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PSEUDOCOMPLEMENTED AND IMPLICATIVE SEMILATTICES

C. S. HOO

1. Introduction. Let L be a semilattice and let $a \in L$. We refer the reader to Definitions 2.2, 2.4, 2.5 and 2.12 below for the terminology. If L is a-implicative, let C_a be the set of a-closed elements of L, and let D_a be the filter of a-dense elements of L. Then C_a is a Boolean algebra. If $a = 0$, then C_0 and D_0 are the usual closed algebra and dense filter of L. If L is a-admissible and $f : C_a \times D_a \to D_a$ is the corresponding admissible map, we can form a quotient semilattice $C_a \times D_a/f$. In case $a = 0$, Murty and Rao [**4**] have shown that $C_0 \times D_0/f$ is isomorphic to L, and hence that $C_0 \times D_0/f$ is 0-admissible. In case L is in fact implicative, Nemitz [**5**] has shown that $C_0 \times D_0/f$ is isomorphic to L, and that $C_0 \times D_0/f$ is also implicative. We shall show a more general result as follows. Let C_aD_a denote the set of all products cd, where $c \in C_a$ and $d \in D_a$. Then C_aD_a is a subsemilattice of L containing the filter $[a)$. If $a = 0$, then C_0D_0 will be L. We shall show that, in general, $C_a \times D_a/f$ is isomorphic to C_aD_a. It will also be shown that C_aD_a is b-admissible for all $b \in C_a$. As a corollary, it will follow that if L is 0-admissible, it will also be a-admissible for all $a \in C_0$. The major consequence is that a bounded semilattice L is implicative if and only if L is 0-admissible and D_0 is implicative.

2. Preliminaries. Let L be a meet semilattice. We shall denote the greatest lower bound of two elements a, b of L by ab, and the least upper bound, if it exists, by $a + b$. A non-empty subset F of L is a filter provided that $xy \in F$ if and only if $x \in F$ and $y \in F$. Given an element a of L, let $[a) = \{x \in L | x \geqq a\}$. Then $[a)$ is a filter, called the *principal filter generated by* a.

A *semi-ideal* of L is a non-empty subset I of L such that $b \in I$ and $a \leqq b$ imply that $a \in I$. We call I an *ideal* if further whenever $a + b$ exists, where $a, b \in I$, then $a + b \in I$. Given an element a of L, let $(a] = \{x \in L | x \leqq a\}$. Then $(a]$ is the *principal ideal generated by* a.

A semilattice L is *distributive* if $z \geqq xy$ (where $x, y, z \in L$) implies the existence in L of elements x_1, y_1 such that $x_1 \geqq x$, $y_1 \geqq y$ and $z = x_1y_1$. A semilattice L is *modular* if whenever $y \geqq z \geqq xy$, where $x, y, z \in L$,

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then there exists an element $x_1 \in L$ such that $x_1 \geqq x$ and $z = x_1 y$. Clearly a distributive semilattice is modular.

Definition 2.1. Let x, a be elements of a semilattice L. The *annihilator* $\langle x, a \rangle = \{y \in L | xy \leqq a\}$. It is easily checked that $\langle x, a \rangle$ is always a semi-ideal of L.

Definition 2.2. Let L be a semilattice and let $a \in L$. Then L is *a-implicative* if $\langle x, a \rangle$ is a principal ideal. We shall denote this principal ideal by $(x * a]$. A semilattice L is implicative if and only if it is a-implicative for all $a \in L$. We observe that $x * a$ is a relative pseudocomplement.

If L has a least element 0, then L is 0-implicative means that L is pseudocomplemented, and it is customary to denote $x * 0$ by x^*. We observe that an a-implicative semilattice always has a greatest element 1 since $x \in \langle a, a \rangle$ for all elements x, and hence $a * a = 1$.

If the semilattice L is a-implicative, then the following results hold (see [**6**]):

(1) $x * a \geqq a$
(2) $x \leqq a$, $x * a = 1$ and $x * a \geqq x$ are equivalent
(3) $1 * a = a$
(4) $x * a = x$ if and only if $x = a = 1$
(5) $(x * a) * a \geqq x, a$
(6) if $x \leqq y$, then $x * a \geqq y * a$ and $(x * a) * a \leqq (y * a) * a$
(7) $((x * a) * a) * a = x * a$
(8) $(x * a)((x * a) * a) = a$
(9) $x(x * a) = xa$.

If L is implicative, then the following also hold (see [**5**]):

(10) if $x \leqq y$, then $z * x \leqq z * y$
(11) $x * (y * z) = (xy) * z$
(12) $x * (yz) = (x * y)(x * z)$
(13) $x * (y * z) = (x * y) * (x * z)$
(14) $x \leqq y$ if and only if $x * y = 1$.

In [**6**], Varlet proved the following result.

THEOREM 2.3. *If x and y are elements of an a-implicative semilattice, then*

$$((xy) * a) * a = ((x * a) * a)((y * a) * a).$$

From this, it is also quite easy to verify that

$$(xy) * a = (((x * a) * a)((y * a) * a)) * a.$$

Definition 2.4. An element x of an a-implicative semilattice L is called *a-closed* if $(x * a) * a = x$. The set of a-closed elements of L will be denoted by C_a. If $a = 0$, 0-closed means closed. From Theorem 2.3 we see that C_a is a subsemilattice of L.

