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# A FUNCTIONAL EQUATION ARISING FROM IVORY'S THEOREM IN GEOMETRY

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1. **Introduction.** In previous papers (see [1, 2, 3, 4]), we solved the following functional equation:

$$|f(x+y)-f(x-y)| = |f(x+\bar{y})-f(x-\bar{y})|, \tag{1}$$

where $f=f(z)$ is an entire function of a complex variable $z$ and $x$, $y$ are complex variables.

The following theorem was proved in these papers [1, 2, 3, 4]:

THEOREM A. *The only solutions of* (1) *are*

$$f(z) = a \sin \alpha z+b \cos \alpha z+c,$$

*and*

$$f(z) = a \sinh \alpha z+b \cosh \alpha z+c,$$

*and*

$$f(z) = az^2+bz+c,$$

*where $a$, $b$, $c$ are arbitrary complex constants and $\alpha$ is an arbitrary real constant.*

Equation (1) is closely related to the following Ivory's Theorem (see [1, 2, 3, 4, 6]) in geometry:

For a family of confocal conics, let $P$, $Q$, $R$, $S$ be the four vertices of a curviinear rectangle formed by any four members of this family arbitrarily chosen. Then $\overline{PR}=\overline{QS}$ holds.

Geometrically speaking, (1) is the above Ivory's property and the three solutions of (1) characterize the confocal conics.

If we put $x=y=\frac{1}{2}(s+it)$ in (1) where $s$, $t$ are real variables and put $g(z)=f(z)-f(0)$, then we have

$$|g(s+it)| = |g(s)-g(it)|, \tag{2}$$

where $g=g(z)$ is an entire function of $z$ and $s$, $t$ are real variables.

The following theorem was proved in [2]:

THEOREM B. *The only solutions of* (2) *are*

$$g(z) = a \sin \alpha z+b \cos \alpha z-b,$$

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*and*

$$g(z) = a \sinh \alpha z + b \cosh \alpha z - b,$$

*and*

$$g(z) = az^2 + bz,$$

*where $a$, $b$ are arbitrary complex constants and $\alpha$ is an arbitrary real constant.*

Now, we shall consider the following functional equation:

$$g(x+y)h(x-y) = (g(x)-g(y))(h(x)-h(-y)), \tag{3}$$

where $g=g(z)$, $h=h(z)$ are unknown entire functions of $z$ and $x$, $y$ are complex variables.

In Section 2 we shall prove that (3) is a generalization of (2). It is obvious that (3) is also a generalization of the following functional equation (see [5]) which is a generalization of sine functional equations:

$$f(x+y)f(x-y) = (f(x)-f(y))(f(x)-f(-y)), \tag{4}$$

where $f=f(z)$ is an entire function of $z$ and $x$, $y$ are complex variables.

The following theorem was proved in [5]:

THEOREM C. *The only solutions of* (4) *are*

$$f(z) = a \sin 2pz + b \sin^2 pz,$$

*and*

$$f(z) = az + bz^2,$$

*where $a$, $b$, $p$ are arbitrary complex constants.*

The purpose of this paper is to solve (3), in Section 4, with the help of Theorem C, i.e., to prove the following theorem and moreover, to prove Theorems A and B, in Section 5, by using it.

THEOREM. *The only systems of solutions of* (3) *are*

(i) $g(z) \equiv 0, \qquad h(z) = \text{arbitrary},$

*and*

(ii) $g(z) = \text{arbitrary}, \qquad h(z) \equiv 0,$

*and*

(iii) $g(z) = a \sin 2pz + b \sin^2 pz, \qquad h(z) = c \sin 2pz + d \sin^2 pz,$

*and*

(iv) $g(z) = az + bz^2, \qquad h(z) = cz + dz^2,$

*where $a$, $b$, $c$, $d$, $p$ are arbitrary complex constants.*

2. **Proof that (2) implies (3).** Squaring both sides of (2) and using the formula $|\lambda|^2 = \lambda\bar{\lambda}$ where $\lambda$ is an arbitrary complex number yields

$$g(s+it)\overline{g(s+it)} = (g(s)-g(it))(\overline{g(s)}-\overline{g(it)}). \tag{5}$$

