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Trigonometric Multipliers on $H_{2\pi}$

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Abstract. In this paper we consider multipliers on the real Hardy space $H_{2\pi}$. It is known that the Marcinkiewicz and the Hörmander–Mihlin conditions are sufficient for the corresponding trigonometric multiplier to be bounded on $L^p_{2\pi}$, $1 < p < \infty$. We show among others that the Hörmander–Mihlin condition extends to $H_{2\pi}$ but the Marcinkiewicz condition does not.

1 Introduction

By $H_{2\pi}$ we will mean the real Hardy space of 2π periodic functions. $H_{2\pi}$ is equivalent to the Hardy space $H(T)$, which is the collection of those complex valued functions defined on the torus that have zero Fourier coefficients for negative indices. This equivalence will often be used in the proofs.

$H_{2\pi}$ is defined as the family of real 2π periodic integrable functions, the trigonometric conjugate of which is also integrable. The trigonometric conjugate, in other words the periodic Hilbert transform, of an $f \in L^1_{2\pi}$ will be denoted by $\tilde{f}$. The $H_{2\pi}$ norm is given by $\|f\|_{H_{2\pi}} = \|f\|_{L^1_{2\pi}} + \|\tilde{f}\|_{L^1_{2\pi}}$.

Let $\varphi = \{\varphi(k)\}_{k=-\infty}^{\infty}$ be a sequence of complex numbers and let the transformed Fourier series be defined by $S^\varphi f = \sum_{k=-\infty}^{\infty} \varphi(k)\widehat{f}(k)e_k$ $(f \in H_{2\pi})$, where $e_k(t) = e^{ikt}$ and $\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,dt$ is the kth Fourier coefficient of f $(k \in \mathbb{Z})$. Then φ is called a bounded multiplier on $H_{2\pi}$ if the operator T^φ defined by

$$\widehat{T^\varphi f}(k) = \varphi(k)\widehat{f}(k) \qquad (k \in \mathbb{Z},\ f \in H_{2\pi})$$

is bounded from $H_{2\pi}$ to itself. We note that f real implies $\widehat{f}(-k) = \overline{\widehat{f}(k)}$ $(k \in \mathbb{N})$. Consequently, for a multiplier φ to be bounded on $H_{2\pi}$ it is necessary that $\varphi(-k) = \overline{\varphi(k)}$ $(k \in \mathbb{N})$.

The atomic structure of $H_{2\pi}$ plays a fundamental role in the proofs. To facilitate the use of the atomic decomposition we will modify the concept of intervals on $[0, 2\pi)$. We will identify $[0, 2\pi)$ with the unit circle. The sets corresponding to intervals on the unit circle will be called intervals and their collection will be denoted by $\mathcal{I}$. By definition a function $\mathrm{a}\colon [0, 2\pi) \mapsto \mathbb{R}$ is an atom if it is the constant 1 or there exists $I \in \mathcal{I}$ such that

(i) $\operatorname{supp} \mathrm{a} \subset I$,
(ii) $\|\mathrm{a}\|_{L^\infty_{2\pi}} \le |I|^{-1}$,
(iii) $\int_0^{2\pi} \mathrm{a} = 0$,

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where $|A|$ stands for the Lebesgue measure of the measurable set A. Then f belongs to $H_{2\pi}$ if and only if there exist α_k real numbers with $\sum_{k=0}^{\infty}|\alpha_k| < \infty$ and a_k atoms such that

$$f = \sum_{k=0}^{\infty} \alpha_k \, \mathrm{a}_k, \tag{1.1}$$

and the convergence in the decomposition is understood in the $L^1_{2\pi}$ norm. Moreover

$$\|f\|_{H_{2\pi}} \approx \inf \sum_{k=0}^{\infty} |\alpha_k|$$

with taking the infimum over all decompositions of the form (1.1).

The difference sequence $\Delta\varphi$ is defined as $\Delta\varphi(k) = \varphi(k) - \varphi(k+1)$ $(k \geq 0)$, and $\Delta\varphi(k) = \varphi(k) - \varphi(k-1)$ $(k < 0)$. Throughout the paper C will denote an absolute positive constant not necessarily the same in different occurrences.

2 Results

Theorem 2.1 *Suppose that $\varphi = \{\varphi(k)\}_{-\infty}^{\infty}$ is a bounded sequence and satisfies $\varphi(-k) = \overline{\varphi(k)}$, $k \in \mathbb{N}$. Let $r > 1$. If*

$$2^j \Big(\sum_{k=2^j}^{2^{j+1}-1} \frac{|\Delta\varphi(k)|^r}{2^j} \Big)^{1/r} \leq C \quad (j \in \mathbb{N}), \tag{2.1}$$

then φ is a bounded multiplier on $H_{2\pi}$.