Definition 2.5. Let a be an element of a semilattice L. An element $x \in L$ is *a-dense* if $\langle x, a\rangle \subset (a]$, that is, if $\langle x, a\rangle = (a]$.

If $a = 0$, 0-dense means dense. The set of all a-dense elements will be denoted by D_a. Then D_a is either empty or is a filter. If L is a-implicative, then $x \in D_a$ if and only if $x * a = a$, or equivalently, $(x * a) * a = 1$.

We observe that $a \in C_a$ since $(a * a) * a = 1 * a = a$. Also $(1 * a) * a = a * a = 1$, so that $1 \in C_a$. Obviously, $x \in C_a$ if and only if there exists an element $y \in L$ such that $x = y * a$. For each $x \in C_a$, we have $a \leqq (x * a) * a = x$. Thus a is the smallest element of C_a, and 1 is the largest. We can define an operation $\oplus$ in C_a by

$$x \oplus y = ((x * a)(y * a)) * a, \quad \text{for } x, y \in C_a.$$

If $x \in C_a$, we define $x' = x * a \in C_a$. Then

$$x \oplus a = ((x * a)(a * a)) * a = (x * a) * a = x.$$

THEOREM 2.6. *Let L be an a-implicative semilattice. Then C_a is a Boolean algebra with join $\oplus$ and complementation $'$ as above.*

Proof. We verify that $x \oplus y$ is the least upper bound of x and y in C_a. We have that $x * a \geqq (x * a)(y * a)$. Hence

$$x = (x * a) * a \leqq [(x * a)(y * a)] * a = x \oplus y.$$

Similarly, $y \leqq x \oplus y$. Now suppose that $x \leqq z$ and $y \leqq z$, where $z \in C_a$. Then

$$x * a \geqq z * a \quad \text{and} \quad y * a \geqq z * a.$$

Hence $(x * a)(y * a) \geqq z * a$. Thus

$$x \oplus y = [(x * a)(y * a)] * a \leqq (z * a) * a = z.$$

This proves that $x \oplus y$ is the least upper bound of x and y in C_a. Next, we observe that $x' = x * a$ is the complement of x in C_a. For, we have $x'x = (x * a)x \leqq a$. Since a is the smallest element of C_a, this means that $x'x = a$. Also

$$x' \oplus x = [(x' * a)(x * a)] * a.$$

But $x' * a = (x * a) * a = x$. Hence

$$x' \oplus x = [x(x * a)] * a = (xa) * a = 1.$$

Thus, C_a is a complemented lattice. Finally, we show that C_a is a complemented distributive lattice, that is, a Boolean algebra. According to [**2**] (Lemma 4.10, page 30), we need only verify the inequality

$$(x \oplus y)z \leqq x \oplus (yz) \quad \text{for } x, y, z \in C_a.$$

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We have that $xz \leqq x \oplus (yz)$ and $yz \leqq x \oplus (yz)$. Hence

$$xz[(x \oplus (yz)) * a] = (xz)[x \oplus (yz)]' = a$$

and

$$yz[(x \oplus (yz)) * a] = (yz)[x \oplus (yz)]' = a$$

since a is the least element in C_a. Thus

$$z[(x \oplus (yz)) * a] \leqq x * a \quad \text{and} \quad z[(x \oplus (yz)) * a] \leqq y * a.$$

This means that $z[(x \oplus (yz)) * a] \leqq (x * a)(y * a)$ and hence

$$z[(x \oplus (yz)) * a][\{(x * a)(y * a)\} * a] = a.$$

Thus

$$z(x \oplus y) \leqq [(x \oplus (yz)) * a] * a.$$

But $x \oplus (yz) \in C_a$ and hence

$$[(x \oplus (yz)) * a] * a = x \oplus (yz).$$

Thus $z(x \oplus y) \leqq x \oplus (yz)$, and hence C_a is a Boolean algebra.

THEOREM 2.7. *If L is an implicative semilattice, then for each $x \in L$ and each $a \in L$, we have*

$$x = [(x * a) * a]\{[(x * a) * a] * x\}.$$

If $x \geqq a$, then $[(x * a) * a] * x \in D_a$ and is the greatest element $d \in D_a$ satisfying

$$x = [(x * a) * a]d.$$

Proof. Let $b = (x * a) * a$, $y = b * x$. We wish to show that $x = by$. We have $x \leqq b$ and $x \leqq y$. Hence $x \leqq by$. But $\langle b, x\rangle = (b * x] = (y]$, and hence $by \leqq x$. Thus $x = by$. Suppose that $x \geqq a$. We wish to show that $y \in D_a$. Since $\langle y, a\rangle = (y * a]$, we need only show that $y * a \leqq a$. Since $y \geqq x$, we have $y * a \leqq x * a$. On the other hand, since $x \geqq a$, we have that

$$(x * a)[(x * a) * a] = a \leqq x.$$

Thus $(x * a)b \leqq x$, that is,

$$x * a \in \langle b, x\rangle = (b * x] = (y].$$

This means that $x * a \leqq y$. Hence $y * a \leqq (x * a) * a$ and hence

$$y * a \leqq (x * a)[(x * a) * a] = a.$$

Finally, suppose that $d \in D_a$ and $x = [(x * a) * a]d$. Then we have

$x = bd = by$. This means that

$$d \in \langle b, by \rangle = (b * (by)] = (b * x].$$

Thus $d \leqq b * x = [(x * a) * a] * x$.