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If we put $h(z)=\overline{g(\bar{z})}$, then $h=h(z)$ is an entire function of $z$. Furthermore, by (5) we have for all real $s$, $t$

$$g(s+it)h(s-it) = (g(s)-g(it))(h(s)-h(-it)). \tag{6}$$

By (6) and by the Identity Theorem we have for all complex $x$, $y$

$$g(x+y)h(x-y) = (g(x)-g(y))(h(x)-h(-y)).$$

Hence we conclude that (2) implies (3) with $h(z)=\overline{g(\bar{z})}$ which is an entire function of $z$.

3. **Preliminary considerations.** We split each of the functions $g(z)$ and $h(z)$ into an even and odd part:

$$g_0(z) = \tfrac{1}{2}(g(z)+g(-z)),\ g_1(z) = \tfrac{1}{2}(g(z)-g(-z)), \tag{7}$$

$$h_0(z) = \tfrac{1}{2}(h(z)+h(-z)),\ h_1(z) = \tfrac{1}{2}(h(z)-h(-z)), \tag{8}$$

with

$$g(z) = g_0(z)+g_1(z),\ h(z) = h_0(z)+h_1(z). \tag{9}$$

We shall prove that $g_1(z)$, $h_0(z)$ and $g_0(z)$, $h_1(z)$ and $g_1(z)$, $h_1(z)$ and $g_0(z)$, $h_0(z)$ are four systems of solutions of (3), i.e., for all complex $x$, $y$,

$$g_1(x+y)h_0(x-y) = (g_1(x)-g_1(y))(h_0(x)-h_0(-y)), \tag{10}$$

$$g_0(x+y)h_1(x-y) = (g_0(x)-g_0(y))(h_1(x)-h_1(-y)), \tag{11}$$

$$g_1(x+y)h_1(x-y) = (g_1(x)-g_1(y))(h_1(x)-h_1(-y)), \tag{12}$$

$$g_0(x+y)h_0(x-y) = (g_0(x)-g_0(y))(h_0(x)-h_0(-y)). \tag{13}$$

We shall give a proof of (10) only, because (11), (12), (13) are similarly proved.

By (7), (8) we have

$$\begin{aligned} g_1(x+y)h_0(x-y) &= \tfrac{1}{4}(g(x+y)-g(-x-y))(h(x-y)+h(-x+y)) \\ &= \tfrac{1}{4}(g(x+y)h(x-y)+g(x+y)h(-x+y) \\ &\quad -g(-x-y)h(x-y)-g(-x-y)h(-x+y)). \end{aligned} \tag{14}$$

By (3) we have

$$g(x+y)h(x-y) = (g(x)-g(y))(h(x)-h(-y)), \tag{15}$$

$$g(x+y)h(-x+y) = (g(y)-g(x))(h(y)-h(-x)), \tag{16}$$

$$g(-x-y)h(x-y) = (g(-y)-g(-x))(h(-y)-h(x)), \tag{17}$$

$$g(-x-y)h(-x+y) = (g(-x)-g(-y))(h(-x)-h(y)). \tag{18}$$

Substituting (15), (16), (17), (18) into the right-hand side of (14) and rearranging the resulting expression yields

$$\begin{aligned} g_1(x+y)h_0(x-y) = (\tfrac{1}{2}(g(x)-g(-x))-\tfrac{1}{2}(g(y) \\ -g(-y)))(\tfrac{1}{2}(h(x)+h(-x))-\tfrac{1}{2}(h(-y)+h(y))), \end{aligned}$$

and so, (10).

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4. **Proof of the theorem.** Putting $y=-x$ in (3) yields

$$h(x+y)g(x-y) = (h(x)-h(y))(g(x)-g(-y)). \tag{19}$$

We may assume that $g(z)\not\equiv 0$ and $h(z)\not\equiv 0$. Putting $y=x$ in (3) and (19) and taking into account the facts that $g(z)\not\equiv 0$ and $h(z)\not\equiv 0$ yields

$$g(0) = 0,\ h(0) = 0. \tag{20}$$

We discuss nine cases (Case A–Case I).