Remark 2.2 We note that a similar result holds for $H(T)$. We also note that it follows from the duality relation between $H_{2\pi}$ and $BMO_{2\pi}$ that (2.1) is sufficient for a multiplier to be bounded on $BMO_{2\pi}$.

In the case that $r = 1$, condition (2.1) is the well-known Marcinkiewicz condition for $L^p_{2\pi}$ $(1 < p < \infty)$ multipliers. The following theorem, in particular, means that the Marcinkiewicz condition does not extend to $H_{2\pi}$. (See historical notes.)

Theorem 2.3 *There exists a sequence φ with $\varphi(-k) = \overline{\varphi(k)}$ $(k \in \mathbb{N})$ which is of bounded variation,* i.e., *$\sum_{k=-\infty}^{\infty} |\Delta\varphi(k)| < \infty$, but the corresponding multiplier operator T^{φ} is not bounded from $H_{2\pi}$ to $L^1_{2\pi}$.*

Corollary 2.4 *There exists a bounded φ with $\varphi(-k) = \overline{\varphi(k)}$ $(k \in \mathbb{N})$ that satisfies the Marcinkiewicz condition*

$$\sum_{k=2^j}^{2^{j+1}-1} |\Delta\varphi(k)| \leq C \quad (j \in \mathbb{N}), \tag{2.2}$$

but T^{φ} is not bounded from $H_{2\pi}$ to $L^1_{2\pi}$.

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3 Historical Comments

J. Marcinkiewicz [Ma] published the multiplier theorem that bears his name in 1939. His argument proves directly that for a bounded sequence φ satisfying

$$\sup_{n\geq 0} \sum_{2^n\leq |k|<2^{n+1}} |\varphi(k+1)-\varphi(k)| < \infty,$$

the corresponding multiplier operator is bounded on $L^p_{2\pi}$ for $1<p<\infty$, by showing that certain weighted partial sums of the Fourier series of the function on dyadic blocks that arise from partial summation have appropriate $L^p_{2\pi}$ bounds. This is the proof which most of us learned. It is the one found in Zygmund's *Trigonometric Series* [Z]. The estimates of Marcinkiewicz do not hold for $L^1_{2\pi}$ and no mention is made of *weak*$(1,1)$ results nor for the Hardy space $H_{2\pi}$. In their monograph, Edwards and Gaudry [EG] studied the interrelationship between the Marcinkiewicz condition for multipliers and Littlewood–Paley square function decompositions for L^p, $1<p<\infty$, for various groups. In particular, for the circle and the dyadic group, they show that for $1<p<\infty$ the validity of the Marcinkiewicz Multiplier Theorem for L^p is equivalent to the Littlewood–Paley square function decomposition of L^p. As the square function decomposition extends to the Hardy space $H_{2\pi}$, the natural question to ask is whether the Marcinkiewicz Theorem extends to the Hardy space. If not, is there then a variant that extends to the Hardy space? This is the prime motivation for this work.

Our Corollary 2.4 gives a counter-example. In fact, the multiplier constructed satisfies the stronger condition that it is of bounded variation. On the positive side, our Theorem 2.1 gives an appropriate replacement: if φ is bounded and satisfies

$$\sup_{n\geq 0} 2^{n\varepsilon} \sum_{2^n\leq |k|<2^{n+1}} |\varphi(k+1)-\varphi(k)|^{1+\varepsilon} < \infty$$

for any $\varepsilon>0$, then T_φ is bounded on $H_{2\pi}$. The authors [DF] considered these questions initially in the context of the dyadic group and Walsh series and proved results analagous to the ones contained here.

The only positive information concerning the Marcinkiewicz condition for $H_{2\pi}$ was that found in a statement by S. V. Kislyakov [Ki]: for bounded φ, the condition

$$\sup_{n\geq 0} R^n \sum_{R^n\leq |k|<R^{n+1}} |\varphi(k+1)-\varphi(k)|^2 < \infty$$

implies the multiplier operator T_φ is bounded on $H_{2\pi}$. He directs the reader to follow the proof of Hörmander [H] for multliplier operators on $L^p(\mathbb{R}^n)$, $1<p<\infty$; however, no proof is given. The Sidon type inequality in Lemma 4.2 plays an essential role in our proof of Theorem 2.1. No mention of this inequality appears in Kislyakov nor is one needed in the proof of Hörmander.

