ON RIGID UNDIRECTED GRAPHS

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By an *undirected graph* we mean a couple (X, R), where X is a set and R is a subset of $X \times X$ such that $(x, y) \in R$ implies $(y, x) \in R$. The cardinal of X, denoted by $|X|$, will be called the cardinal of the graph.

A mapping $f:X \to X$ is called an *endomorphism* of (X, R) if $(x, y) \in R$ implies that $(f(x), f(y)) \in R$ for all $x, y \in R$.

An undirected graph (X, R) is called *rigid* if there is only one endomorphism of (X, R), namely the identity mapping of X.

P. Erdös communicated orally that, using probability methods, it is possible to prove that almost all finite undirected graphs are rigid.

The aim of this note is to prove the following statement: Let n be a natural number, $n \geqslant 2$. There exists a rigid undirected graph with the cardinality n if and only if $n > 7$.

We remark that there exist rigid undirected graphs with any infinite cardinal, as follows from **(1** and **3)**. Undirected graphs rigid under automorphisms were studied in **(2)**.

The proof of the statement will be done in two sections. First, we prove that there is no rigid undirected graph (X, R) with $2 \leqslant |X| \leqslant 7$. In the second part we construct a rigid undirected graph with any prescribed cardinality greater than 7.

Proof of the necessity. As there are only finitely many non-isomorphic graphs (X, R) such that $2 \leqslant |X| \leqslant 7$, the proof of the necessity can be given by checking all possible cases. Actually, we did not find any other proof. The only improvement is that we can give an approach which makes the checking simple enough.

Further, we shall always assume that all graphs under consideration are undirected. We shall use the following notation and conventions:
If (X, R) is a graph, put

$$R(x) = \{y | (x, y) \in R\}, \qquad N(x) = (R(x), R \cap (R(x) \times R(x))),$$
$$i(x) = |R(x)|, \qquad i(X) = \max \{i(x) | x \in X\}.$$

The expression $f = \{x_1 \to y_1, \ldots, x_m \to y_m\}$ denotes that $f:X \to X$ is a mapping defined by $f(x_i) = y_i$ $(i = 1, 2, \ldots, m)$, $f(x) = x$ otherwise. The symbol $\{x \to y, y \to x, \ldots\}$ is usually abbreviated by $\{x \leftrightarrow y, \ldots\}$. We usually say that x is joined with y if $(x, y) \in R$.

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LEMMA 1. *Let* (X, R) *be a rigid graph* $,|X| = n > 1$. *Then*
(a) $(x, x) \notin R$ *and* $i(x) \neq 0$ *for all* $x \in X$,
(b) *if* $i(x_0) = n - 1$, *then* $N(x_0)$ *is rigid,*
(c) $i(x) \neq n - 2$ *for all* $x \in X$,
(d) $i(x) \neq 1$ *for all* $x \in X$,
(e) $i(X) \neq 2$,
(f) *if* (X, R) *is k-colourable, i.e. it is possible to colour* (X, R) *by k colours, then it does not contain a complete k-vertex subgraph.*

The proof is simple. We prove, for example, assertion (c). Let $i(x_0) = n - 2$. Then there is only one vertex $y \neq x_0$ such that $(x_0, y) \notin R$. Bearing in mind that $(y, y) \notin R$ and $(x_0, x_0) \notin R$, by (a), we readily see that $f = \{y \to x_0\}$ is an endomorphism of the graph, which yields a contradiction.

From Lemma 1 we obtain immediately:

LEMMA 2. *If* $2 \leqslant |X| \leqslant 5$, *then* (X, R) *is not rigid.*

LEMMA 3. *There is no rigid graph* (X, R) *such that* $|X| = 6$.

Proof. Let $|X| = 6$, (X, R) being rigid. By Lemma 1, $i(X) = 3$. Let $x_0 \in X$ be such that $i(x_0) = 3$. We shall consider $N(x_0)$. There are only four non-isomorphic graphs with three vertices, namely the graphs of Figure 1. We may designate the vertices of $R(x_0)$ by u, v, w in such a way thar $N(x_0)$ is one of the graphs I-IV. Let $\{y, z\} = X\backslash(R(x_0) \cup \{x_0\})$. Evidently, $(y, z) \in R$; otherwise $\{y \to x_0\}$ is an endomorphism.

