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On Strongly Convex Indicatrices in Minkowski Geometry

Min Ji and Zhongmin Shen

Abstract. The geometry of indicatrices is the foundation of Minkowski geometry. A strongly convex indicatrix in a vector space is a strongly convex hypersurface. It admits a Riemannian metric and has a distinguished invariant—(Cartan) torsion. We prove the existence of non-trivial strongly convex indicatrices with vanishing mean torsion and discuss the relationship between the mean torsion and the Riemannian curvature tensor for indicatrices of Randers type.

1 Introduction

An indicatrix Σ in a vector space $\mathbf{V}^{n+1}$ is an embedded C^∞ hypersurface such that every ray issuing from the origin intersects Σ at most one point. To study the geometric properties of Σ, we consider the open cone over Σ,

$$\mathcal{C}(\Sigma) := \{\lambda y; \lambda > 0, y \in \Sigma\}.$$

The *defining function* L of Σ is the positive function on $\mathcal{C}(\Sigma)$ with $L(\lambda y) = \lambda^2 L(y)$, $\forall \lambda > 0$ such that $L^{-1}(1) = \Sigma$. Differentiating L yields a family of bilinear forms on $\mathbf{V}^{n+1}$, $g = \{g_y\}_{y \in \mathcal{C}(\Sigma)}$,

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [L(y + su + tv)]\Big|_{s=t=0}. \tag{1}$$

Σ is said to be *strongly convex* (resp. *non-degenerate*) if g_y is positive definite (resp. non-degenerate) for any $y \in \Sigma$. If a strongly convex indicatrix Σ is closed (compact without boundary) so that $\mathcal{C}(\Sigma) = \mathbf{V}^{n+1} - \{0\}$, then $\|y\| := \sqrt{L(y)}$ is a (non-reversible) norm on $\mathbf{V}^{n+1}$. Such a norm is called a *Minkowski norm* in Minkowski geometry. One is referred to [Tho] for a systematic study on classical Minkowski geometry.

A Finsler manifold is a manifold whose tangent spaces carry a norm varying smoothly with the base point. The length of a curve in the manifold is defined by the integral of the norm of its tangent vectors. Thus, the geometry of indicatrices is the foundation of Finsler geometry [BCS].

Given a strongly convex indicatrix Σ in $\mathbf{V}^{n+1}$. Via the natural identification $T_y\mathcal{C}(\Sigma) = \mathbf{V}^{n+1}$, g induces a Riemannian metric $\hat{g}$ on $\mathcal{C}(\Sigma)$ and hence a Riemannian metric $\bar{g} := \hat{g}|_\Sigma$ on Σ. Therefore, every strongly convex indicatrix admits a

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standard Riemannian metric. Besides the Riemannian invariants, there are other two important geometric invariants: torsion C and distortion τ (see (5) and (11)). A simple fact is that the torsion $C = 0$ if and only if

$$\Sigma = \{y = y^i e_i \mid \sqrt{a_{ij} y^i y^j} = 1\}$$

where (a_{ij}) is a positive definite matrix. Such Σ is said to be *quadratic*.

There are many interesting indicatrices in a vector space. An interesting indicatrix is constructed by G. Asanov in his Finslerian generalization of relativity theories [As]. Let $(\mathbf{V}^n, |\cdot|)$ be an Euclidean space.

$$\Sigma_\lambda := \left\{ (\rho, y) \in \mathrm{R} \times \mathbf{V}^n, |\rho| \varphi\left(\frac{|y|}{|\rho|}\right) = 1, \rho \neq 0 \right\}, \tag{2}$$

where $\varphi(\xi) := \sqrt{\xi^2 + 2\lambda\xi + 1}\exp[-\frac{\lambda}{\sqrt{1-\lambda^2}}\tan^{-1}(\frac{\sqrt{1-\lambda^2}\xi}{\lambda\xi+1})]$ and $|\lambda| < 1$. Asanov [As] shows that the induced Riemannian metric on $\Sigma_\lambda \subset \mathrm{R} \times \mathbf{V}^n$ has constant curvature $K = 1 - \lambda^2$. Note that Σ_λ consists of two identical hypersurfaces sharing a common boundary in the hyperplane $\{0\} \times \mathbf{V}^n$.

Let S^n denote the unit sphere in an Euclidean space $(\mathbf{V}^{n+1}, |\cdot|)$. For any vector $\mathbf{v} \in \mathbf{V}^{n+1}$ with $|\mathbf{v}| < 1$, the shifted unit sphere $\mathrm{S}^n_{\mathbf{v}} := \mathrm{S}^n - \{\mathbf{v}\}$ is also an indicatrix. Randers studied a special class of non-reversible norms in electron optics, whose unit spheres are just shifted unit spheres $\mathrm{S}_{\mathbf{v}}$, see [AIM]. Thus we call $\mathrm{S}_{\mathbf{v}}$ a *Randers indicatrix*. We have the following:

Theorem 1.1 *Let $\mathrm{S}^n_{\mathbf{v}}$ be a Randers indicatrix in the Euclidean space $(\mathbf{V}^{n+1}, |\cdot|)$ associated with a vector $\mathbf{v}$ with $|\mathbf{v}| < 1$. The following hold:*

(a) *For any $y \in \mathrm{S}^n_{\mathbf{v}}$, the mean torsion $I = \mathrm{trace}(C)$ satisfies the bound*

$$\|I_y\| < \frac{n+2}{\sqrt{2}}. \tag{3}$$

(b) *For any plane $P \subset T_y\mathrm{S}^n_{\mathbf{v}}$, the sectional curvature of $\bar{g}$ satisfies*

$$0 < \bar{K}(P) \leq 1. \tag{4}$$

Moreover, $\lim_{|\mathbf{v}| \to 1^-} \min \bar{K} = 0$.