4 Proofs

In the proofs we will often consider series with respect to the non-negative complex trigonometric system $\{e_k\}_{k=0}^{\infty}$. The corresponding Dirichlet kernels D_k are defined by

$$D_n = \sum_{j=0}^{n} e_k = \frac{e_{n+1}-1}{e_1-1} \quad (n \geq 1). \tag{4.1}$$

We will need the following Sidon type inequality. The full trigonometric version was proved by Móricz in [Mó], and the Walsh version by Daly and Fridli in [DF].

Lemma 4.1 *Let $\delta > 0$, $1 < q \leq 2$, and $n \in \mathbb{N}$. Suppose that $\sum_{k=1}^{n} c_k = 0$. Then*

$$\int_{\delta}^{2\pi-\delta} \Big| \sum_{k=0}^{n} c_k D_k(x) \Big| \, dx \leq C\, \delta^{(1/q)-1} \Big(\sum_{k=0}^{n} |c_k|^q \Big)^{1/q}. \tag{4.2}$$

Proof Without loss of generality we may assume $n > \delta^{-1}$. It follows from (4.1) and from the assumption $\sum_{k=1}^{n} c_k = 0$ that

$$\int_{\delta}^{2\pi-\delta} \Big| \sum_{k=0}^{n} c_k D_k \Big| = \int_{\delta}^{2\pi-\delta} \frac{1}{|e_1-1|} \Big| \sum_{k=0}^{n} c_k e_{k+1} \Big| \, dx.$$

Using Hölder inequality we obtain that this integral can be dominated by:

$$\Big\| \frac{\chi_{[\delta,2\pi-\delta]}}{e_1-1} \Big\|_q \Big\| \sum_{k=1}^{n} c_k e_{k+1} \Big\|_p,$$

where $1/q + 1/p = 1$. Since $|e_1(x)-1| = O(1/x)$ $(\delta \leq x \leq \pi)$ and $|e_1(x)-1| = |e_1(2\pi - x) - 1|$, we have that the first factor is of order $\delta^{1/q-1}$. Then the proof can be finished by applying Hausdorff-Young inequality to the second factor. ■

Proof of Theorem 2.1

Using the conjugate function characterization or the natural identification of $H_{2\pi}$ with $H(T)$, we may restrict our attention to the nonnegative part of the Fourier series. More precisely, $f \in H_{2\pi}$ if and only if $\sum_{k=0}^{\infty} \widehat{f}(k) e_k$ represents an integrable function g. Moreover, $\|f\|_{H_{2\pi}} \approx \|g\|_{L^1_{2\pi}}$.

The boundedness of φ implies that if $h \in L^2_{2\pi}$ then $\sum_{k=0}^{\infty} \varphi(k)\widehat{h}(k) e_k$ converges to a function denoted by $T^{\varphi^+} h$ in $L^2_{2\pi}$ norm. This, in particular, is true for any atom. Consequently, using the atomic structure of $H_{2\pi}$ and the previous comments we can reduce the problem to showing that there exists a $C > 0$ for which

$$\| T^{\varphi^+} \mathrm{a} \|_1 \leq C \tag{4.3}$$

holds for all atoms a.

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In the proof of (4.3) we may assume that $\operatorname{supp} \mathrm{a} \subset [0, 2^{-N}]$, $\|\mathrm{a}\|_\infty \le 2^{N+1}$, and $\int_0^{2\pi} \mathrm{a} = 0$. To apply Lemma 4.1 and separate the lower frequency terms from the higher, we define the sequence of trigonometric polynomials $U_n \mathrm{a}$ $(N \le n < \infty)$ as

$$U_n \mathrm{a} = \sum_{k=1}^{2^{N+1}} \lambda(k)\varphi(k)\widehat{\mathrm{a}}(k)e_k + \sum_{k=2^N}^{2^n} \mu(k)\varphi(k)\widehat{\mathrm{a}}(k)e_k,$$

where

$$\lambda(k) = \begin{cases} 1 & \text{if } 0 \le k \le 2^N, \\ 1 - \dfrac{k-2^N}{2^N} & \text{if } 2^N < k \le 2^{N+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mu(k) = \begin{cases} \dfrac{k-2^N}{2^N} & \text{if } 2^N \le k \le 2^{N+1}, \\ 1 & \text{if } 2^{N+1} < k \le 2^{n-1}, \\ 1 - \dfrac{k-2^{n-1}}{2^{n-1}} & \text{if } 2^{n-1} < k \le 2^n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\sum_{k=2^N}^{2^n} \Delta\mu(k) = 0$ as $\mu(2^N) = \mu(2^n) = \mu(2^n+1) = 0$. So we may apply our lemma to the sequence $\{\Delta\mu(k)\}$. Also $\lambda(k) + \mu(k) = 1$ for $2^N \le k \le 2^{N+1}$ to compensate for the linear increase in $\mu(k)$. Descriptively, we use two ramp functions.