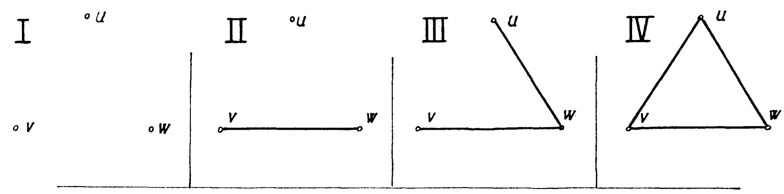

FIGURE 1

In Case IV, $\{y \leftrightarrow z\}$ is an endomorphism, as $i(X) = 3$. In Case III, (X, R) is evidently 3-colourable and contains a three-vertex complete graph. The same holds in Case II. In Case I, it is easy to show that we may redesignate the vertices in $R(x_0)$ in such a way that $\{u \to v\}$ is an endomorphism.

LEMMA 4. *If* $|X| = 7$, $i(X) = 4$, *then* (X, R) *is never rigid.*

Proof. Let $|X| = 7$, (X, R) being rigid, $i(X) = 4$. Let $i(x_0) = 4$. The elements in $R(x_0)$ are denoted by t, u, v, w; the elements in $X\backslash(R(x_0) \cup \{x_0\})$

by y, z. We may denote the vertices in $R(x_0)$ in such a way that $N(x_0)$ is one of the graphs I–XI of Figure 2.

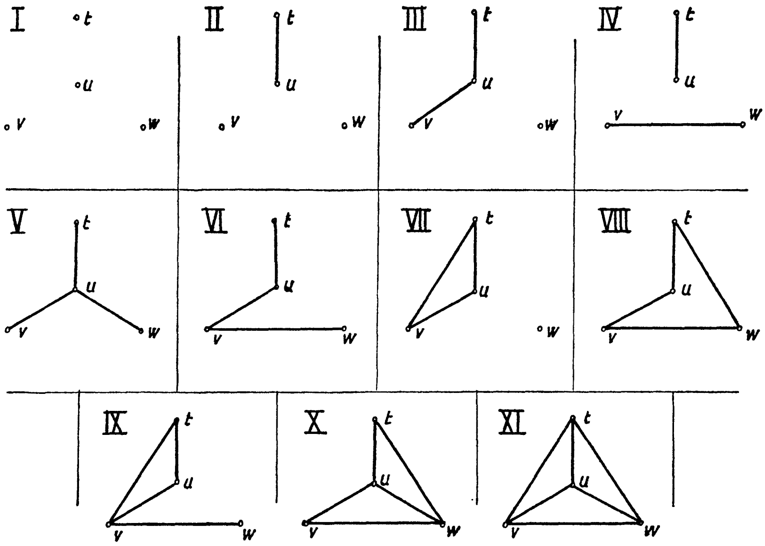

FIGURE 2

Similarly, as in the proof of Lemma 3, $(y, z) \in R$. Cases I and XI cannot occur (see Cases I and IV in the previous lemma). Considering colouring in the cases V, IX, and X we easily obtain a contradiction to Lemma 1 (f).

Case II. Evidently (X, R) is 4-colourable; hence, the vertices t, u, y, z cannot form a complete graph. Thus, we may assume that $(y, u) \notin R$. We define a function χ on X by $\chi(x_0) = \chi(z) = 1$, $\chi(u) = \chi(y) = 2$, $\chi(x) = 3$ otherwise; χ is a colouring and we obtain a contradiction to Lemma 1(f).

In Case VII, u cannot be joined to both y and z, as $i(u) \leqslant 4$. Then we use the same idea as in the previous case. We dispose of Case III similarly.

Case VIII. The graph cannot be 3-colourable by Lemma 1(f). Therefore clearly $i(y) \geqslant 3$ and $i(z) \geqslant 3$. Since $i(X) = 4$, it follows that $i(y) = i(z) = 3$ and hence that each of the vertices t, u, v, w is joined to exactly one of the vertices y, z. It is now easy to see that the graph admits a non-identical automorphism.

Case VI is the most complicated. It is easy to show (considering colouring and symmetries) that we may designate the vertices in such a way that $(y, t) \in R$ and $(y, u) \in R$. Further, considering colouring, it is possible to show that $(z, w) \in R$—using now the same designation—and that either $(v, z) \in R$ or $(t, z) \in R$.