Note. $d = 1 \Leftrightarrow [(x * a) * a] * x = 1 \Leftrightarrow (x * a) * a = x \Leftrightarrow x \in C_a$. The referee has pointed out to us that in this proof b and y may be interchanged.

We also have the following result.

THEOREM 2.8. *Suppose that L is an a-implicative semilattice which is also modular. Then for each $x \geqq a$, there exists an element $d \in D_a$ such that*

$$x = [(x * a) * a]d.$$

Proof. We first observe that

$$(x * a)[(x * a) * a] = a \leqq x \leqq (x * a) * a.$$

Since L is modular, there exists $d \geqq x * a$ such that

$$x = [(x * a) * a]d.$$

It remains only to show that $d \in D_a$. We have $x \leqq d$ and $x * a \leqq d$. Hence $x * a \geqq d * a$ also, and $(x * a) * a \geqq d * a$. Thus

$$(x * a)[(x * a) * a] = a \geqq d * a \geqq a,$$

that is, $d * a = a$. Thus $d \in D_a$.

The following are easy corollaries.

LEMMA 2.9. *Suppose that L is implicative (or a-implicative and modular). Let $x, y \geqq a$. Then $(x * a) * a = (y * a) * a$ if and only if there exists $d \in D_a$ such that $xd = yd$.*

LEMMA 2.10. *Let L be an implicative semilattice and let $x \geqq a$. Then $x \in D_a$ if and only if $x = [(y * a) * a] * y$ for some $y \geqq a$.*

THEOREM 2.11. *Let L be an implicative semilattice. Then the following holds in L:*

$$\{(x * y) * a\} * a = \{(x * a) * a\} * \{(y * a) * a\} \quad \textit{if } y \geqq a.$$

Proof. For simplicity, let us generally write z^* for $z * a$ and z^{**} for $(z * a) * a$, and so on. Since $y \geqq a$, by Theorem 2.7, we can find $d \in D_a$ such that $y = y^{**}d$. Since $d \in D_a$ and $d \leqq x * d$, it follows that $x * d \in D_a$ also because D_a is a filter. Hence

$$(x * y)^{**} = \{x * (y^{**}d)\}^{**} = \{(x * y^{**})(x * d)\}^{**} = (x * y^{**})^{**}(x * d)^{**}$$

by Theorem 2.3. But since $x * d \in D_a$, we have that $(x * d)^{**} = 1$. Hence

$$(x * y)^{**} = (x * y^{**})^{**} = x * y^{**}$$

since it can be easily verified that $(xy)^* = x * y^*$ in general. Hence

$$(x * y)^{**} = (xy^*)^{***} = (x^{**}y^{***})^*$$

by Theorem 2.3 again. Thus $(x * y)^{**} = (x^{**}y^*)^* = x^{**} * y^{**}$, proving our result.

Definition 2.12. $L = (L, \cdot, *, a, 1)$ is an *a-admissible semilattice* if

(i) L is an a-implicative semilattice.

(ii) For each $x \in L$ such that $x \geqq a$, there exists $d \in D_a$ such that $x = [(x * a) * a]d$.

(iii) There exists a function $f : C_a \times D_a \to D_a$ such that for each $x \in L$, we have $x \leqq f(c, d)$ if and only if $xc \leqq d$, that is, if and only if $x \in \langle c, d \rangle$. This means that $c * d$ exists and is in D_a for $c \in C_a$ and $d \in D_a$.

We observe that if L has a least element 0, then 0-admissible means admissible in the sense of [**3**]. If L is implicative, then for each $a \in L$, L is a-admissible, for we can define f by $f(c, d) = c * d$. We observe also that in this case, D_a is also implicative.

Definition 2.13. Let A be a Boolean algebra and let D be a meet semilattice with 1. A map $f: A \times D \to D$ is *admissible* if

(1) $f(ab, d) = f(a, f(b, d))$.

(2) For each $a \in A$, $f_a : D \to D$ given by $f_a(d) = f(a, d)$, is an endomorphism.

(3) $a \leqq b$ implies that $f(b, d) \leqq f(a, d)$.

(4) $f(1, d) = d$.

LEMMA 2.14. *If L is an a-admissible semilattice, then the corresponding map $f : C_a \times D_a \to D_a$ satisfies $d \leqq f(c, d)$, $cf(c, d) = cd$.*

Proof. Since $cd \leqq d$, we have $d \leqq f(c, d)$. Also $f(c, d) \leqq f(c, d)$ and hence $cf(c, d) \leqq d$. This means that $cf(c, d) \leqq cd$. Thus $cd \leqq cf(c, d) \leqq cd$, that is, $cf(c, d) = cd$.