By the Parseval equality, a simple comparison of the Fourier coefficients of $T^{\varphi^+} \mathrm{a}$ and $U_n \mathrm{a}$ shows that the later converges to the first in $L^2_{2\pi}$, and so in $L^1_{2\pi}$ as well. Therefore, $\lim_{n\to\infty} \|T^+_\varphi \mathrm{a} - U_n \mathrm{a}\|_1 = 0$. Consequently, (4.3) can be proved by showing that

$$\|U_n \mathrm{a}\|_1 \le C \quad (n \in \mathbb{N}). \tag{4.4}$$

The usual L^2 argument shows that

$$\int_{-2^{-N+1}}^{2^{-N+1}} |U_n \mathrm{a}| < C. \tag{4.5}$$

Indeed, it follows from $\operatorname{supp} \mathrm{a} \subset [0, 2^{-N}]$, and $\|\mathrm{a}\|_\infty \le 2^{N+1}$ that $\|\mathrm{a}\|_2 \le 2^{N/2+1}$. Also, since $|\lambda(k)| \le 1$, $|\mu(k)| \le 1$ $(k \in \mathbb{N})$, and φ is bounded we have $\|U_n \mathrm{a}\|_2 \le C\|\mathrm{a}\|_2$. Thus $\int_{-2^{-N+1}}^{2^{-N+1}} |U_n \mathrm{a}| \le \left(2^{-N+2}\right)^{1/2} \|U_n \mathrm{a}\|_2 \le C$ by the Cauchy–Schwartz inequality.

In order to show $\int_{2^{-N+1}}^{2\pi - 2^{-N+1}} |U_n \mathrm{a}| < C$, let R and Q_n be the kernel functions that correspond to the sequences $\{\lambda(k)\}$ and $\{\mu(k)\}$, respectively. More precisely, let

$$R = \sum_{k=0}^{2^{N+1}} \lambda(k)\varphi(k)e_k, \quad Q_n = \sum_{k=2^N}^{2^n} \mu(k)\varphi(k)e_k.$$

Then by definition

$$U_n\,\mathrm{a}(x) = \int_0^{2\pi} R(x-y)\,\mathrm{a}(y)\,dy + \int_0^{2\pi} Q_n(x-y)\,\mathrm{a}(y)\,dy.$$

Since $\int_0^{2\pi} \mathrm{a} = 0$, and supp a $\subset [0, 2^{-N}]$, we can write

$$U_n\,\mathrm{a}(x) = \int_0^{2^{-N}} \big(R(x-y) - R(x)\big)\,\mathrm{a}(y)\,dy + \int_0^{2^{-N}} Q_n(x-y)\,\mathrm{a}(y)\,dy.$$

Hence

$$\begin{aligned}(4.6)\qquad \int_{2^{-N+1}}^{2\pi-2^{-N+1}} |U_n\,\mathrm{a}(x)|\,dx \le \int_0^{2^{-N}} |\,\mathrm{a}(y)| \int_{2^{-N+1}}^{2\pi-2^{-N+1}} |Q_n(x-y)|\,dx\,dy \\ + \int_0^{2^{-N}} |\,\mathrm{a}(y)| \int_{2^{-N+1}}^{2\pi-2^{-N+1}} |R(x-y)-R(x)|\,dx\,dy.\end{aligned}$$

We will show that the integrals of the kernel functions R, and Q_n in (4.6) are bounded. Let us start with the integral for R. For a fixed $y \in [0, 2^{-N}]$ set

$$\psi(0) = 0, \quad \text{and} \quad \psi(k) = \lambda(k)\varphi(k)(e_k(-y) - 1) \quad (k \ge 1).$$

Then $R(x-y) - R(x) = \sum_{k=0}^{2^{N+1}} \psi(k)e_k(x)$. Summation by parts yields

$$R(x-y) - R(x) = \sum_{k=0}^{2^{N+1}-1} \Delta\psi(k)D_k + \psi(2^{N+1})D_{2^{N+1}}.$$