Case A. Suppose that $g$ is odd and $h$ is odd. (Then we have $g(z)\equiv g_1(z)$ and $h(z)\equiv h_1(z)$.)

Differentiating both sides of (3) with respect to $y$, putting $y=0$ and using (20) yields

$$g'(x)h(x)-g(x)h'(x) = -g'(0)h(x)+h'(0)g(x). \tag{21}$$

Differentiating both sides of (3) with respect to $y$, putting $y=x$ and using (20) and the oddness of $h$ yields

$$h'(0)g(2x) = 2g'(x)h(x). \tag{22}$$

Differentiating both sides of (3) with respect to $y$, putting $y=-x$, and using (20) and the oddness of $g$ yields

$$g'(0)h(2x) = 2g(x)h'(x). \tag{23}$$

By (21), (22), (23) we have

$$h'(0)g(2x)-g'(0)h(2x) = 2(h'(0)g(x)-g'(0)h(x)). \tag{24}$$

If we put

$$P(x) = h'(0)g(x)-g'(0)h(x), \tag{25}$$

then $P(x)$ is an entire function of $x$ and moreover, by (24), we have

$$P(2x) = 2P(x). \tag{26}$$

By using the power series expansion of $P(x)$ and by (26) we see that

$$P(x) = Kx, \tag{27}$$

where $K$ is a complex constant.

Differentiating both sides of (25) and putting $x=0$ yields

$$P'(0) = 0,$$

and so, by (27), we have $K=0$. Hence, by (25), (27) we have

$$h'(0)g(x)-g'(0)h(x) \equiv 0. \tag{28}$$

By our assumption we have $g(z)\not\equiv 0$ and $h(z)\not\equiv 0$. By (21), (28) the Wronskian $W(g, h)(z)=g(z)h'(z)-g'(z)h(z)$ is 0 in $|z|<+\infty$. Hence we see that

$$h(z) = Ag(z) \tag{29}$$

holds in $|z|<+\infty$, $A$ being a non-zero complex constant.

Substituting (29) into (3), dividing both sides of the resulting equality by $A(\neq 0)$ and using the oddness of $g$ yields

$$g(x+y)g(x-y) = (g(x)-g(y))(g(x)+g(y)).$$

This equation is the so-called sine functional equation. Since $g$ is odd, $g(z)$ is a solution of (4). Selecting odd functions from the solutions of (4) (see Theorem C) and using (29) yields

$$(30) \qquad g(z) = a \sin 2pz, \qquad h(z) = c \sin 2pz,$$

or

$$(31) \qquad g(z) = az, \qquad h(z) = cz,$$

where $a$, $c(=Aa)$, $p$ are non-zero complex constants.

Case B. Suppose that $g$ is even and $h$ is even. (Then we have $g(z)\equiv g_0(z)$ and $h(z)\equiv h_0(z)$.)

Since $g$, $h$ are even by our assumption, we have

$$(32) \qquad g'(0) = 0, \qquad h'(0) = 0.$$

Differentiating both sides of (3) with respect to $y$, putting $y=0$ and using (32) yields

$$(33) \qquad g'(x)h(x)-g(x)h'(x) \equiv 0.$$

By our assumption we have $g(z)\not\equiv 0$ and $h(z)\not\equiv 0$. By (33) the Wronskian $W(g, h)(z)=g(z)h'(z)-g'(z)h(z)$ is 0 in $|z|<+\infty$. Hence we see that

$$(34) \qquad h(z) = Bg(z)$$

holds in $|z|<+\infty$, $B$ being a non-zero complex constant.

Substituting (34) into (3), dividing both sides of the resulting equality by $B(\neq 0)$ and using the evenness of $g$ yields

$$g(x+y)g(x-y) = (g(x)-g(y))^2.$$

Since $g$ is even, $g(z)$ is a solution of (4). Selecting even functions from the solutions of (4) and using (34) yields

$$(35) \qquad g(z) = b \sin^2 pz, \qquad h(z) = d \sin^2 pz,$$

or

$$(36) \qquad g(z) = bz^2, \qquad h(z) = dz^2,$$

where $b$, $d(=Bb)$, $p$ are non-zero complex constants.