THEOREM 2.15. *If L is an a-admissible semilattice, then the corresponding map $f: C_a \times D_a \to D_a$ is admissible.*

Proof. We have seen that C_a is a Boolean algebra and that D_a is a filter, that is, a meet semilattice with 1.

(i) To show that $f(bc, d) = f(b, f(c, d))$. Since $f(bc, d) \leqq f(bc, d)$, we have $bcf(bc, d) \leqq d$. Hence $bf(bc, d) \leqq f(c, d)$ and hence

$$f(bc, d) \leqq f(b, f(c, d)).$$

On the other hand,

$$bcf(b, f(c, d)) = cbf(b, f(c, d)) = cbf(c, d) = bcf(c, d) = bcd \leqq d.$$

Thus $f(b, f(c, d)) \leqq f(bc, d)$.

(ii) Let $b \in C_a$. To show that $f_b \colon C_a \to C_a$ is an endomorphism, that is, $f_b(de) = f_b(d)f_b(e)$ for $d, e \in D_a$. We have

$$bf(b, de) = bde \leqq d.$$

Hence $f(b, de) \leqq f(b, d)$. Similarly, $f(b, de) \leqq f(b, e)$. Hence

$$f(b, de) \leqq f(b, d)f(b, e).$$

On the other hand,

$$bf(b, d)f(b, e) = bf(b, d)bf(b, e) = bdeb = bde \leqq de.$$

Hence $f(b, d)f(b, e) \leqq f(b, de)$.

(iii) Suppose that $b \leqq c$. To show that $f(c, d) \leqq f(b, d)$. We have

$$bf(c, d) \leqq cf(c, d) = cd \leqq d.$$

Hence $f(c, d) \leqq f(b, d)$.

(iv) $f(1, d) = 1 \cdot f(1, d) = 1 \cdot d = d$.

Thus f is an admissible map.

3. Admissible semilattices. Let A be a Boolean algebra and let D be a meet semilattice with 1. Let $f : A \times D \to D$ be an admissible map. We can define an equivalence relation $\sim$ on $A \times D$ by $(a, d) \sim (b, e)$ if and only if $a = b$ and $f(a, d) = f(a, e)$. Then we can form the quotient $A \times D/f$ consisting of all the equivalence classes $[a, d]$.

We can define $[a, d]\,[b, e] = [ab, de]$, and $[a, d] \leqq [b, e]$ if and only if $[a, d]\,[b, e] = [a, d]$. Then $A \times D/f$ is a meet semilattice. As in [**5**], we have the following observations.

(i) Suppose that $b \leqq a$ and $f(a, d) = f(a, e)$, that is, $[a, d] = [a, e]$. Then $f(b, d) = f(b, e)$, that is, $[b, d] = [b, e]$.

(ii) $[a, d] \leqq [b, e]$ if and only if $a \leqq b$ and $f(a, d) \leqq f(a, e)$.

If $f(0, d) = 1$ for all $d \in D$, then $A \times D/f$ has $[0, 1]$ as its zero.

We can define a pseudocomplementation on $A \times D/f$ by $[a, d]^* = [a', 1]$. Then $A \times D/f$ is a pseudocomplemented semilattice. The 0-closed algebra of $A \times D/f$ is isomorphic to A via $[a, 1] \leftrightarrow a$. That is, the 0-closed elements consist of all $[a, 1]$. The 0-dense filter of $A \times D/f$ is isomorphic to D via $[1, d] \leftrightarrow d$. That is, the 0-dense elements are all the elements $[1, d]$. Then, each element $[a, d]$ of $A \times D/f$ can be written as $[a, d] = [a, 1]\,[1, d]$, with $[a, d]^{**} = [a, 1]$ being 0-closed, and $[1, d]$ being 0-dense. In fact, $A \times D/f$ is 0-admissible. The corresponding admissible map f_1 is

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given by

$$f_1([a, 1], [1, d]) = [1, f(a, d)].$$

It can be verified that f_1 satisfies the requirements, making $A \times D/f$ a 0-admissible semilattice.

If L is a 0-admissible semilattice and $f : C_0 \times D_0 \to D_0$ is the corresponding admissible map, then $L \cong C_0 \times D_0/f$. The required isomorphism $g : C_0 \times D_0/f \to L$ may be defined by $g[a, d] = ad$. The above is a summary of the results obtained in [**5**] and [**4**]. In fact, we claim the following result.

