - 8\alpha\mu + \gamma^2 - \tfrac{c(c-\mu^2)}{\mu^2+c}\right]\Big\} \\
=& \tfrac{\kappa_2}{\mu^2+c}\Big\{\mu(4\alpha + \gamma - 2\mu)\left(\gamma - \tfrac{c\mu}{\mu^2+c}\right) + (\mu - 4\alpha)\Big[4\alpha\gamma - 8\alpha\mu + \gamma^2 \\
&\qquad - \tfrac{c(c-\mu^2)}{\mu^2+c}\Big]\Big\}.
\end{aligned} \tag{4.7}$$

In the following, we shall calculate the expression of $\xi(\kappa_1) - U(\kappa_3)$ in two different ways. On the one hand, from (4.4) and (4.7), we have

$$
\begin{aligned}
\xi(\kappa_1) - U(\kappa_3) = \tfrac{\kappa_2}{\mu^2+c}\Big\{&\mu(4\alpha+\gamma-2\mu)\Big(\gamma - \tfrac{c\mu}{\mu^2+c}\Big)\\
&+ (\mu-4\alpha)\Big[4\alpha\gamma - 8\alpha\mu + \gamma^2 - \tfrac{c(c-\mu^2)}{\mu^2+c}\Big]\\
&- (\gamma-\mu)\Big[4c + \gamma^2 + 4\alpha\gamma + 2\gamma\mu - \tfrac{c^2(c+3\mu^2)}{\mu^2(\mu^2+c)}\Big]\Big\}\\
= \tfrac{\kappa_2}{\mu^2+c}\Big\{&\gamma\mu(4\alpha+\gamma-2\mu) + (\mu-4\alpha)\Big(4\alpha\gamma - 8\alpha\mu + \gamma^2\Big)\\
&+ (\mu-\gamma)(4c + \gamma^2 + 4\alpha\gamma + 2\gamma\mu)\\
&+ (\mu-\gamma)\tfrac{c\mu^4 - c^2(c+3\mu^2)}{\mu^2(\mu^2+c)} + (\mu-4\alpha)\tfrac{c\mu^4 - c\mu^2(c-\mu^2)}{\mu^2(\mu^2+c)}\Big\}.
\end{aligned}
\tag{4.8}
$$

By using (3.3), we have

$$
\begin{aligned}
&\gamma\mu(4\alpha+\gamma-2\mu) + (\mu-4\alpha)(4\alpha\gamma-8\alpha\mu+\gamma^2)\\
&\quad + (\mu-\gamma)(4c+\gamma^2+4\alpha\gamma+2\gamma\mu)\\
&\quad = 12\alpha\gamma\mu + \gamma^2\mu - 8\alpha\mu^2 - 16\alpha^2\gamma + 32\alpha^2\mu - 8\alpha\gamma^2 + 4(\mu-\gamma)c - \gamma^3\\
&\quad = 8\alpha(4\alpha\mu+\gamma\mu-\mu^2) + \gamma(4\alpha\mu+\gamma\mu-\mu^2) + \mu^2\gamma + 4(\mu-\gamma)c - \gamma(\gamma+4\alpha)^2\\
&\quad = -8\alpha c - 5\gamma c + 4\mu c + \mu^2\gamma - \gamma(\gamma+4\alpha)^2
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
&(\mu-\gamma)\tfrac{c\mu^4 - c^2(c+3\mu^2)}{\mu^2(\mu^2+c)} + (\mu-4\alpha)\tfrac{c\mu^4 - c\mu^2(c-\mu^2)}{\mu^2(\mu^2+c)}\\
&\quad = \tfrac{c}{\mu^2(\mu^2+c)}\Big[3\mu^5 - (\mu-\gamma)c^2 - 4c\mu^3 - \gamma\mu^4 + 3c\gamma\mu^2 - 8\alpha\mu^4 + 4\alpha\mu^2 c\Big]\\
&\quad = \tfrac{c}{(\mu^2+c)}\Big[3\mu^3 - (\mu-\gamma)(\mu-\gamma-4\alpha)^2 - 4c\mu - \gamma\mu^2 + 3c\gamma - 8\alpha\mu^2\\
&\qquad + 4\alpha c\Big]\\
&\quad = \tfrac{c}{(\mu^2+c)}\Big[\mu^3 - 4\alpha(4\alpha\mu+\gamma\mu-\mu^2) - \mu(4\alpha\mu+\gamma\mu-\mu^2)\\
&\qquad - 3\gamma(4\alpha\mu+\gamma\mu-\mu^2) + \gamma(\gamma+4\alpha)^2 - 4c\mu + 3c\gamma + 4\alpha c\Big]\\
&\quad = \tfrac{c}{(\mu^2+c)}\Big[\mu^3 + 8\alpha c - 3\mu c + 6\gamma c + \gamma(\gamma+4\alpha)^2\Big].
\end{aligned}
\tag{4.10}
$$