Deike's [De] proves that for a *closed* strongly convex indicatrix $\Sigma \subset \mathbf{V}^{n+1}$, $I = 0$ if and only if it is quadratic. See also [Bk], [BCS]. A natural problem is whether or not there are non-quadratic strongly convex indicatrices with $I = 0$. In one dimension $(n = 1)$, every indicatrix with $I = 0$ is quadratic. But in higher dimensions, $I = 0$ does not imply that $C = 0$. More precisely, we have:

Theorem 1.2 *In R^{n+1}, there are infinitely many non-quadratic strongly convex indicatrices with vanishing mean torsion $I = 0$.*

Strongly convex indicatrices with vanishing mean torsion have special curvature properties. See more details in Section 3 below.

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2 Torsion and Distortion

Let $\Sigma \subset \mathbf{V}^{n+1}$ be a strongly convex indicatrix and L the defining function of Σ. Differentiating L yields a family of trilinear forms on $\mathbf{V}^{n+1}$, $C = \{C_y\}_{y\in\mathcal{C}(\Sigma)}$,

$$C_y(u,v,w) := \frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}[L(y+ru+sv+tw)]_{r=s=t=0}. \tag{5}$$

C is called the *(Cartan) torsion* of Σ. Let $\mathbf{V}^{n+1} = \operatorname{span}\{e_i\}_{i=1}^{n+1}$. Define a linear form $I_y \colon \mathbf{V}^{n+1} \to \mathrm{R}$ by

$$I_y(u) = \sum_{i=1}^{n+1} g^{ij}(y)C_y(u,e_i,e_j), \tag{6}$$

where $g_{ij}(y) := g_y(e_i,e_j)$ and $\big(g^{ij}(y)\big) = \big(g_{ij}(y)\big)^{-1}$. The family $I = \{I_y\}$ is called the *mean (Cartan) torsion* of Σ. We claim that

$$I_y(u) = u^k \frac{\partial}{\partial y^k}\Big[\ln\sqrt{\det\big(g_{ij}(y)\big)}\Big], \tag{7}$$

where $u = u^i e_i \in \mathbf{V}^{n+1}$ and $g_{ij}(y) := g_y(e_i,e_j)$.

To prove (7), we let $I_i(y) := I_y(e_i)$ and $C_{ijk}(y) := C_y(e_i,e_j,e_k)$. By definition,

$$C_{ijk}(y) = \frac{1}{4}\frac{\partial^3 L}{\partial y^i\partial y^j\partial y^k}(y) = \frac{1}{2}\frac{\partial g_{jk}}{\partial y^i}(y)$$

and

$$I_i(y) = g^{jk}(y)C_{ijk}(y). \tag{8}$$

Observe

$$\frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det\big(g_{jk}(y)\big)}\Big] = \frac{1}{2}g^{jk}(y)\frac{\partial g_{jk}}{\partial y^i}(y) = g^{jk}(y)C_{ijk}(y) = I_i(y).$$

This gives (7).

Define $C_y \colon \mathbf{V}^{n+1}\times\mathbf{V}^{n+1} \to \mathbf{V}^{n+1}$ and $I_y \in \mathbf{V}^{n+1}$ by

$$g_y\big(C_y(u,v),w\big) = C_y(u,v,w),$$
$$g_y(I_y,u) := I_y(u).$$

It follows from (7) that

$$I_y = \sum_{ij=1}^{n+1} g^{ij}(y)C_y(e_i,e_j).$$

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The norm of I_y is defined in a natural way

$$\|I_y\| := \sqrt{\sum_{ij=1}^{n+1} g^{ij}(y) I_y(e_i) I_y(e_j)} = \sqrt{g_y(I_y, I_y)}. \tag{9}$$

Assume that the Euclidean volume of $\mathcal{C}(\Sigma)$ is finite,

$$\sigma := \mathrm{Vol}\{\lambda(y^i) \in \mathrm{R}^{n+1}, y = y^i e_i \in \Sigma, 0 < \lambda < 1\} < \infty. \tag{10}$$

Then the following quantity is independent of the choice of $\{e_i\}_{i=1}^{n+1}$.

$$\tau(y) := \ln \frac{\sqrt{\det\big(g_{ij}(y)\big)}}{\sigma}. \tag{11}$$

τ is called the *distortion* of Σ. It follows from (7) that

$$I_y(u) = \frac{d}{dt}[\tau(y + tu)]\big|_{t=0} = u^k \frac{\partial}{\partial y^k}\left[\ln \frac{\sqrt{\det\big(g_{ij}(y)\big)}}{\sigma}\right]. \tag{12}$$

Therefore we obtain the following.

Lemma 2.1 *For a strongly convex indicatrix Σ and its defining function L, the following conditions are equivalent:*

(a) $I = 0$;
(b) $\tau =$ *constant;*
(c) $\det(g_{ij}) =$ *constant.*

3 Gauss Equation for Indicatrices

Let Σ be a strongly convex indicatrix in $\mathbf{V}^{n+1}$. Identifying $T_y\mathbf{V}^{n+1} = \mathbf{V}^{n+1}$ in a natural way, we obtain a Riemannian metric $\hat{g} = \{\hat{g}_y\}$ on $\mathcal{C}(\Sigma)$ by setting

$$\hat{g}_y(u, v) := g_y(u, v), \quad u, v \in T_y\mathcal{C}(\Sigma) = \mathbf{V}^{n+1}.$$

For each $y \in \mathcal{C}(\Sigma)$, define $\hat{C}_y\colon T_y\mathcal{C}(\Sigma) \times T_y\mathcal{C}(\Sigma) \to T_y\mathcal{C}(\Sigma)$ by

$$\hat{C}_y(u, v) := C_y(u, v), \quad u, v \in T_y\mathcal{C}(\Sigma) = \mathbf{V}^{n+1}.$$

We obtain the so-called Cartan torsion tensor $\hat{C} = \{\hat{C}_y\}$ on $\mathcal{C}(\Sigma)$.