Since $\psi(0) = \psi(2^{N+1}) = 0$ we have by Lemma 4.1 that

$$(4.7)\qquad \int_{2^{-N+1}}^{2\pi-2^{-N+1}} |R(x-y)-R(x)|\,dx \le C\big(2^{N-1}\big)^{1-1/r}\Big(\sum_{k=0}^{2^{N+1}-1} |\Delta\psi(k)|^r\Big)^{1/r}$$

as long as $r \le 2$. Without loss of generality we may assume so. It is easy to check by direct calculation that

$$\Delta\psi(k) = \alpha_k\Delta\varphi(k) + \beta_k\varphi(k+1),$$

where

$$\alpha_k = \lambda(k)(e_k(-y) - 1) \quad (1 \le k < 2^{N+1}),$$

and

$$\beta_k = \begin{cases} e_k(-y)(1 - e_1(-y)) & \text{if } 1 \le k < 2^N, \\ \lambda(k)(e_k(-y) - 1) - \lambda(k+1)(e_{k+1}(-y) - 1) & \text{if } 2^N \le k \le 2^{N+1}. \end{cases}$$

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Since $0 \leq \lambda(k) \leq 1$, and $0 \leq y \leq 2^{-N}$ we have $|\alpha_k| \leq Ck2^{-N}$. Similarly, $|\beta_k| \leq C2^{-N}$ is obvious from $0 \leq y \leq 2^{-N}$ if $1 \leq k < 2^N$. For $2^N \leq k < 2^{N+1}$ let us rewrite β_k as

$$\beta_k = \Bigl(1 - \frac{k-2^N}{2^N}\Bigr) e_k(-y)(1-e_1(-y)) + \frac{1}{2^N}(e_{k+1}(-y)-1).$$

Since both terms are of order 2^{-N} we obtain $|\beta_k| \leq C2^{-N}$ for $2^N \leq k < 2^{N+1}$ as well. Consequently,

$$\Bigl(\sum_{k=0}^{2^{N+1}-1} |\Delta\psi(k)|^r\Bigr)^{1/r} \leq |\Delta\psi(0)| + C\sum_{j=0}^{N} \frac{2^{j+1}}{2^N}\Bigl(\sum_{k=2^j}^{2^{j+1}-1} |\Delta\varphi(k)|^r\Bigr)^{1/r} + C\frac{1}{2^N}\Bigl(\sum_{k=1}^{2^{N+1}-1} |\varphi(k)|^r\Bigr)^{1/r}.$$

By the construction, $|\Delta\psi(0)| = |\psi(1)| \leq C2^{-N}$. Let us use (2.1) for the first sum to obtain

$$\sum_{j=0}^{N} \frac{2^{j+1}}{2^N}\Bigl(\sum_{k=2^j}^{2^{j+1}-1} |\Delta\varphi(k)|^r\Bigr)^{1/r} \leq C\sum_{j=0}^{N} \frac{2^{j+1}}{2^N}(2^j)^{-1+1/r} \leq C(2^N)^{-1+1/r}.$$

The same estimate holds for the second sum since φ is bounded. Consequently, $\bigl(\sum_{k=0}^{2^{N+1}-1} |\Delta\psi(k)^r|\bigr)^{1/r} \leq C(2^N)^{-1+1/r}$, which by (4.7) implies

$$\int_{2^{-N+1}}^{2\pi-2^{-N+1}} |R(x-y) - R(x)|\,dx \leq C. \tag{4.8}$$

The integral for Q_n in (4.6) can be estimated in basically the same manner. Since $0 \leq y \leq 2^{-N}$ we have that

$$\int_{2^{-N+1}}^{2\pi-2^{-N+1}} |Q_n(x-y)|\,dx \leq \int_{2^{-N}}^{2\pi-2^{-N+1}} |Q_n(x)|\,dx. \tag{4.9}$$

Set

$$\psi(k) = \mu(k)\varphi(k).$$

Then $\psi(2^N) = \psi(2^n) = 0$. Hence $\sum_{k=2^N}^{2^n} \Delta\psi(k) = 0$. Summation by parts yields $Q_n(x) = \sum_{k=2^N}^{2^n-1} \Delta\psi(k)D_k$. Consequently, by Lemma 4.1 we have

$$\int_{2^{-N}}^{2\pi-2^{-N+1}} |Q_n(x)|\,dx \leq C(2^N)^{1-1/r}\Bigl(\sum_{k=2^N}^{2^n-1} |\Delta\psi(k)|^r\Bigr)^{1/r}. \tag{4.10}$$

Similarly to the previous case, $\Delta\psi(k)$ can be decomposed as $\Delta\psi(k) = \alpha_k\Delta\varphi(k) + \beta_k\varphi(k+1)$, with

$$\alpha_k = \mu(k), \quad \text{and} \quad \beta_k = \begin{cases} -2^{-N} & \text{if } 2^N \leq k < 2^{N+1}, \\ 2^{-n+1} & \text{if } 2^{n-1} \leq k < 2^n. \end{cases}$$