THEOREM 3.1. *Let A be a Boolean algebra, D be a meet semilattice with* 1, *and $f : A \times D \to D$ be an admissible map. Then $A \times D/f$ is a $[b, 1]$-admissible semilattice for all elements $b \in A$, that is $A \times D/f$ is an x-admissible semilattice for all elements x in the* 0*-closed algebra of $A \times D/f$.*

Proof. Let $b \in A$. We first show that $A \times D/f$ is $[b, 1]$-implicative. We define $[a, b] * [b, 1] = [a' \vee b, 1]$ for each $[a, d] \in A \times D/f$, where $\vee$ denotes the join operation in the Boolean algebra A. This is obviously well-defined. We have to show that $[a_1, d_1]\ [a, d] \leqq [b, 1]$ if and only if $[a_1, d_1] \leqq [a' \vee b, 1]$, that is, $[aa_1, dd_1] \leqq [b, 1]$ if and only if $[a_1, d_1] \leqq [a' \vee b, 1]$. This reduces to the statement $aa_1 \leqq b$ if and only if $a_1 \leqq a' \vee b$, and this is trivially true since A is a Boolean algebra. Next, suppose that $[a, d] \geqq [b, 1]$, that is, $b \leqq a$ and $f(b, d) = 1$. We have

$$\begin{aligned}([a, d] * [b, 1]) * [b, 1] &= [a' \vee b, 1] * [b, 1] \\ &= [(a' \vee b)' \vee b, 1] = [ab' \vee b, 1].\end{aligned}$$

Now, $(ab' \vee b)' = (ab')'b' = (a' \vee b)b' = a'b'$. Hence $ab' \vee b = (a'b')' = a \vee b = a$ since $b \leqq a$. Thus

$$([a, d] * [b, 1]) * [b, 1] = [a, 1].$$

Also $[1, d] * [b, 1] = [b, 1]$, that is, $[1, d]$ is $[b, 1]$-dense. Thus, for each $[a, d] \geqq [b, 1]$, we have

$$[a, d] = [a, 1]\ [1, d] = \{([a, d] * [b, 1]) * [b, 1]\}[1, d].$$

Finally, to obtain the corresponding admissible map

$$g : \{[b, 1]\text{-closed elements}\} \times \{[b, 1]\text{-dense elements}\} \\ \to \{[b, 1]\text{-dense elements}\},$$

we observe the following facts. We have that

$$\begin{aligned}&[a, d] \text{ is } [b, 1]\text{-dense} \\ &\quad \Leftrightarrow [a, d] * [b, 1] = [b, 1] \\ &\quad \Leftrightarrow [a' \vee b, 1] = [b, 1] \\ &\quad \Leftrightarrow a' \leqq b.\end{aligned}$$

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Also

$$[a, d] \text{ is } [b, 1]\text{-closed}$$
$$\Leftrightarrow ([a, d] * [b, 1]) * [b, 1] = [a, d]$$
$$\Leftrightarrow [a, d] = [a, 1] \quad \text{with } b \leqq a.$$

We define g by

$$g([a_1, 1], [a_2, d_2]) = [a_1' \vee a_2, f(a_1a_2, d_2)]$$

where $b \leqq a_1$ and $a_2' \leqq b$. We note that $[a_1' \vee a_2, f(a_1a_2, d_2)]$ is $[b, 1]$-dense since

$$(a_1' \vee a_2)' = a_1a_2' \leqq b.$$

This map g satisfies the requirements. In fact,

$$[a, d] \leqq g([a_1, 1], [a_2, d_2])$$
$$\Leftrightarrow [a, d] \leqq [a_1' \vee a_2, f(a_1a_2, d_2)]$$
$$\Leftrightarrow a \leqq a_1' \vee a_2$$

and

$$f(a, d) \leqq f(aa_1a_2, d_2)$$
$$\Leftrightarrow aa_1 \leqq a_2$$

and

$$f(a, d) \leqq f(aa_1a_2, d_2)$$
$$\Leftrightarrow aa_1 \leqq a_2$$

and

$$f(aa_1, d) \leqq f(aa_1, d_2)$$
$$\Leftrightarrow [aa_1, d] \leqq [a_2, d_2]$$
$$\Leftrightarrow [a, d]\,[a_1, 1] \leqq [a_2, d_2].$$

Thus $A \times D/f$ is $[b, 1]$-admissible for all $b \in A$.

Definition 3.2. If L is an a-implicative semilattice, then

$$C_aD_a = \{cd | c \in C_a, d \in D_a\}.$$

We observe that C_aD_a is a subsemilattice of L containing the filter $[a)$. If $a = 0$, the $C_0D_0 = L$.

Definition 3.3. Let L be a semilattice, then $\mathscr{A}(L)$ will denote the set of all elements $a \in L$ such that L is a-implicative.

LEMMA 3.4. *$\mathscr{A}(L)$ is a subsemilattice of L.*

Proof. Let $a, b \in \mathscr{A}(L)$. We define $x * (ab) = (x * a)(x * b)$ for each $x \in L$. It is easily verified that this shows that $ab \in \mathscr{A}(L)$.

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LEMMA 3.5. *If* $a \in \mathscr{A}(L)$, *then* $b * a \in \mathscr{A}(L)$ *for all* $b \in L$.

Proof. Define $x * (b * a) = (xb) * a$. It is easily checked that this satisfies the requirements.

LEMMA 3.6. *If* $a \in \mathscr{A}(L)$ *and* $d \in D_a$, *then* $(bd) * a = b * a$ *for all* $b \in L$.

Proof. $bd \leqq b$. Hence $b * a \leqq (bd) * a$. But $bd((bd) * a) \leqq a$. Hence

$$b((bd) * a) \leqq d * a = a$$

and hence $(bd) * a \leqq b * a$.