First, assume that $(v, z) \in R$. If the graph has only the edges described so far, $\{y \leftrightarrow z, t \leftrightarrow w, u \leftrightarrow v\}$ is an automorphism. Hence, R must contain at least one other edge. Considering $i(u)$ and $i(v)$, we see that there are only two possible edges, namely (t, z) and (y, w). If both of them belong to R, then the described mapping remains an automorphism. By symmetry, we may assume that $(t, z) \in R$. Then $\{x_0 \leftrightarrow u, t \leftrightarrow v, w \leftrightarrow y\}$ is an automorphism.

If $(v, z) \notin R$, it is easy to exclude all possibilities in a similar way.

It remains to investigate Case IV. First, considering colouring, we can show that either $(t, y) \in R$, $(u, y) \in R$ or $(v, y) \in R$, $(w, y) \in R$. The same must hold if y is replaced by z. We may assume that $(t, y) \in R$, $(u, y) \in R$. If $(t, z) \in R$ and $(u, z) \in R$, $i(t) = 4$ and $N(t)$ is isomorphic to IX. Hence, $(v, z) \in R$, $(w, z) \in R$. If no other edges are present, the graph is not rigid. Thus, we may assume that $(t, z) \in R$ and $(u, z) \notin R$. Then $i(t) = 4$ and $N(t)$ is isomorphic to VI. The proof is complete.

LEMMA 5. *If* $|X| = 7$, $i(X) = 3$, *then* (X, R) *is not rigid.*

Proof. If $i(X) = 3$, then there exists a vertex $x_0 \in X$ such that $i(x_0) \leqslant 2$. By Lemma 1, $i(x_0) = 2$.

The proof will be divided into sections according to the minimal length of a cycle containing x_0 (if such a cycle exists).

Let $(x_0, y) \in R$, $(x_0, z) \in R$, $z \neq y$. First, let $(y, z) \in R$. Considering $i(y)$ and $\{x_0 \leftrightarrow y\}$, we see that there is exactly one $t, t \neq x_0, t \neq z$ such that $(y, t) \in R$. Similarly, we find a vertex u for z. If $u = t$, then $\{y \leftrightarrow z\}$ is an automorphism. If $u \neq t$, $(u, t) \notin R$, the graph is 3-colourable. If $(u, t) \in R$, the graph is evidently not rigid.

If (x_0, y, t, z) is the minimal cycle containing x_0, $\{x_0 \rightarrow t\}$ is an endomorphism.

Let (x_0, y, t, u, z) be a minimal cycle containing x_0, the remaining vertices being v, w. If $(v, w) \notin R$, then one of the previous cases occurs, where v or w plays the role of x_0. Hence, $(v, w) \in R$. Now, we can easily investigate all the remaining possibilities. (Note that the minimality of (x_0, y, t, u, z) implies that (y, u), (y, z), (t, z) cannot belong to R, that (v, y), (v, z) cannot belong to R, and that (w, y), (w, z) cannot both belong to R.)

Let the minimal cycle containing x_0 consist of six vertices. Let w be the remaining vertex. Evidently, the graph is 3-colourable. Hence, w cannot be joined with two joined vertices. Consequently, $i(w) = 2$. Replacing x_0 by w, we obtain one of the previous cases.

If either the minimal cycle is of length seven or if there is no cycle containing x_0, then the graph (X, R) is evidently not rigid. The proof is complete.

The preceding lemmas imply:

THEOREM 1. *There is no rigid undirected graph* (X, R) *such that* $2 \leqslant |X| \leqslant 7$.

Proof of the sufficiency.

THEOREM 2. *If* n *is a natural number*, $n \geqslant 8$, *then there exists a rigid undirected graph* (X, R) *such that* $|X| = n$.

Proof. The proof will be based on properties of endomorphisms of cycles in undirected graphs without loops **(1)**.