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Then

$$\Big(\sum_{k=2^N}^{2^n-1}|\Delta\psi(k)|^r\Big)^{1/r} \le \Big(\sum_{j=N}^{n-1}\sum_{k=2^j}^{2^{j+1}-1}|\mu(k)\Delta\varphi(k)|^r\Big)^{1/r} + 2^{-N}\Big(\sum_{k=2^N+1}^{2^{N+1}}|\varphi(k)|^r\Big)^{1/r} + 2^{-n+1}\Big(\sum_{k=2^{n-1}+1}^{2^n}|\varphi(k)|^r\Big)^{1/r}.$$

Since $|\mu(k)| \le 1$ we have from the conditions upon φ that

$$\Big(\sum_{j=N}^{n-1}\sum_{k=2^j}^{2^{j+1}-1}|\mu(k)\Delta\varphi(k)|^r\Big)^{1/r} \le C\Big(\sum_{j=N}^{n-1}(2^j)^{1-r}\Big)^{1/r} \le C(2^N)^{-1+1/r},$$

and

$$2^{-\ell}\Big(\sum_{k=2^\ell}^{2^{\ell+1}-1}|\varphi(k)|^r\Big)^{1/r} \le C2^{-\ell}(2^\ell)^{1/r} \le C(2^N)^{-1+1/r} \qquad (\ell = N, n).$$

Consequently, it follows from (4.9) and (4.10) that

$$\int_{2^{-N+1}}^{2\pi-2^{-N+1}}|Q_n(x-y)|\,dx \le C. \tag{4.11}$$

Then (4.8) and (4.11) together imply that we can continue the estimate in (4.6) as follows:

$$\int_{2^{-N+1}}^{2\pi-2^{-N+1}}|U_n\,\mathrm{a}(x)|\,dx \le C\int_0^{2^{-N}}|\,\mathrm{a}(y)|\,dy.$$

Since a is an atom supported on $[0, 2^{-N}]$ we have that $\int_{2^{-N+1}}^{2\pi-2^{-N+1}}|U_n\,\mathrm{a}(x)|\,dx \le C$. This, along with (4.5), means that (4.4) holds. ■

Proof of Theorem 2.3 We will use the correspondence between $H(T)$ and $H_{2\pi}$ described in the beginning of the proof of Theorem 2.1. Namely, we will construct a function $f \in H(T)$, and a multiplier $\varphi = \{\varphi\}_{k=0}^{\infty}$ that is of bounded variation but no integrable function exists whose Fourier series is $S^\varphi f$. We start with the following elementary inequalities that are needed in our construction:

$$\text{(4.12)}\qquad \text{(i)}\quad \int_a^b|D_n(x)|\,dx \le \pi\ln\frac{b}{a} \qquad (0 < a < b \le \pi),$$

$$\text{(ii)}\quad \int_a^b|D_n(x)|\,dx \le n(b-a) \qquad (0 \le a < b \le 2\pi),$$

$$\text{(iii)}\quad \int_a^b|D_n(x)|\,dx \ge \frac{2}{\pi}\ln\frac{b}{a+\frac{\pi}{n+1}} \qquad (0 \le a < b < 2\pi).$$

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The first two follow easily from $|D_n(x)| \le n$ $(0 \le x \le 2\pi)$, and

$$|D_n(x)| = \left| \frac{e^{i(n+1)x} - 1}{e^{ix} - 1} \right| \le \frac{2}{2\sin\frac{x}{2}} \le \frac{\pi}{x} \quad (0 < x \le \pi).$$

For the proof of the third one we use the lower estimate

$$|D_n(x)| \ge \frac{|\sin(n+1)x|}{2\sin\frac{x}{2}} \ge \frac{|\sin(n+1)x|}{x} \quad (0 < x \le 2\pi).$$

After a change of variable we obtain

$$\int_a^b |D_n(x)|\,dx \ge \int_{(n+1)a}^{(n+1)b} \frac{|\sin x|}{x} dx.$$

We may suppose that $(n+1)a + \pi < (n+1)b$. Let α, β be real with $\beta - \alpha > \pi$, and let ℓ denote the greatest integer for which $\alpha + \ell\pi \le \beta$ holds. Then

$$\begin{aligned}\int_\alpha^\beta \frac{|\sin x|}{x}\,dx &\ge \sum_{j=1}^{\ell} \frac{1}{\alpha + j\pi} \int_{\alpha+(j-1)\pi}^{\alpha+j\pi} |\sin x|\,dx = 2\sum_{j=1}^{\ell} \frac{1}{\alpha + j\pi} \\ &\ge \frac{2}{\pi} \int_{\alpha+\pi}^{\alpha+(\ell+1)\pi} \frac{1}{x}\,dx \\ &\ge \frac{2}{\pi} \ln \frac{\beta}{\alpha+\pi}.\end{aligned}$$

Then (4.12)(iii) follows by choosing $\alpha = (n+1)a$, $\beta = (n+1)b$.