THEOREM 3.7. *Let* L *be an* a*-admissible semilattice and let* $f : C_a \times D_a \rightarrow D_a$ *be the corresponding admissible map. Then*

$$C_a \times D_a/f \cong C_aD_a.$$

Proof. Let $g : C_a \times D_a/f \rightarrow L$ be given by $g([b, d]) = bd$. This is well-defined. For if $[b, d] = [b_1, d_1]$, then $b = b_1$ and $f(b, d) = f(b, d_1)$. Hence

$$bd = bf(b, d) = bf(b, d_1) = bd_1 = b_1d_1.$$

Now g is a homomorphism, for

$$g([b,d][b_1,d_1]) = g([bb_1,dd_1]) = bb_1dd_1 = bdb_1d_1 = g([b,d])g([b_1,d_1]).$$

Also g is one to one. For if $bd = b_1d_1$, then

$$\{(bd) * a\} * a = \{(b_1d_1) * a\} * a,$$

that is,

$$\{(b * a) * a\}\{(d * a) * a\} = \{(b_1 * a) * a\}\{(d_1 * a) * a\}.$$

Hence $b = b_1$ since $b, b_1 \in C_a$, and $d, d_1 \in D_a$. Hence

$$bf(b, d) = bd = b_1d_1 \leqq d_1,$$

and hence $f(b, d) \leqq f(b, d_1)$. Similarly

$$bf(b, d_1) = bd_1 = b_1d_1 = bd \leqq d$$

and hence $f(b, d_1) \leqq f(b, d)$. Thus $f(b, d) = f(b, d_1)$ and hence $[b, d] = [b_1, d_1]$. Clearly, the image of g is contained in C_aD_a. On the other hand, it is also obvious that the image of g contains C_aD_a. Thus

$$g : C_a \times D_a/f \cong C_aD_a.$$

COROLLARY 3.8. *Let* L *be a* 0*-admissible semilattice and let* $f : C_0 \times D_0 \rightarrow D_0$ *be the corresponding admissible map. Then*

$$C_0 \times D_0/f \cong L.$$

Proof. $C_0D_0 = L$.

The following is easily verified.

LEMMA 3.9. *Let L, L_1 be semilattices and let $a \in L$, $a_1 \in L_1$. Suppose that L is a-implicative and L_1 is a_1-implicative, and suppose that $g: L \to L_1$ is an isomorphism such that*

$$g(a) = a_1 \quad \text{and} \quad g(x * a) = g(x) * a_1$$

for all $x \in L$. If L is a-admissible, then L_1 is a_1-admissible.

THEOREM 3.10. *Suppose that L is an a-admissible semilattice for some $a \in L$. Then C_aD_a is a b-admissible semilattice for all $b \in C_a$.*

Proof. By Theorem 3.7, $g : C_a \times D_a/f \to C_aD_a$, given by $g([b, d]) = bd$, is an isomorphism, where $f : C_a \times D_a \to D_a$ is the corresponding admissible map. By Theorem 3.1, $C_a \times D_a/f$ is $[b, 1]$-admissible for all $b \in C_a$. Now $g([b, 1]) = b$ for each $b \in C_a$. Also, if $b_1 \in C_a$, we have $(b_1 * a) * a = b_1$. Since $a \in \mathscr{A}(L)$, it follows by Lemma 3.5 that $(b_1 * a) * a \in \mathscr{A}(L)$, that is, $b_1 \in \mathscr{A}(L)$. We have, for $b, b_1 \in C_a$, $d \in D_a$ that

$$(bd) * b_1 = g([b, d]) * g([b_1, 1]).$$

Thus

$$g([b, d]) * g([b_1, 1]) = (bd) * b_1 = (bd) * \{(b_1 * a) * a\}$$
$$= \{(bd)(b_1 * a)\} * a = (b_1 * a) * \{(bd) * a\} = (b_1 * a)*(b * a)$$

since $d \in D_a$. Thus

$$g([b, d]) * g([b_1, 1]) = \{b(b_1 * a)\} * a.$$

On the other hand.

$$g([b, d] * [b_1, 1]) = g([b' \vee b_1, 1]) = b' \vee b_1.$$

But in C_a,

$$b' \vee b_1 = \{(b' * a)(b_1 * a)\} * a = \{b(b_1 * a)\} * a$$

since $b' = b * a$ in C_a. Thus g satisfies

$$g([b, d] * [b_1, 1]) = g([b, d]) * g([b_1, 1]) = g([b, d]) * b_1$$

for all $b, b_1 \in C_a$, $d \in D_a$. The proof is completed by applying Lemma 3.9.