For a vector field V on $\mathcal{C}(\Sigma)$, we can view it as a vector-valued function $V\colon \mathcal{C}(\Sigma) \to \mathbf{V}^{n+1}$ by setting $V(y) := V_y \in T_y\mathcal{C}(\Sigma) = \mathbf{V}^{n+1}$. Thus $dV|_y\colon T_y\mathcal{C}(\Sigma) = \mathbf{V}^{n+1} \to T_{y'}\mathbf{V}^{n+1} = \mathbf{V}^{n+1}$, where $y' = V(y)$, is a linear map. The Levi-Civita connection $\hat{\nabla}$ of $\hat{g}$ is given by

$$\hat{\nabla}_u V = dV(u) + \hat{C}(u, v), \quad u, v \in T_y\mathcal{C}(\Sigma) = \mathbf{V}^{n+1},$$

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where V is a vector field on $\mathcal{C}(\Sigma)$ with $V_y = v$. Moreover, the Riemann curvature tensor of $\hat{g}$ is given by

$$\hat{R}(u,v)w = \hat{C}\big(v, \hat{C}(u,w)\big) - \hat{C}\big(u, \hat{C}(v,w)\big), \quad u,v,w \in T_y\mathcal{C}(\Sigma). \tag{13}$$

See [Ki] and Section 14.2 in [BCS] for related discussion.

For each $y \in \mathcal{C}(\Sigma)$, let $\hat{I}_y = I_y \in T_y\mathcal{C}(\Sigma) = \mathbf{V}^{n+1}$. We have

$$\hat{I} := \sum_{ij=1}^{n+1} \hat{g}^{ij}\hat{C}(e_i, e_j), \tag{14}$$

where $\hat{g}_{ij} := \hat{g}_y(e_i, e_j)$ and $(\hat{g}^{ij}) := (\hat{g}_{ij})^{-1}$. Then the Ricci curvature of $\hat{g}$ is given by

$$\widehat{\mathrm{Ric}}(u,v) = \sum_{ij=1}^{n+1} \hat{g}^{ij}\hat{g}\big(\hat{C}(u,e_i), \hat{C}(v,e_j)\big) - \hat{g}\big(\hat{C}(u,v), \hat{I}\big). \tag{15}$$

From (15), we see that if $I = 0$, then the Ricci curvature of $\big(\mathcal{C}(\Sigma), \hat{g}\big)$ satisfies

$$\widehat{\mathrm{Ric}}(v,v) \geq 0.$$

Equality holds if and only if Σ is quadratic.

Let $\bar{g}$ denote the induced Riemannian metric on Σ. Let $\overline{\nabla}$ denote the Levi-Civita connection of $\bar{g}$. For each $y \in \Sigma$, identify $T_y\Sigma$ with a hyperplane $\mathrm{W}_y \subset \mathbf{V}^{n+1}$, where

$$\mathrm{W}_y := \{u \in \mathbf{V}^{n+1}, g_y(u,y) = 0\}.$$

Then for any vectors $u, v \in T_y\Sigma = \mathrm{W}_y$,

$$\nabla_u \tilde{V} = \overline{\nabla}_u V - \bar{g}(u,v), \tag{16}$$

where V is a vector field on Σ and $\tilde{V}$ is a vector field on $\mathbf{V}^{n+1}$ with $\tilde{V}|_\Sigma = V$ and $V_y = v$. This means that Σ is umbilical in $\big(\mathcal{C}(\Sigma), \hat{g}\big)$. Observe that for $y \in \Sigma$,

$$\hat{g}_y\big(\hat{C}_y(u,v), y\big) = C_y(u,v,y) = 0.$$

Thus $\hat{C}_y(u,v) \in T_y\Sigma$. Let $\bar{C}_y := \hat{C}_y|_{T_y\Sigma}$. We obtain a tensor $\bar{C} = \{\bar{C}_y\}_{y\in\Sigma}$ on Σ. It follows from (13) and (16) that the Riemann curvature of $\bar{g}$ satisfies the following Gauss equation

$$\bar{R}(u,v)w = \bar{C}\big(v, \bar{C}(u,w)\big) - \bar{C}\big(u, \bar{C}(v,w)\big) + \bar{g}(v,w)u - \bar{g}(u,w)v. \tag{17}$$

See [Kaw] and Section 14.6 in [BCS] for related discussions.

Observe that for $y \in \Sigma$,

$$\hat{g}_y(\hat{I}_y, y) = I_y(y) = 0.$$

Thus $\hat{I}_y \in T_y\Sigma$. Let $\bar{I}_y := \hat{I}_y$ for $y \in \Sigma$. We obtain a vector field $\bar{I} = \{\bar{I}_y\}$ on Σ. From (17), the Ricci curvature of $\bar{g}$ is given by

$$\overline{\mathrm{Ric}}(u, v) = \sum_{ij=1}^{n} \bar{g}^{ij}\bar{g}\big(\bar{C}(u, e_i), \bar{C}(v, e_j)\big) - \bar{g}\big(\bar{C}(u, v), \bar{I}\big) + (n-1)\bar{g}(u, v), \tag{18}$$

where $\{e_i\}_{i=1}^n$ is a basis for $T_y\Sigma = \mathrm{W}_y$ and $\bar{g}_{ij} := \bar{g}(e_i, e_j)$. We obtain the following.

Proposition 3.1 *For a strongly convex indicatrix $\Sigma \subset \mathbf{V}^{n+1}$, if $I = 0$, then the Ricci curvature of $(\Sigma, \bar{g})$ satisfies*

$$\overline{\mathrm{Ric}}(v, v) \geq (n-1)\bar{g}(v, v).$$

Equality holds if and only if Σ is quadratic.