Now we continue with the construction of $f \in H(T)$. Let us define the trigonometric polynomial f_k as

$$f_k = \sum_{j=0}^{2^{n_k}} c_j^{(k)} e_{2^{n_k}+j} \quad (k \in \mathbb{N}), \tag{4.13}$$

where $n_k = 2^{5k}$, and

$$c_j^{(k)} = \begin{cases} j2^{-n_k+1} & \text{if } 0 \le j \le 2^{n_k-1}, \\ 2 - j2^{-n_k+1} & \text{if } 2^{n_k-1} < j \le 2^{n_k}. \end{cases}$$

Then f will be of the form $\sum_{k=1}^{\infty} \alpha_k f_k$. We show, under a simple condition on α_k's, that f will belong to $H(T)$. For this purpose we use summation by parts in (4.13). Since $c_0^{(k)} = c_{2^{n_k}}^{(k)} = 0$, we have

$$f_k = e_{2^{n_k}} \sum_{j=0}^{2^{n_k}} c_j^{(k)} e_j = e_{2^{n_k}} \sum_{j=0}^{2^{n_k}-1} \Delta c_j^{(k)} D_j.$$

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Clearly

$$\sum_{j=0}^{2^{n_k}-1} \Delta c_j^{(k)} = 0, \quad \text{and} \quad \Delta c_j^{(k)} = \begin{cases} -2^{-n_k+1} & \text{if } 0 \le j < 2^{n_k-1}, \\ 2^{-n_k+1} & \text{if } 2^{n_k-1} \le j < 2^{n_k}. \end{cases}$$

At this point we can use the following Sidon type inequality (see Schipp [Sch]):

$$\int_0^{2\pi} \Big| \sum_{j=0}^{n} a_j D_j(x) \Big| \, dx \le C(n+1) \max_{0 \le j \le n} |a_j|$$

provided $\sum_{j=0}^n a_j = 0$. Hence $\int_0^{2\pi} |f_k(x)| \, dx \le C 2^{n_k} 2^{-n_k-1} \le C$ $(k \in \mathbb{N})$. Consequently, if $\sum_{k=0}^\infty |\alpha_k| < \infty$ then $f = \sum_{k=1}^\infty \alpha_k f_k \in H_{2\pi}$.

Let us define the multiplier φ as follows:

$$\varphi(\ell) = \begin{cases} \dfrac{\beta_k}{c^{(k)}_{\ell - 2^{n_k}}} & \text{if } 2^{n_k} + 2^{n_k-2} \le \ell \le 2^{n_k} + 3 \cdot 2^{n_k-2}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta_k > 0$ with $\sum_{k=1}^\infty \beta_k < \infty$. We will specify β_k later. By definition $c_j^{(k)}$ is increasing in j from 0 to 2^{n_k-1} and decreasing from 2^{n_k-1} to 2^{n_k}. Therefore,

$$\begin{aligned}\sum_{\ell=2^{n_k}}^{2^{n_k+1}-1} |\Delta\varphi(\ell)| =& \varphi(2^{n_k} + 2^{n_k-2}) + |\varphi(2^{n_k} + 2^{n_k-2}) - \varphi(2^{n_k} + 2^{n_k-1})| \\ & + |\varphi(2^{n_k} + 2^{n_k-1}) - \varphi(2^{n_k} + 3 \cdot 2^{n_k-2})| + \varphi(2^{n_k} + 3 \cdot 2^{n_k-2}) \\ =& \beta_k (2 + |2-1| + |1-2| + 2) \\ =& 6\beta_k. \end{aligned}$$

On the other hand, $\sum_{\ell=2^j}^{2^{j+1}-1} |\Delta\varphi(\ell)| = 0$ if j is not one of the n_k's. Consequently, φ is of bounded variation.

Let us apply the multiplier φ to the Fourier series of $f = \sum_{k=1}^\infty \alpha_k f_k$. For the Fourier partial sums we obtain

$$\begin{aligned} S_j^\varphi f &= \sum_{\ell=0}^{j} \varphi(\ell) \widehat{f}(\ell) e_\ell = \sum_{i=1}^{k} \alpha_i \beta_i \sum_{j=2^{n_i}+2^{n_i-2}}^{2^{n_i}+3\cdot 2^{n_i-2}} e_j = \sum_{i=1}^{k} e_{2^{n_i}+2^{n_i-2}} \alpha_i \beta_i D_{2^{n_i-1}} \\ &= \sum_{i=1}^{k} g_i \quad (2^{n_k+1} \le j \le 2^{n_{k+1}-1}, k \in \mathbb{N}). \end{aligned}$$

If $S^\varphi f$ was the Fourier series of an integrable function, then the de la Vallée Poussin means of the partial sums would converge to that function in norm. We will show that the sequence of these means is not bounded in $L^1_{2\pi}$. Then we can conclude that the multiplier φ is not bounded from $H_{2\pi}$ to $L^1_{2\pi}$.