Case C. Suppose that $g$ is odd and $h$ is even. (Then we have $g(z)\equiv g_1(z)$ and $h(z)\equiv h_0(z)$.)

Since $g$ is odd and $h$ is even by our assumption, we have

$$(37) \qquad g''(0) = 0, \qquad h'(0) = 0.$$

Differentiating both sides of (3) with respect to $y$, putting $y=0$ and using (20), (37) yields

(38) $$g'(x)h(x)-g(x)h'(x) = -g'(0)h(x).$$

Differentiating both sides of (3) twice with respect to $y$, putting $y=0$ and using (20), (37) yields

(39) $$g''(x)h(x)-2g'(x)h'(x)+g(x)h''(x) = -h''(0)g(x).$$

Differentiating both sides of (3) twice with respect to $y$, putting $y=x$ and using (20), (37) and the evenness of $h(h'$ odd) yields

(40) $$h''(0)g(2x) = 2g'(x)h'(x).$$

Differentiating both sides of (3) with respect to $y$, putting $y=-x$ and using (20) and the oddness of $g$ yields

(41) $$g'(0)h(2x) = 2g(x)h'(x).$$

Differentiating both sides of (38) yields

(42) $$g''(x)h(x)-g(x)h''(x) = -g'(0)h'(x).$$

Subtracting (39) from (42) side by side yields

(43) $$2g'(x)h'(x)-2g(x)h''(x) = -g'(0)h'(x)+h''(0)g(x).$$

Differentiating both sides of (41) yields

(44) $$2g(x)h''(x) = 2g'(0)h'(2x)-2g'(x)h'(x).$$

Substituting (44) into (43) yields

(45) $$4g'(x)h'(x)-2g'(0)h'(2x) = -g'(0)h'(x)+h''(0)g(x).$$

By (40), (45) we have

(46) $$2(h''(0)g(2x)-g'(0)h'(2x)) = h''(0)g(x)-g'(0)h'(x).$$

If we put

(47) $$Q(x) = h''(0)g(x)-g'(0)h'(x),$$

then $Q(x)$ is an entire function of $x$ and moreover, by (46), we have

(48) $$2Q(2x) = Q(x).$$

By using the power series expansion of $Q(x)$ and by (48) we see that for all complex $x$

$$Q(x) = 0.$$

Hence, by (47) we have for all complex $x$

(49) $$h''(0)g(x)-g'(0)h'(x) = 0.$$

By our assumption we have $g(z)\not\equiv 0$. Since by our assumption we have $h(z)\not\equiv 0$ and by (20) we have $h(0)=0$, we see that $h'(z)\not\equiv 0$. By (43), (49) the Wronskian $W(g, h')(z)=g(z)h''(z)-g'(z)h'(z)$ is 0 in $|z|<+\infty$. Hence we see that

$$g(z) = Ch'(z) \tag{50}$$

holds in $|z|<+\infty$, $C$ being a non-zero complex constant.

Substituting (50) into (38) and using $C\neq 0$ yields

$$h''(x)h(x)-h'(x)^2 = (-g'(0)/C)h(x). \tag{51}$$

Since $h(x)\not\equiv 0$ by our assumption, there exist a circular neighbourhood $N$ and a regular function $k(x)$ in $N$ such that

$$h(x) = k(x)^2 \qquad (\neq 0) \tag{52}$$

holds in $N$.

Substituting (52) into (51) and using the fact that $k(x)\neq 0$ in $N$, we have in $N$

$$2k''(x)k(x)-2k'(x)^2 = -g'(0)/C. \tag{53}$$

Differentiating both sides of (53) and using the fact that $k(x)\neq 0$ in $N$, we have in $N$

$$(k''(x)/k(x))' = 0,$$

or

$$k''(x) = Dk(x), \tag{54}$$

where $D$ is a complex constant.