4 Randers Indicatrices

In this section, we consider a special class of indicatrices—Randers indicatrices. Let S^n be a unit sphere in an Euclidean space $(\mathbf{V}^{n+1}, |\cdot|)$ and $\mathbf{v}$ a vector with $b := |\mathbf{v}| < 1$. $\mathrm{S}_{\mathbf{v}} := \mathrm{S}^n - \{\mathbf{v}\}$ is a Randers indicatrix associated with $\mathbf{v}$. To find the defining function, let $\mathbf{V}^n$ denote the orthogonal complement of $\mathbf{v}$ so that $\mathbf{V}^{n+1} = \mathrm{R}\cdot\mathbf{v} \oplus \mathbf{V}^n$. Define

$$\alpha(y) := \sqrt{\Big(\frac{b}{1-b^2}\Big)^2\lambda^2 + \frac{1}{1-b^2}|w|^2}, \quad \beta(y) := \frac{b^2}{1-b^2}\lambda,$$

where $y = \lambda\mathbf{v} + w \in \mathrm{R}\cdot\mathbf{v} \oplus \mathbf{V}^n$. Then $\|\beta\| := \sup_{\alpha(y)=1} \beta(y) = b$. Let

$$F(y) := \alpha(y) + \beta(y).$$

Note that for a vector $y = \lambda\mathbf{v} + w \in \mathrm{R}\cdot\mathbf{v} \oplus \mathbf{V}^n$, the following are equivalent:

(i) $F(y) = 1$;
(ii) $(1+\lambda)^2 b^2 + |w|^2 = 1$;
(iii) $y + \mathbf{v} = (1+\lambda)\mathbf{v} + w \in \mathrm{S}^n$;
(iv) $y \in \mathrm{S}^n_{\mathbf{v}}$.

Thus $L(y) := F^2(y)$ is the defining function of $\mathrm{S}^n_{\mathbf{v}}$.

We have the following:

Lemma 4.1 **(Matsumoto [Ma])** *The Cartan torsion of any Randers indicatrix $\mathrm{S}^n - \{\mathbf{v}\}$ is reducible, namely,*

$$C_y(u, v) = \frac{1}{n+2}\{h_y(u, v)I_y + h_y(v)I_y(u) + h_y(u)I_y(v)\}, \tag{19}$$

where $h_y(u) := u - F^{-2}(y)g_y(y, u)y$ and $h_y(u, v) := g_y\big(h_y(u), v\big)$.

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Thus for Randers indicatrices, $I = 0$ if and only if $C = 0$.

Lemma 4.2 *For any $y \in \mathrm{S}^n - \{\mathbf{v}\}$, the norm of $I_y \colon \mathbf{V}^{n+1} \to \mathrm{R}$ satisfies*

$$\|I_y\| \le \frac{n+2}{\sqrt{2}} \sqrt{1 - \sqrt{1 - b^2}}. \tag{20}$$

Proof Fix a basis $\{e_i\}_{i=1}^{n+1}$ for $\mathbf{V}^{n+1}$. Let $\alpha(y) = \sqrt{a_{ij} y^i y^j}$ and $\beta(y) = b_i y^i$. It is known that

$$\det(g_{ij}) = \left(\frac{F}{\alpha}\right)^{n+2} \det(a_{ij}).$$

See [Ma]. Thus by (7), $I_y(u) = I_i(y) u^i$ is given by

$$I_i(y) = \frac{n+2}{2} \frac{\partial}{\partial y^i} \left[\ln \frac{F(y)}{\alpha(y)}\right] = \frac{n+2}{2F(y)} \left\{ b_i - \frac{\beta(y)}{\alpha(y)} y_i \right\}, \tag{21}$$

where $y_i = \alpha_{y^i} = a_{ij} y^j / \alpha(y)$. See [Ma] or (11.2.8) in [BCS]. Let $g_{ij}(y) := g_y(e_i, e_j)$ and $\big(g^{ij}(y)\big) := \big(g_{ij}(y)\big)^{-1}$. Let $a_{ij} = \langle e_i, e_j \rangle$ and $(a^{ij}) = (a_{ij})^{-1}$.

$$g_{ij} = \frac{F}{\alpha} a_{ij} + b_i b_j + \frac{1}{\alpha}(b_i y_j + b_j y_i) - \beta \alpha^3 y_i y_j, \tag{22}$$

$$g^{ij} = \frac{\alpha}{F} a^{ij} - \frac{\alpha}{F^2}(b^i y^j + b^j y^i) + \frac{\alpha b^2 + \beta}{\alpha^3} y^i y^j, \tag{23}$$

where $y_i := a_{ik} y^k$ and $b^i := a^{ik} b_k$. Observe that

$$\left(b_i - \frac{\beta}{\alpha} y_i\right) a^{ij} \left(b_i - \frac{\beta}{\alpha} y_i\right) = b^2 - \left(\frac{\beta}{\alpha}\right)^2$$

$$\left(b_i - \frac{\beta}{\alpha} y_i\right) (b^i y^j + b^j y^i) \left(b_i - \frac{\beta}{\alpha} y_i\right) = 0$$

$$\left(b_i - \frac{\beta}{\alpha} y_i\right) y^i y^j \left(b_i - \frac{\beta}{\alpha} y_i\right) = 0.$$

Thus by (21)

$$\|I_y\|^2 = I_i(y) I_j(y) g^{ij}(y) = \left(\frac{n+2}{2F(y)}\right)^2 \frac{\alpha(y)}{F(y)} \left\{ b^2 - \left(\frac{\beta(y)}{\alpha(y)}\right)^2 \right\}. \tag{24}$$