COROLLARY 3.11. *Suppose that L is 0-admissible. Then L is also b-admissible for all 0-closed elements b.*

We may, of course, iterate the situation described in Theorem 3.7. That is to say, suppose that L is as described in Theorem 3.7. Then

$$g : C_a \times D_a/f \to C_aD_a$$

is an isomorphism given by $g([b, d]) = bd$. Then by Theorem 3.10, $C_a \times D_a/f$ is $[b, 1]$-admissible for each $b \in C_a$, and C_aD_a is b-admissible for each $b \in C_a$. For each $b \in C_a$, let $C[b, 1]$, $D[b, 1]$ denote the $[b, 1]$-closed and $[b, 1]$-dense elements of $C_a \times D_a/f$ respectively. Let

$$h : C[b, 1] \times D[b, 1] \to D[b, 1]$$

be the corresponding admissible map. Then by Theorem 3.7,

$$C[b, 1] \times D[b, 1]/h \cong C[b, 1]D[b, 1].$$

We recall that $C[b, 1]$ consists of all $[c, 1]$, $c \in C_a$, and $b \leqq c$ and $D[b, 1]$ consists of all $[c, d]$ with $c' \leqq b$. Thus $C[b, 1]D[b, 1]$ consists of all elements $[cc_1, d]$ of $C_a \times D_a/f$ with $c_1' \leqq b \leqq c$, that is, of all elements $[c, d]\ [c_1, 1]$ with $b' \leqq c_1$ and $b \leqq c$. Thus, $C[b, 1]D[b, 1]$ consists of all the products $[c, d]\ [c_1, 1]$ with $[b', 1] \leqq [c_1, 1]$ and $b \leqq c$, that is, $C[b, 1]D[b, 1]$ is the filter of $C_a \times D_a/f$ generated by

$$[b', 1]\ [b, d] = [(b * a)b, d] = [a, d].$$

In case $a = 0$, then for each $b \in C_0$, $C[b, 1] \cong C_0$ via $[a, 1] \leftrightarrow a$, and $D[b, 1]$ consists of all products dx, $d \in D_0$, $x \geqq b' = b^*$, that is, the set of all products $D_0[b') = D_0[b^*)$. Thus

$$C[b, 1]D[b, 1] \cong C_0 D[b^*) = \text{principal filter of } L \text{ generated by } b^*.$$

Thus we have the following result.

THEOREM 3.12. *Let L be an a-admissible semilattice and let $f : C_a \times D_a \to D_a$ be the corresponding admissible map. For each $b \in C_a$, let $C[b, 1]$ be the set of all $[b, 1]$-closed elements of $C_a \times D_a/f$, and let $D[b, 1]$ be the set of all $[b, 1]$-dense elements of $C_a \times D_a/f$. Then $C[b, 1]D[b, 1]$ is the filter of $C_a \times D_a/f$ generated by $[a, d]$ for all $d \in D_a$. In case $a = 0$, then*

$$C[b, 1]D[b, 1] \cong [b^*) \quad \text{for each } b \in C_0.$$

LEMMA 3.13. *Let L be an a-admissible semilattice and let $f : C_a \times D_a \to D_a$ be the corresponding admissible map. If D_a is implicative, then f satisfies*

$$f(b, d_1 * d_2) = f(b, d_1) * f(b, d_2)$$

for all $b \in C_a$, $d_1, d_2 \in D_a$.

Proof.

$$f(b, d_1)f(b, d_1 * d_2) = f(b, d_1(d_1 * d_2)) = f(b, d_1 d_2) \leqq f(b, d_2).$$

Hence, $f(b, d_1 * d_2) \leqq f(b, d_1) * f(b, d_2)$. On the other hand, since

$$b\{f(b, d_1) * f(b, d_2)\} \leqq f(b, d_1) * f(b, d_2),$$

we have

$$bf(b, d_1)\{f(b, d_1) * f(b, d_2)\} \leqq f(b, d_2).$$

Thus

$$bd_1\{f(b, d_1) * f(b, d_2)\} \leqq f(b, d_2)$$

and hence

$$bd_1\{f(b, d_1) * f(b, d_2)\} \leqq d_2.$$

This gives

$$b\{f(b, d_1) * f(b, d_2)\} \leqq d_1 * d_2$$

and hence

$$f(b, d_1) * f(b, d_2) \leqq f(b, d_1 * d_2).$$

THEOREM 3.14. *Let L be an a-admissible semilattice and let $f: C_a \times D_a \to D_a$ be the corresponding admissible map. If D_a is implicative, then $C_a \times D_a/f$ is $[1, d]$-implicative for each $d \in D_a$.*

Proof. Let $d \in D_a$ and let $[a_1, d_1] \in C_a \times D_a/f$. We define

$$[a_1, d_1] * [1, d] = [1, f(a_1, d_1 * d)].$$

This makes $C_a \times D_a/f$ into an $[1, d]$-implicative semilattice. For, let $[a_2, d_2] \in C_a \times D_a/f$. Then

$$\begin{aligned}
[a_2, d_2] \leqq [1, f(a_1, d_1 * d)] & \\
&\Leftrightarrow f(a_2, d_2) \leqq f(a_2a_1, d_1 * d)\\
&\Leftrightarrow f(a_2a_1, d_2) \leqq f(a_2a_1, d_1 * d) = f(a_2a_1, d_1) * f(a_2a_1, d)\\
&\Leftrightarrow f(a_2a_1, d_2)f(a_2a_1, d_1) \leqq f(a_2a_1, d)\\
&\Leftrightarrow f(a_2a_1, d_2d_1) \leqq f(a_2a_1, d)\\
&\Leftrightarrow [a_2a_1, d_2d_1] \leqq [1, d]\\
&\Leftrightarrow [a_2, d_2][a_1, d_1] \leqq [1, d].
\end{aligned}$$