Solving (54), observing (52), (50) and taking (20) into account yields

$$g(z) = a \sin 2pz, \qquad h(z) = d \sin^2 pz, \tag{55}$$

or

$$g(z) = az, \qquad h(z) = dz^2, \tag{56}$$

where $a$, $d$, $p$ are non-zero complex constants.

Case D. Suppose that $g$ is even and $h$ is odd. (Then we have $g(z)\equiv g_0(z)$ and $h(z)\equiv h_1(z)$.)

Observing (19) and interchanging the roles of $g$ and $h$ in Case C yields

$$g(z) = b \sin^2 pz, \qquad h(z) = c \sin 2pz, \tag{57}$$

or

$$g(z) = bz^2, \qquad h(z) = cz, \tag{58}$$

where $b$, $c$, $p$ are non-zero complex constants.

Case E. Suppose that $g$ is neither odd nor even and $h$ is neither odd nor even. (Then we have $g_1(z)\not\equiv 0$, $g_0(z)\not\equiv 0$, $h_1(z)\not\equiv 0$, $h_0(z)\not\equiv 0$.)

Since $g_1(z)(\not\equiv 0)$ is odd and $h_0(z)(\not\equiv 0)$ is even and (10) holds for all complex

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$x, y$, by the result obtained above in Case C we have

$$(59) \qquad g_1(z) = a \sin 2pz, \qquad h_0(z) = d \sin^2 pz,$$

or

$$(60) \qquad g_1(z) = az, \qquad h_0(z) = dz^2,$$

where $a$, $d$, $p$ are non-zero complex constants.

Since $g_1(z)(\not\equiv 0)$, $h_1(z)(\not\equiv 0)$ are odd and (12) holds for all complex $x, y$, by the result obtained above in Case A $h_1(z)$ is a non-zero constant multiple of $g_1(z)$. Hence, by (59), (60) we have

$$(61) \qquad g_1(z) = a \sin 2pz, \qquad h_1(z) = c \sin 2pz,$$

or

$$(62) \qquad g_1(z) = az, \qquad h_1(z) = cz,$$

where $c$ is a non-zero complex constant.

Since $g_0(z)(\not\equiv 0)$, $h_0(z)(\not\equiv 0)$ are even and (13) holds for all complex $x, y$, by the result obtained above in Case B $g_0(z)$ is a non-zero constant multiple of $h_0(z)$. Hence, by (59), (60) we have

$$(63) \qquad g_0(z) = b \sin^2 pz, \qquad h_0(z) = d \sin^2 pz,$$

or

$$(64) \qquad g_0(z) = bz^2, \qquad h_0(z) = dz^2,$$

where $b$ is a non-zero complex constant.

By (9), (61), (62), (63), (64) we have in this case

$$(65) \qquad g(z) = a \sin 2pz + b \sin^2 pz, \qquad h(z) = c \sin 2pz + d \sin^2 pz,$$

or

$$(66) \qquad g(z) = az + bz^2, \qquad h(z) = cz + dz^2,$$

where $a$, $b$, $c$, $d$, $p$ are non-zero complex constants.

Case F. Suppose that $g$ is odd and $h$ is neither odd nor even. (Then we have $g(z) \equiv g_1(z)$, $h_1(z) \not\equiv 0$, $h_0(z) \not\equiv 0$.)

By a similar argument to that in Case E (use (10), (12)) we have

$$(67) \qquad g(z) = a \sin 2pz, \qquad h(z) = c \sin 2pz + d \sin^2 pz,$$

or

$$(68) \qquad g(z) = az, \qquad h(z) = cz + dz^2,$$

where $a$, $c$, $d$, $p$ are non-zero complex constants.

Case G. Suppose that $g$ is neither odd nor even and $h$ is odd. (Then we have $g_1(z) \not\equiv 0$, $g_0(z) \not\equiv 0$, $h(z) \equiv h_1(z)$.)

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By a similar argument to that in Case E (use (11), (12)) we have

(69) $$g(z) = a \sin 2pz + b \sin^2 pz, \qquad h(z) = c \sin 2pz,$$

or

(70) $$g(z) = az + bz^2, \qquad h(z) = cz,$$

where $a, b, c, p$ are non-zero complex constants.