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By definition, the nth de la Vallée Poussin mean $V_n S^\varphi f$ of $S^\varphi f$ is

$$V_n S^\varphi f = \frac{1}{n} \sum_{j=n}^{2n-1} S_j^\varphi f.$$

Then it follows from the construction that

$$V_{2^{n_k+1}} S^\varphi f = \sum_{j=1}^{k} g_j.$$

We will show that $\int_0^{2\pi} \left| \sum_{j=1}^k g_j(x) \right| dx \to \infty$ as $k \to \infty$. To accomplish this, write

$$\begin{aligned}
\int_0^{2\pi} \Big| \sum_{j=1}^{k} g_j(x) \Big| \, dx &\geq \sum_{\ell=1}^{k} \int_{2^{-n_\ell}}^{2^{-n_{\ell-1}}} \Big| \sum_{j=1}^{k} g_j(x) \Big| \, dx \\
&\geq \sum_{\ell=1}^{k} \Big(\int_{2^{-n_\ell}}^{2^{-n_{\ell-1}}} |g_\ell(x)| \, dx - \sum_{j=1}^{\ell-1} \int_{2^{-n_\ell}}^{2^{-n_{\ell-1}}} |g_j(x)| \, dx \\
&\qquad - \sum_{j=\ell+1}^{k} \int_{2^{-n_\ell}}^{2^{-n_{\ell-1}}} |g_j(x)| \, dx \Big).
\end{aligned}$$

By (4.12) we have

$$\begin{aligned}
A_\ell &= \int_{2^{-n_\ell}}^{2^{-n_{\ell-1}}} |g_\ell(x)| \, dx \geq \frac{2}{\pi} |\alpha_\ell \beta_\ell| \ln \frac{2^{n_\ell}}{(2\pi + 1) 2^{n_{\ell-1}}} \\
&\geq \frac{2}{\pi} |\alpha_\ell \beta_\ell| \ln 2^{n_\ell - n_{\ell-1} - 3}, \\
B_\ell &= \sum_{j=1}^{\ell-1} \int_{2^{-n_\ell}}^{2^{-n_{\ell-1}}} |g_j(x)| \, dx \leq 2^{-n_{\ell-1}} \sum_{j=1}^{\ell-1} |\alpha_j \beta_j| 2^{n_j - 1}, \\
C_\ell &= \sum_{j=\ell+1}^{k} \int_{2^{-n_\ell}}^{2^{-n_{\ell-1}}} |g_j(x)| \, dx \leq \pi \sum_{j=\ell+1}^{k} |\alpha_j \beta_j| \ln 2^{n_\ell - n_{\ell-1}}.
\end{aligned} \tag{4.14}$$

Set $\alpha_j = \beta_j = n_j^{-1/2} = 2^{-5j/2}$, $1 \leq j < \infty$. Substituting these values into (4.14) we obtain

$$\begin{aligned}
A_\ell &\geq \frac{2 \ln 2}{\pi} \Big(1 - \frac{n_{\ell-1} + 3}{n_\ell} \Big) \geq \frac{7 \ln 2}{4\pi} > \frac{7}{8\pi} > 0.25, \\
B_\ell &< 2^{-n_{\ell-1}} \alpha_1 \beta_1 2^{n_{\ell-1}} = \frac{1}{32}, \\
C_\ell &< \pi \ln 2 \, n_\ell \sum_{j=\ell+1}^{k} \frac{1}{n_j} < 2\pi \ln 2 \frac{n_\ell}{n_{\ell+1}} = \frac{\pi \ln 2}{16} < \frac{\pi}{16} < 0.2.
\end{aligned}$$

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Consequently, $A_\ell - B_\ell - C_\ell \geq 0.01$. This implies

$$\int_0^{2\pi} \left| V_{2^{2^{5k}+1}} S^\varphi f(x) \right| \, dx \geq \sum_{\ell=1}^{k} (A_\ell - B_\ell - C_\ell) > 0.01\, k \quad (k > 1).$$

We conclude that there is no integrable function whose Fourier series is $S^\varphi f$. ■

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