Since $|\beta(y)| \le b\alpha(y)$, we can write $\beta(y) = b\alpha(y)\cos\theta$, where $0 \le \theta \le 2\pi$. For $y \in \Sigma$, $F(y) = \alpha(y) + \beta(y) = 1$,

$$\alpha(y) = 1 - \beta(y) = 1 - b\alpha(y)\cos\theta.$$

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This gives

$$\alpha(y) = \frac{1}{1 + b\cos\theta}.$$

Plugging it into (24) yields

$$\|I_y\|^2 = \left(\frac{n+1}{2}\right)^2 \frac{b^2 \sin^2\theta}{1 + b\cos\theta} \le \frac{(n+2)^2}{2}\big(1 - \sqrt{1-b^2}\big). \tag{25}$$

■

Remark 4.3 Define

$$\|C_y\| := \sup_{g_y(v,v)=1} |C_y(v,v,v)|.$$

It follows from (19) and (20) that for any unit vector $y \in \mathbf{V}^{n+1}$ $(F(y) = 1)$,

$$\|C_y\| \le \frac{3}{\sqrt{2}}\sqrt{1 - \sqrt{1-b^2}} < \frac{3}{\sqrt{2}}. \tag{26}$$

Namely, the torsion is uniformly bounded by $3/\sqrt{2}$. The bound (26) for two-dimensional Randers indicatrices is given in Exercise 11.2.6 in [BCS] which is suggested by Brad Lackey. But (20) does not follow from (6) and (26) directly.

We now estimate the sectional curvature of the induced Riemannian metric on a Randers indicatrix.

Lemma 4.4 *Let Σ be a Randers indicatrix. For any plane $P = \mathrm{span}\{u, v\} \subset T_y\Sigma$, where u, v are $\bar{g}$-orthonormal, the sectional curvature of $\bar{g}$ satisfies*

$$\bar{K}(P) = \bar{g}\big(\bar{R}(u,v)v, u\big) = 1 - \frac{1}{(n+2)^2}\{\bar{I}(u)^2 + \bar{I}(v)^2 + \|\bar{I}\|^2\}. \tag{27}$$

Proof Note that

$$\bar{g}(u,v) = h_y(u,v), \quad \bar{I}(u) = I_y(u), \quad \bar{C}(u,v) = C_y(u,v).$$

(19) implies

$$\bar{C}(u,v) = \frac{1}{n+2}\{\bar{g}(u,v)\bar{I} + \bar{I}(u)v + \bar{I}(v)u\}. \tag{28}$$

Applying (28) to (17) we obtain

$$\begin{aligned}\bar{R}(u,v)w = \frac{1}{(n+2)^2}\Big\{ &\big(\bar{g}(u,w)\bar{I}(v) - \bar{g}(v,w)\bar{I}(u)\big)\,\bar{I} \\ &+ \big(\bar{g}(u,w)v - \bar{g}(v,w)u\big)\,\|\bar{I}\|^2 + \big(\bar{I}(u)v - \bar{I}(v)u\big)\,\bar{I}(w)\Big\} \\ &+ \bar{g}(v,w)u - \bar{g}(u,w)v,\end{aligned} \tag{29}$$

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where $\|\bar{I}\|^2 := \sum_{ij=1}^{n} \bar{g}^{ij}\bar{I}(e_i)\bar{I}(e_j)$. From (29), we obtain (27). ∎

Proof of Theorem 1.1 Note that

$$0 \le \bar{I}(u)^2 + \bar{I}(v)^2 \le \|\bar{I}\|^2.$$

By (27), we obtain

$$1 - \frac{2}{(n+2)^2}\|\bar{I}\|^2 \le \bar{K}(P) \le 1 - \frac{1}{(n+2)^2}\|\bar{I}\|^2. \tag{30}$$

Since $I_y(y) = 0$ and $g_y(y,u) = 0$ for $u \in T_y\Sigma$, we have

$$\|I_y\| = \|\bar{I}\|.$$

By Lemma 4.2,

$$\|\bar{I}\| \le \frac{n+2}{\sqrt{2}}\sqrt{1-\sqrt{1-b^2}}. \tag{31}$$

Plugging (31) into (30) yields

$$0 < \sqrt{1-b^2} \le \bar{K}(P) \le 1. \tag{32}$$

It follows from (25) that there is a point $y_o \in \Sigma$ such that

$$\|I_{y_o}\| = \frac{n+2}{\sqrt{2}}\sqrt{1-\sqrt{1-b^2}}.$$

There is a unit vector $u_o \in T_{y_o}\Sigma$ such that $I_{y_o}(u_o) = \|I_{y_o}\|$. In virtue of (27), for any section $P = \operatorname{span}\{u_o, v_o\} \subset T_{y_o}\Sigma$,

$$\bar{K}(P) = 1 - \frac{2}{(n+2)^2}\|I_{y_o}\|^2 = \sqrt{1-b^2}.$$

Thus $\lim_{b\to 1^-} \min \bar{K} = 0$. ∎

From (29), we obtain the Ricci curvature

$$\overline{\mathrm{Ric}}(v,v) = -\frac{1}{(n+2)^2}\{(n-2)\bar{I}(v)^2 + n\bar{g}(v,v)\|\bar{I}\|^2\} + (n-1)\bar{g}(v,v). \tag{33}$$

This implies

$$1 - 2\left(\frac{\|\bar{I}\|}{n+2}\right)^2 \le \frac{\overline{\mathrm{Ric}}}{n-1} \le 1 - \frac{n}{n-1}\left(\frac{\|\bar{I}\|}{n+2}\right)^2. \tag{34}$$

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By (31), we obtain

$$0 < \sqrt{1-b^2} \le \frac{\overline{\mathrm{Ric}}}{n-1} \le 1. \tag{35}$$

(35) also follows from (32). By (33), we obtain a formula for the scalar curvature

$$\bar{S} = \frac{n-1}{n+2}\big(n(n+2) - \|\bar{I}\|^2\big). \tag{36}$$

Using (31), we obtain

$$\frac{n-2}{2n} < \frac{n-2}{2n} + \frac{n+2}{2n}\sqrt{1-b^2} \le \frac{\bar{S}}{n(n-1)} \le 1. \tag{37}$$

Thus, for $n > 2$, the scalar curvature is bounded below by a positive number.