Let $p = \frac{1}{2}(n - 4)$ if n is even and let $p = \frac{1}{2}(n - 5)$ if n is odd; thus $p \geqslant 2$. Put $C = \{0, 1, 2, \ldots, 2p\}$. The elements of C will be considered as representatives of residue classes (mod $2p + 1$). The symbols $i + j$, $-i$, etc., should be understood in this sense. Put $X = C \cup \{a, b, c\}$ if n is even and

$$X = C \cup \{a, b, c, d\}$$

if n is odd, where a, b, c, d are different elements, which are not contained in C. Let R be a symmetric relation generated by: $(i, i + 1) \in R$, $(c, i) \in R$ for all $i \in C$, $(a, 0) \in R$, $(a, 2) \in R$, $(a, 3) \in R$, $(a, b) \in R$, $(b, 0) \in R$, $(b, 1) \in R$ (moreover, $(d, a) \in R$, $(d, b) \in R$, $(d, 2) \in R$ if n is odd); see Figure 3.

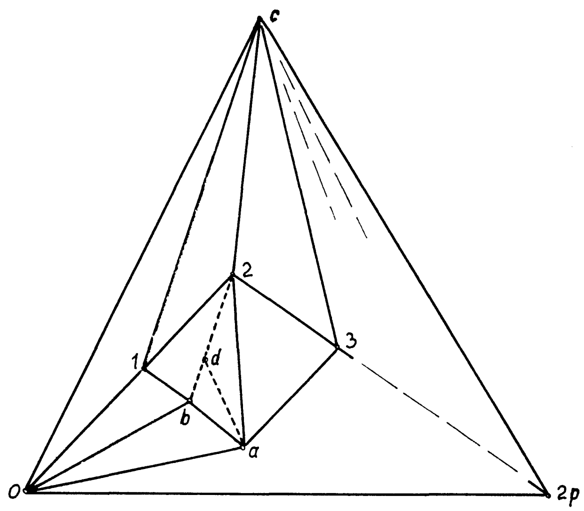

FIGURE 3

We are going to prove that (X, R) is rigid. Let f be an endomorphism of (X, R). Thus $f(R(x)) \subset R(f(x))$ for all $x \in X$. There is only one $x \in X$ such that $R(x)$ contains the set of vertices of a proper cycle of odd length, namely $x = c$. We have $R(c) = C$ and C is the set of vertices of a proper cycle of odd length. Moreover, no proper subset of C has this property. As the length is odd **(1)** and $f(R(c)) \subset R(f(c))$, we obtain $f(c) = c, f(C) = C$, and there is a $k \in C$ such that either $f(i) = k + i$ for all $i \in C$ or $f(i) = k - i$ for all $i \in C$. If $f(a) \in C$ ($f(b) \in C$), we have $f(\{a, 2, 3\}) \subset C$ ($f(\{0, 1, b\}) \subset C$), which is not possible owing to the length of the cycles. Consequently, $f(a) \neq c$, $f(b) \neq c$. If $f(a) = b$ (or $f(a) = d$ if n is odd), we have

$$f(\{0, 2, 3\}) = f(R(a) \cap C) \subset R(b) \cap C = \{0, 1\}$$

(or ... $\subset R(d) \cap C = \{2\}$). As f is 1-1 on C, we obtain a contradiction. Similarly, we can show that $f(b) \neq d$. Hence, $f(a) = a$. As $(a, b) \in R$, we have $f(b) \neq a$ and, hence, $f(b) = b$.

Further, $f(0) = 0$ because $R(a) \cap R(b) \cap C = \{0\}$. Thus, either $f(i) = i$ or $f(i) = -i$ for all $i \in C$. $f(i) = -i$ implies that $f(1) = 2p$, but $(1, b) \in R$, and $(2p, b) \notin R$. We infer that $f(i) = i$ for all $i \in C$.

If n is odd, $\{d\} = R(a) \cap R(b) \cap R(2)$ and we obtain $f(d) = d$. Hence, (X, R) is rigid. If n is even, $|X| = 2p + 1 + 3 = n$ and, if n is odd,

$$|X| = 2p + 1 + 4 = n.$$

COROLLARY *Let X be a set. The following assertions are equivalent:*

(1) *there exists a symmetric relation R on X such that (X, R) is rigid,*

(2) *either $|X| = 1$ or $|X| \geqslant 8$.*

Proof. If $|X|$ is finite, the statement is an immediate consequence of the preceding theorems. By **(1** and **3)**, the statement holds for any infinite $|X|$.

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