Case H. Suppose that $g$ is even and $h$ is neither odd nor even. (Then we have $g(z) \equiv g_0(z)$, $h_1(z) \not\equiv 0$, $h_0(z) \not\equiv 0$.)

By a similar argument to that in Case E (use (11), (13)) we have

(71) $$g(z) = b \sin^2 pz, \qquad h(z) = c \sin 2pz + d \sin^2 pz,$$

or

(72) $$g(z) = bz^2, \qquad h(z) = cz + dz^2,$$

where $b, c, d, p$ are non-zero complex constants.

Case I. Suppose that $g$ is neither odd nor even and $h$ is even. (Then we have $g_1(z) \not\equiv 0$, $g_0(z) \not\equiv 0$, $h(z) \equiv h_0(z)$.)

By a similar argument to that in Case E (use (10), (13)) we have

(73) $$g(z) = a \sin 2pz + b \sin^2 pz, \qquad h(z) = d \sin^2 pz,$$

or

(74) $$g(z) = az + bz^2, \qquad h(z) = dz^2,$$

where $a, b, d, p$ are non-zero complex constants.

Summing up, by all results obtained in Case A–Case I ((30), (31); (35), (36); (55), (56); (57), (58); (65), (66); (67), (68); (69), (70); (71), (72); (73), (74)) we have

(75) $$g(z) = a \sin 2pz + b \sin^2 pz, \qquad h(z) = c \sin 2pz + d \sin^2 pz,$$

or

(76) $$g(z) = az + bz^2, \qquad h(z) = cz + dz^2,$$

where $a, b, c, d, p$ are complex constants with $|a|+|b|>0$, $|c|+|d|>0$, $p \neq 0$.

Direct substitution shows that (75), (76) satisfy our original equation (3). Since all possible cases were taken into consideration, the theorem is proved.

## 5. Proofs of Theorems A and B.

**Proof of Theorem B.** In Section 2 we proved that (2) implies (3) with

(77) $$h(z) = \overline{g(\bar{z})}.$$

We discuss two cases.

Case 1. $g(z) = a \sin 2pz + b \sin^2 pz$, $h(z) = c \sin 2pz + d \sin^2 pz$ where $a, b, c, d, p$ are complex constants; then by (77) we have

(78) $$c \sin 2pz + d \sin^2 pz = \bar{a} \sin 2\bar{p}z + \bar{b} \sin^2 \bar{p}z.$$

Differentiating both sides of (78) $j$ times $(j=1, 2, 3, 4)$ and putting $z=0$ yields in turn

$$cp = \bar{a}\bar{p}, \tag{79}$$

$$dp^2 = \bar{b}\bar{p}^2, \tag{80}$$

$$cp^3 = \bar{a}\bar{p}^3, \tag{81}$$

$$dp^4 = \bar{b}\bar{p}^4. \tag{82}$$

We may assume that

$$|a|+|b|+|c|+|d| > 0. \tag{83}$$

By (79), (80), (81), (82), (83) we have

$$p^2 = \bar{p}^2.$$

Hence $p$ is real or purely imaginary. Hence, by using the formula $\sin^2 z = \frac{1}{2}(1-\cos 2z)$ and changing letters denoting constants we see that $g(z)$ must be of the form

$$g(z) = a \sin \alpha z + b \cos \alpha z - b, \tag{84}$$

or

$$g(z) = a \sinh \alpha z + b \cosh \alpha z - b, \tag{85}$$

where $a$, $b$ are complex constants and $\alpha$ is a real constant.

Direct substitution shows that (84), (85) satisfy (2).

Case 2. $g(z)=az+bz^2$, $h(z)=cz+dz^2$ where $a$, $b$, $c$, $d$ are complex constants; then direct substitution shows that $g(z)=az+bz^2$ satisfies (2). Interchanging $a$, $b$ yields a solution $g(z)=az^2+bz$ of (2). (See Theorem B.)

Since all possible cases were taken into consideration, Theorem B is proved.

By the relation between (1) and (2) which is shown in Section 1, Theorem A readily follows from Theorem B.

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