THEOREM 3.15. *Let L be an a-admissible semilattice and let $f: C_a \times D_a \to D_a$ be the corresponding admissible map. If D_a is implicative, then $C_a \times D_a/f$ is implicative.*

Proof. We have seen that $C_a \times D_a/f$ is $[c, 1]$-admissible for all $c \in C_a$, and hence is $[c, 1]$-implicative for all $c \in C_a$. By Theorem 3.14, $C_a \times D_a/f$ is $[1, d]$-implicative for all $d \in D_a$. Since $[c, d] = [c, 1]\,[1, d]$, it follows by Lemma 3.4 that $C_a \times D_a/f$ is $[c, d]$-implicative for all $[c, d] \in C_a \times D_a/f$.

COROLLARY 3.16. *Let L be a 0-admissible semilattice. If D_0 is implicative, then L is implicative.*

Remark. Let L be a-admissible and let $f: C_a \times D_a \to D_a$ be the corresponding admissible map. Suppose that D_a is implicative. Then by

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Theorem 3.15, $C_a \times D_a/f$ is implicative. The implication $*$ is given by

$$\begin{aligned}[a_1, d_1] * [c, d] &= [a_1, d_1] * ([c, 1]\,[1, d]) \\ &= ([a_1, d_1] * [c, 1])([a_1, d_1] * [1, d]) \\ &= [a_1' \vee c, 1]\,[1, f(a_1, d_1 * d)] = [a_1' \vee c, f(a_1, d_1 * d)].\end{aligned}$$

THEOREM 3.17. *Let L be a bounded semilattice. Then L is an implicative semilattice if and only if L is 0-admissible and D_0 is implicative.*

Proof. If L is implicative, then of course L is 0-admissible and D_0 is implicative. The converse follows from Theorem 3.15.

Remark. On a constructive level, we can say the following. Suppose that L is 0-admissible and D_0 is implicative. Let $f: C_0 \times D_0 \to D_0$ be the corresponding admissible map. The implication in L can be described by

$$x * y = (x^* \vee y^{**})f(x^{**}, d * e)$$

where $x = x^{**}d$, $y = y^{**}e$, $x, y \in L$, $d, e \in D_0$. For let $t \in \langle x, y\rangle$, that is, $tx \leqq y$. Then $txy^* \leqq yy^* = 0$. Hence

$$tx^{**}y^* \leqq t^{**}x^{**}(y^*)^{**} = (txy^*)^{**} = 0.$$

Thus $t \leqq (x^{**}y^*)^* = x^* \vee y^{**}$. Also, since $tx \leqq y$, we have

$$tx^{**}d \leqq y^{**}e \leqq e.$$

Hence

$$tx^{**} \leqq d * e \quad \text{and} \quad t \leqq f(x^{**}, d * e).$$

Thus

$$t \leqq (x^* \vee y^{**})f(x^{**}, d * e),$$

that is,

$$\langle x, y\rangle \subset ((x^* \vee y^{**})f(x^{**}, d * e)].$$

On the other hand, suppose that $t \leqq (x^* \vee y^{**})f(x^{**}, d * e)$. Then

$$tx \leqq (x^* \vee y^{**})f(x^{**}, d * e)x.$$

But

$$(x^* \vee y^{**})x \leqq y^{**} \quad \text{for } x^*, y^{**}, x^{**} \in C_0,$$

a Boolean algebra, and

$$(x^* \vee y^{**})x \leqq (x^* \vee y^{**})x^{**} \in C_0,$$

and hence

$$(x^* \vee y^{**})x \leqq x^*x^{**} \vee y^{**}x^{**} \leqq y^{**}.$$

Thus

$$tx \leqq f(x^{**}, d * e)y^{**} \quad \text{and}$$
$$txe \leqq f(x^{**}, d * e)y^{**}e = f(x^{**}, d * e)y \leqq y.$$

But we also have $t \leqq f(x^{**}, d * e)$, and hence $tx^{**} \leqq d * e$. Thus $tx^{**}d \leqq d(d * e) \leqq e$, and hence $tx \leqq e$. But $txe \leqq y$. Hence $tx \leqq txe \leqq y$ and hence $t \in \langle x, y \rangle$. This proves our claim. This argument holds more generally, and in fact, we have the following result.

LEMMA 3.18. *Suppose that L is a-admissible and D_a is implicative. Then $[a)$ is implicative. In fact, for $x, y \geqq a$, we have*

$$x * y = (x^* \vee y^{**})f(x^{**}, d * e)$$

where $f : C_a \times D_a \rightarrow D_a$ is the corresponding admissible map, $x^* = x * a$, $x^{**} = (x * a) * a$, $y^{**} = (y * a) * a$, $x = x^{**}d$, $y = y^{**}e$, where $d, e \in D_a$.

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University of Alberta,
Edmonton, Alberta