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Quasisymmetrically rigid self-similar carpets

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Abstract. We define balanced self-similar quasi-round carpets and compare the carpet moduli of some path families relating to such a carpet. Then, using some known results on quasiconformal geometry of carpets, we prove that the group of quasisymmetric self-homeomorphisms of every balanced self-similar quasi-round carpet is finite. Furthermore, we prove that some balanced self-similar carpets in the unit square with strong geometric symmetry are quasisymmetrically rigid by using the quasisymmetry of weak tangents of carpets.

Key words: balanced self-similar quasi-round carpet, quasisymmetric rigidity, carpet modulus, weak tangent
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1. *Introduction*

Bonk, Kleiner, and Merenkov established a nice quasiconformal geometry of carpets, by showing that quasisymmetric maps between carpets behave like conformal maps between regions [**1**–**5**, **10**, **11**]. We shall state some results of this theory in §2.

Motivated by the quasisymmetric rigidity of the standard Sierpiński carpet given by Bonk and Merenkov [**4**, **5**], we study the following question.

Question 1.1. Which self-similar Sierpiński carpets are quasisymmetrically rigid?

We start with definitions and notation. A topological homeomorphism $f : (X, d_X) \to (Y, d_Y)$ of metric spaces is called quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right) \quad \text{for all } x, y, z \in X, x \neq z.$$

In the case $Y = X$, we say that f is a quasisymmetric self-homeomorphism of X. Denote by $\mathrm{QS}(X)$ the collection of quasisymmetric self-homeomorphisms of X. It is well known that $\mathrm{QS}(X)$ is a group (cf. [7, 13]). If $\mathrm{QS}(X) = \mathrm{ISO}(X)$, we say that X is quasisymmetrically rigid. Hereafter, $\mathrm{ISO}(X)$ is the group of all isometries of X onto itself.

Let $\mathbb{C}$ be the complex plane. For a subset E of $\mathbb{C}$, we denote by $\mathrm{cl}(E)$, ∂E, $\mathrm{int}(E)$, and $\mathrm{diam}(E)$ its closure, boundary, interior, and diameter, respectively. A subset S of $\mathbb{C}$ is called a carpet if

$$S = D \setminus \bigcup_{j=1}^{\infty} D_j,$$

where D is a closed Jordan region and D_j, $j \geq 1$, are open Jordan regions satisfying:

(1) $\mathrm{cl}(D_j)$, $j \geq 1$, are pairwise disjoint subsets of $\mathrm{int}(D)$;

(2) $\lim_{j\to\infty} \mathrm{diam}(D_j) = 0$; and

(3) $\mathrm{int}(S) = \emptyset$.

In this case, we also say that S is a D-carpet. The boundaries ∂D, ∂D_j, $j \geq 1$, are called peripheral circles of S, among which ∂D is called the outer peripheral circle of S and denoted by O. Denote the family of peripheral circles of S as

$$\mathcal{C}(S) := \{O\} \cup \{\partial D_j : j \geq 1\}.$$

The carpets have the following topological properties (cf. [2, 15]). They are all topologically equivalent. The group of topological self-homeomorphisms of an arbitrarily given carpet is uncountable. Any homeomorphism $h : S \to T$ of carpets S and T sends peripheral circles of S to peripheral circles of T.

We say that a carpet is quasi-round if its peripheral circles are uniform quasicircles. Here, a K-quasicircle is a topological circle in $\mathbb{C}$ satisfying

$$\mathrm{diam}(\gamma_{xy}) \wedge \mathrm{diam}(C \setminus \gamma_{xy}) \leq K|x - y|$$

for all $x, y \in C$, where γ_{xy} is a subarc of C of endpoints x and y. We say that the peripheral circles of a carpet are uniform quasicircles if they are K-quasicircles for some common constant $K \geq 1$.

Definition 1.2. We say that a D-carpet is self-similar, if there is a family Φ consisting of finite contractive similarities of $\mathbb{C}$ such that

$$S = \bigcup_{\phi\in\Phi} \phi(S)$$

and that $\phi(\mathrm{int}(D))$, $\phi \in \Phi$, are pairwise disjoint subsets of $\mathrm{int}(D)$. In this case, the family Φ is called an IFS of S.

Here, a map $\phi : \mathbb{C} \to \mathbb{C}$ is called a contractive similarity, if there is a constant $r \in (0, 1)$ such that $|\phi(x) - \phi(y)| = r|x - y|$ for all $x, y \in \mathbb{C}$. For a self-similar D-carpet with IFS Φ, we have assumed the open set condition for the open set $\mathrm{int}(D)$, so such a carpet is of Hausdorff dimension less than 2 (cf. [6]).