5 Indicatrices with Vanishing Torsion

Let $\mathbb{S}^n$ denote the standard unit ball in the Euclidean space $\mathbb{R}^{n+1}$. Consider an indicatrix Σ in $\mathbb{R}^{n+1}$. Let $\Omega := \mathcal{C}(\Sigma) \cap \mathbb{S}^n$. Then $\mathcal{C}(\Omega) = \mathcal{C}(\Sigma)$. By definition, the defining function of Σ is a function $L\colon \mathcal{C}(\Sigma) \to (0,\infty)$ with $L^{-1}(1) = \Sigma$ and

$$L(\lambda y) = \lambda^2 L(y), \quad \lambda > 0, y \in \mathcal{C}(\Omega). \tag{38}$$

Assume that Σ is strongly convex. Then

$$g_{ij}(y) := \frac{1}{2}\frac{\partial^2 L}{\partial y^i y^j}(y) \text{ is positive definite.} \tag{39}$$

A function $L\colon \mathcal{C}(\Omega) \to (0,\infty)$ satisfying (38) and (39) is called a *Minkowski functional* in $\mathcal{C}(\Omega)$ and $\Sigma := L^{-1}(1)$ is called the *indicatrix* of L. Thus, by Lemma 2.1, to find a strongly convex indicatrix with vanishing mean torsion, we just need to find a Minkowski functional with $\det(g_{ij}) = \text{constant}$. For a domain $\Omega \subset \mathbb{S}^n$, denote by $\lambda_1(\Omega)$ the first eigenvalue of the Laplacian Δ for the Dirichlet problem on Ω, *i.e.*,

$$\lambda_1(\Omega) := \inf_{u \in \mathrm{H}^1_o(\Omega), u \neq 0} \frac{\int |\nabla u|^2\, dv}{\int u^2\, dv}.$$

We have:

Proposition 5.1 *Let $\Omega \subset \mathbb{S}^n$ be an open domain with $\partial\Omega \in C^{2,\alpha}$ and $\lambda_1(\Omega) > 2(n+1)$. There exists $\varepsilon = \varepsilon(n,\Omega) > 0$ such that for any $\phi\colon \partial\Omega \to \mathrm{R}$ satisfying $\phi \in C^{2,\alpha}(\partial\Omega)$ and $\|\phi\|_{2,\alpha} < \varepsilon$, there is a Minkowski functional L on $\mathcal{C}(\Omega)$ satisfying*

$$\begin{cases} \det\big(\frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}\big) = 1 & \text{in } \mathcal{C}(\Omega) \\ L = 1 + \phi & \text{on } \partial\Omega. \end{cases} \tag{40}$$

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Proof To find a Minkowski functional L satisfying (40), we write

$$L(y) = r^2 + r^2 h(\xi),$$

where $r = |y|$ and $\xi \in \Omega$. Let $\varphi = (\varphi^i)\colon \Omega \to \mathrm{R}^{n+1}$ denote the natural embedding and (ξ^a) be a local coordinate system in Ω. Using $y = r\varphi(\xi)$ and (38), we obtain

$$1 + h = \frac{1}{2}L_{rr} = \frac{1}{2}L_{y^iy^j}\varphi^i\varphi^j \tag{41}$$

$$\frac{1}{2}h_{\xi^a} = \frac{1}{4r}L_{r\xi^a} = \frac{1}{2}L_{y^iy^j}\varphi^i_{\xi^a}\varphi^j \tag{42}$$

$$\frac{1}{2}h_{\xi^a\xi^b} = \frac{1}{2r^2}L_{\xi^a\xi^b} = \frac{1}{2}L_{y^iy^j}\varphi^i_{\xi^a}\varphi^j_{\xi^b} + \frac{1}{2}L_{y^iy^j}\varphi^i_{\xi^a\xi^b}\varphi^j. \tag{43}$$

Let $\dot{g}_{ab} := \varphi^i_{\xi^a}\varphi^i_{\xi^b}$ and γ^c_{ab} the Christoffel symbols of $\dot{g} = \dot{g}_{ab}d\xi^a \otimes d\xi^b$. Then

$$\varphi^i_{\xi^a\xi^b} = \gamma^c_{ab}\varphi^i_{\xi^c} - \dot{g}_{ab}\varphi^i. \tag{44}$$

Plugging (44) into (43) and using (41) and (42), we obtain

$$\frac{1}{2}h_{\xi^a\xi^b} = \frac{1}{2}L_{y^iy^j}\varphi^i_{\xi^a}\varphi^j_{\xi^b} + \frac{1}{2}\gamma^c_{ab}h_{\xi^c} - (1+h)\dot{g}_{ab}.$$

Thus

$$\begin{pmatrix}\varphi\\ \varphi_{\xi^a}\end{pmatrix}\begin{pmatrix}\frac{1}{2}L_{y^iy^j}\end{pmatrix}\begin{pmatrix}\varphi\\ \varphi_{\xi^a}\end{pmatrix}^T = \begin{pmatrix}1+h & \frac{1}{2}h_{;b}\\ \frac{1}{2}h_{;a} & (1+h)\dot{g}_{ab} + \frac{1}{2}h_{;a;b}\end{pmatrix}, \tag{45}$$

where $h_{;a} := h_{\xi^a}$ and $h_{;a;b} := h_{\xi^a\xi^b} - \gamma^c_{ab}h_{\xi^c}$. Thus, there exists $\delta > 0$ such that if $h \in C^{2,\alpha}(\bar{\Omega})$ satisfies

$$\|h\|_{C^{2,\alpha}} < \delta, \tag{46}$$

then $L = r^2(1+h)$ is a Minkowski functional on $\mathcal{C}(\Omega)$.