Substituting (4.9) and (4.10) into (4.8), we obtain

$$
\begin{aligned}
\xi(\kappa_1) - U(\kappa_3) &= \tfrac{\kappa_2}{\mu^2+c}\Big\{4\mu c - 5\gamma c - 8\alpha c + \gamma\mu^2 - \gamma(\gamma+4\alpha)^2\\
&\qquad + \tfrac{c}{\mu^2+c}\Big[\mu^3 + 8c\alpha - 3c\mu + 6c\gamma + \gamma(\gamma+4\alpha)^2\Big]\Big\}\\
&= \tfrac{\kappa_2}{\mu^2+c}\Big\{5\mu c - 5\gamma c - 8\alpha c + \gamma\mu^2 - \gamma(\gamma+4\alpha)^2\\
&\qquad + \tfrac{c}{\mu^2+c}\Big[8c\alpha - 4c\mu + 6c\gamma + \gamma(\gamma+4\alpha)^2\Big]\Big\}.
\end{aligned}
\tag{4.11}
$$

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On the other hand, substituting (4.2) into (3.17) and applying (3.3), we obtain

$$\xi(\kappa_1) - U(\kappa_3) = \kappa_2 \frac{c^2}{\mu(\mu^2+c)} = \frac{\kappa_2 c}{\mu^2+c}(\mu - \gamma - 4\alpha). \tag{4.12}$$

From (4.11) and (4.12), we obtain

$$\begin{aligned} 0 = \frac{\kappa_2}{\mu^2+c}\{&5\mu c - 5\gamma c - 8\alpha c + \gamma\mu^2 - \gamma(\gamma+4\alpha)^2 \\ &+ \frac{c}{\mu^2+c}(8c\alpha - 4c\mu + 6c\gamma + \gamma(\gamma+4\alpha)^2) - c(\mu - \gamma - 4\alpha)\}. \end{aligned}$$

Since $\frac{\kappa_2}{\mu^2+c} \neq 0$, we obtain

$$\begin{aligned} 0 =&4\mu c - 4\gamma c - 4\alpha c + \gamma\mu^2 - \gamma(\gamma+4\alpha)^2 \\ &+ \frac{c}{\mu^2+c}\left[8c\alpha - 4c\mu + 6c\gamma + \gamma(\gamma+4\alpha)^2\right]. \end{aligned} \tag{4.13}$$

By using (3.34), we can rewrite (4.13) to obtain

$$\begin{aligned} 0 =&4\mu c - 3\gamma c - c\left(\mu - \frac{c}{\mu}\right) + \gamma\mu^2 - \gamma\left(\mu - \frac{c}{\mu}\right)^2 \\ &+ \frac{c}{\mu^2+c}\left[8c\alpha - 4c\mu + 6c\gamma + \gamma\left(\mu - \frac{c}{\mu}\right)^2\right]. \end{aligned} \tag{4.14}$$

Multiplying both sides of (4.14) by $\mu^2(\mu^2+c)$, we shall obtain

$$0 = c\mu(3\mu^4 + 2c\gamma\mu + c^2 + 8c\alpha\mu).$$

Since $c\mu \neq 0$, we have

$$0 = 3\mu^4 + 2c\gamma\mu + c^2 + 8c\alpha\mu. \tag{4.15}$$

By using (3.3), we can rewrite (4.15) to obtain that

$$0 = 3\mu^4 + c^2 + 2c(\mu^2 - c) = (3\mu^2 - c)(\mu^2 + c). \tag{4.16}$$

It follows from (3.27) that

$$3\mu^2 - c = 0. \tag{4.17}$$

• Taking the derivative of (4.17) along ξ and using (3.10), we obtain

$$\mu\beta\kappa_2 = 0. \tag{4.18}$$

Since $\beta \neq 0$ and $\mu \neq 0$, we obtain that $\kappa_2 \equiv 0$, a contradiction to Lemma 3.3.
We have completed the proof of Theorem 1.1.

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