Note that

$$\left[\det\begin{pmatrix}\varphi\\ \varphi_{\xi^b}\end{pmatrix}\right]^2 = \det\begin{pmatrix}1 & 0\\ 0 & \dot{g}_{ab}\end{pmatrix} = \det(\dot{g}_{ab}).$$

From (45), we obtain

$$\det\begin{pmatrix}\frac{1}{2}L_{y^iy^j}\end{pmatrix} = \sum_{k=0}^{n+1}P_k(D^2h, Dh, h), \tag{47}$$

where $P_k = P_k(\eta, \zeta, \tau)$ is a polynomial of order k in variables $\eta \in \mathrm{R}^{2n}$, $\zeta \in \mathrm{R}^n$ and $\tau \in \mathrm{R}$. P_k's are determined by

$$\sum_{k=0}\lambda^{n+1-k}P_k(\eta,\zeta,\tau) = \det\begin{pmatrix}\lambda+\tau & \frac{1}{2}\zeta_b\\ \frac{1}{2}\zeta_a & \lambda\dot{g}_{ab} + \tau\dot{g}_{ab} + \frac{1}{2}\eta_{ab}\end{pmatrix}. \tag{48}$$

Thus

$$P_0 = 1, \quad P_1 = (n+1)h + \frac{1}{2}\Delta_{\mathbb{S}^n} h.$$

Therefore, (40) is equivalent to the following equation

$$\begin{cases} \Delta_{\mathbb{S}^n} h + 2(n+1)h + \sum_{k=2}^{n+1} P_k(D^2h, Dh, h) = 0 & \text{in } \Omega \\ h = \phi & \text{on } \partial\Omega. \end{cases} \tag{49}$$

Now it suffices to prove the following:

Lemma 5.2 *Let $\Omega \subset \mathbb{S}^n$ with $\partial\Omega \in C^{2,\alpha}$. Suppose that $\lambda_1(\Omega) > 2(n+1)$. Then there exists $\varepsilon > 0$ depending only on n and Ω such that for any $\phi \in C^{2,\alpha}(\partial\Omega)$ with $\|\phi\|_{2,\alpha} < \varepsilon$, the above problem (49) has a solution.*

Proof First, we consider the following linear problem

$$\begin{cases} \Delta f + 2(n+1)f = \chi & \text{in } \Omega \\ f = \phi & \text{in } \partial\Omega, \end{cases} \tag{50}$$

where $\chi \in C^{\alpha}(\bar{\Omega})$. We have the following:

Assertion (50) has a unique solution $f \in C^{2,\alpha}(\bar{\Omega})$ and

$$\|f\|_{2,\alpha} \le C(\|\phi\|_{2,\alpha} + \|\chi\|_{C^{\alpha}}), \tag{51}$$

where C depends on n and Ω. The proof of this assertion is given at the end.

We proceed to prove Lemma 5.2 by granting the above assertion. For $\delta > 0$, let

$$\chi_\delta := \{f \in C^{2,\alpha}(\bar{\Omega}) \mid \|f\|_{2,\alpha} \le \delta, f|_{\partial\Omega} = \phi\}.$$

To find a solution of (49), we define an operator $T\colon \chi_\delta \to C^{2,\alpha}(\bar{\Omega})$ as follows. For $h \in \chi_\delta$, define $T(h) := f$ to be the unique solution of the following linear problem

$$\begin{cases} \Delta_{\mathbb{S}^n} f + 2(n+1)f + \sum_{k=2}^{n+1} P_k(D^2h, Dh, h) = 0 & \text{in } \Omega \\ f = \phi & \text{on } \partial\Omega. \end{cases} \tag{52}$$

By the above claim, the operator is well-defined.

We shall choose $0 < \delta < 1$, $\varepsilon > 0$ such that when $\|\phi\|_{2,\alpha} < \varepsilon$, T maps χ_δ into itself and T is a contraction map.

Observe that

$$\|P_k(D^2h, Dh, h)\|_{C^{\alpha}(\bar{\Omega})} \le C_k \delta^k, \quad \forall h \in \chi_\delta,$$

where C_k are constants depending on the $C^{2,\alpha}$-norm of the coefficients of P_k. By (51), we have that for a constant $C = C(n, \Omega)$,

$$\begin{aligned}\|T(h)\|_{C^{2,\alpha}(\bar\Omega)} &\le C\Big(\|\varphi\|_{2,\alpha} + \sum_{k=2}^{n+1} \|P_k(D^2h, Dh, h)\|_{C^{\alpha}(\bar\Omega)}\Big)\\ &\le C\Big(\|\phi\|_{2,\alpha} + \sum_{k=2}^{n+1} C_k\delta^k\Big)\\ &\le \bar C(\|\phi\|_{2,\alpha} + \delta^2),\end{aligned}$$

where $\bar C$ is a constant depending on n, Ω and P_k, provided that $\delta \le 1$. Take a smaller δ if necessary, so that $\bar C\delta^2 \le \frac{1}{2}\delta$, then take $\varepsilon > 0$ so small that $\varepsilon \le \frac{\delta}{2\bar C}$. We see that if $\|\phi\| \le \varepsilon$, then

$$\|T(h)\|_{C^{2,\alpha}(\bar\Omega)} \le \bar C\varepsilon + \bar C\delta^2 \le \frac{1}{2}\delta + \frac{1}{2}\delta = \delta.$$

Thus T maps χ_δ into itself.

Now we are going to prove that T is a contraction map. Let $f_i := T(h_i)$ where $h_i \in \chi_\delta$, $i = 1, 2$. We have

$$\|P_k(D^2h_1, Dh_1, h_1) - P_k(D^2h_2, Dh_2, h_2)\|_{C^{\alpha}(\bar\Omega)} \le C_k\delta^{k-1}\|h_1 - h_2\|_{C^{2,\alpha}(\bar\Omega)},$$

where C_k is a constant depending only on P_k. Since f_i satisfies (52) with $h = h_i$, $i = 1, 2$, we obtain

$$\begin{aligned}\|f_1 - f_2\|_{C^{2,\alpha}(\bar\Omega)} &\le C\sum_{k=2}^{n+1} C_k\delta^{k-1}\|h_1 - h_2\|_{C^{2,\alpha}(\bar\Omega)}\\ &\le \bar C\delta\|h_1 - h_2\|_{C^{2,\alpha}(\bar\Omega)},\end{aligned}$$

where $\bar C = \bar C(n, \Omega, P_k)$. Thus, if $\bar C\delta < \frac{1}{2}$, then T is a contraction map.

The above arguments show that there is a constant $\bar C$ depending only on n, Ω and P_k such that if

$$\delta \le \min\Big\{\frac{1}{2\bar C}, 1\Big\}, \quad \varepsilon < \frac{\delta}{2\bar C},$$

then $T\colon \chi_\delta \to \chi_\delta$ is a contraction map. Thus there is a function $h \in \chi_\delta$ such that $T(h) = h$. This h is the desired solution to (49). Choosing a smaller $\delta > 0$ if necessarily, we conclude that for the solution h to (49) in χ_δ, the resulting function $L = r^2(1 + h)$ is a Minkowski functional. ■

Proof of Assertion Consider the following functional $J\colon \mathrm{H}^1_o(\Omega) \to R^1$

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u + \nabla\phi|^2 - (n+1)\int_\Omega (u+\phi)^2 + \int_\Omega \chi u, \quad \forall u \in \mathrm{H}^1_o(\Omega),$$

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where $\phi \in C^{2,\alpha}(\bar{\Omega})$ and $\chi \in C^{\alpha}(\bar{\Omega})$ are given in (50). Since $\lambda_1(\Omega) > 2(n+1)$, J has minimum u_o. Then $f := u_o + \phi \in \mathrm{H}^1(\Omega)$ is a weak solution of (50). By the L^2-theory, L^p-theory and Schauder estimates for elliptic equations, we conclude that any weak solution f of (50) must be in $C^{2,\alpha}(\bar{\Omega})$ and

$$\|f\|_{C^{2,\alpha}} \le C(\|f\|_{C^0} + \|\phi\|_{C^{2,\alpha}} + \|\chi\|_{C^\alpha}),$$

where C depends on n and Ω. Now it suffices to show

$$\|f\|_{C^0} \le C(\|\phi\|_{C^0} + \|\chi\|_{C^0}) \tag{53}$$

with C depending on n and Ω. Let $\Omega' \supset \bar{\Omega}$ be an open domain having the property that $\lambda_1(\Omega') = 2(n+1)$ since $\lambda_1(\Omega) > 2(n+1)$. Let w be a first eigenfunction on Ω'. Then

$$\begin{cases} \Delta w + 2(n+1)w = 0 & \text{in } \Omega' \\ w > 0 & \text{in } \Omega' \\ w = 0 & \text{on } \partial\Omega'. \end{cases} \tag{54}$$

Write $f = wg$. From (50) and (54) we see that

$$\begin{cases} \Delta g + 2\frac{\nabla w}{w} \cdot \nabla g = \frac{\chi}{w} & \text{in } \Omega \\ g = \frac{\phi}{w} & \text{on } \partial\Omega, \end{cases}$$

where $w|_{\bar{\Omega}}$ has a positive minimum since $\bar{\Omega} \subset \Omega'$. This implies, by maximum principle, that

$$\|g\|_{C^0} \le C(\|\phi\|_{C^0} + \|\chi\|_{C^0})$$

with the constant C depending on $\inf_{\bar{\Omega}} w$ and $\|\nabla w\|_{C^0}$. Then $f = wg$ satisfies (53). ∎

References

[AIM] P. L. Antonelli, R. Ingarden and M. Matsumoto, *The theory of sprays and Finsler spaces with applications in physics and biology*. Kluwer Academic Publishers, 1993.

[As] G. S. Asanov, *Finslerian metric functions over the product* R $\times M$ *and their potential applications*. Rep. Math. Phys. **40**(1998), 117–131.

[BCS] D. Bao, S. S. Chern and Z. Shen, *An introduction to Riemann-Finsler geometry*. Springer-Verlag, 2000.

[De] A. Deicke, *Über die Finsler-Räume mit* $A_i = 0$. Arch. Math. **4**(1953), 45–51.

[Bk] F. Brickell, *A new proof of Deicke's theorem on homogeneous functions*. Proc. Amer. Math. Soc. **16**(1965), 190–191.

[Kaw] A. Kawaguchi, *On the theory of non-linear connection II. Theory of Minkowski spaces and of non-linear connections in a Finsler space*. Tensor (N.S.) **6**(1956), 165–199.

[Ki] S. Kikuchi, *Theory of Minkowski space and of non-linear connections in Finsler space*. Tensor (N.S.) **12**(1962), 47–60.

[Ma] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*. Kaiseisha Press, Japan, 1986.

[Sh] Z. Shen, *Lectures on Finsler Geometry*. World Scientific Publishing, Singapore, to appear.

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[Tho] A. C. Thompson, *Minkowski Geometry*. Encyclopedia Math. Appl. **63**, Cambridge Univ. Press, Cambridge, 1996.

Graduate School at Beijing
University of Science and Technology of China
and
Institute of Mathematics
Academia Sinica
Beijing 100086
P.R. China
email: jimin@math08.math.ac.cn

Department of Mathematical Sciences
Indiana University-Purdue University
402 N. Blackford Street
Indianapolis, IN 46202-3620
USA
email: zshen@math.iupui.edu