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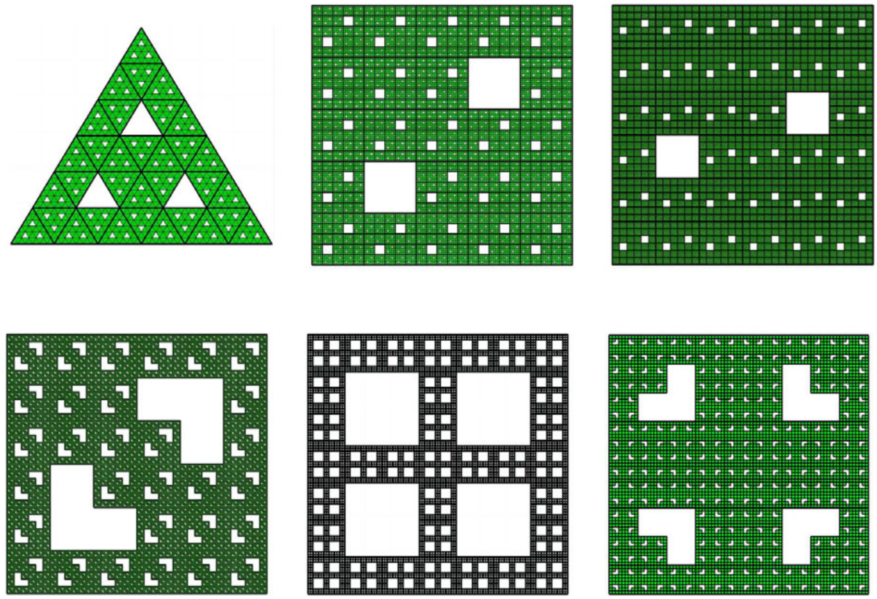

FIGURE 1. Examples of balanced self-similar quasi-round carpets.

Let S be a self-similar D-carpet with IFS Φ. Then, the set

$$\mathrm{int}(D) \setminus \bigcup_{\phi \in \Phi} \phi(D)$$

is open and its bounded components are Jordan domains whose boundaries are called the first generation peripheral circles of S. We denote by $\mathcal{C}_1(S)$ the family of such peripheral circles.

Definition 1.3. We say that a self-similar carpet is balanced if the family $\mathcal{C}_1(S)$ is finite and $\mathrm{mod}_S\Gamma(O, M; S)$ takes the same value for all $M \in \mathcal{C}_1(S)$, where $\mathrm{mod}_S\Gamma(O, M; S)$ is the carpet modulus of the path family $\Gamma(O, M; S)$.

The definition of carpet modulus will be given in §3. Clearly, every self-similar carpet S with $\#\mathcal{C}_1(S) = 1$ is balanced. Let S be a self-similar quasi-round carpet with $\#\mathcal{C}_1(S) > 1$. If for every pair $M, C \in \mathcal{C}_1(S)$ there is an $h \in \mathrm{QS}(S)$ such that $h(O) = O$ and $h(M) = C$, then S is balanced. By this fact and geometric symmetry, we easily construct a lot of balanced self-similar quasi-round carpets. We illustrate six such carpets in Figure 1.

The first main result of this paper is as follows.

THEOREM 1.4. *Let S be a balanced self-similar quasi-round carpet. Then,* $\mathrm{QS}(S)$ *is a finite group.*

Theorem 1.4 generalizes the corresponding results of [**4**, **16**]. The proof is still based on comparing the carpet moduli $\mathrm{mod}_S\Gamma(O, M; S)$ and $\mathrm{mod}_S\Gamma(C_1, C_2; S)$, where $M \in \mathcal{C}_1(S)$

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and $C_1, C_2 \in \mathcal{C}(S)$, with $C_1 \neq C_2$ and $\{C_1, C_2\} \neq \{O, J\}$ for any $J \in \mathcal{C}_1(S)$. At present, we do not know if the inequality

$$\mathrm{mod}_S \Gamma(C_1, C_2; S) < \mathrm{mod}_S \Gamma(O, M; S)$$

is always true, but we prove that it is true for enough of pairs C_1, C_2 in §4. By this discussion and quasiconformal geometry of carpets, we give a complete proof of Theorem 1.4 in §5.

Applying Theorem 1.4 and the quasisymmetry of weak tangents of carpets, we may prove that some self-similar carpets with strong geometric symmetry are quasisymmetrically rigid.

Let $Q := \{z = x + iy : 0 \leq x, y \leq 1\}$ be the unit square. Let γ_0 and γ_1 be the diagonals of Q passing through the vertices 0 and 1, respectively. Let γ_h and γ_v be the horizontal and vertical median lines of Q. Let R_0, R_1, R_h, and R_v be the reflections in γ_0, γ_1, γ_h, and γ_v. Let $r_{\pi/2}$, r_π, and $r_{3\pi/2}$ be the rotations of angles $\pi/2$, π, and $3\pi/2$ around the center of Q. Then,

$$\mathrm{ISO}(Q) = \{\mathrm{id}, r_{\pi/2}, r_\pi, r_{3\pi/2}, R_0, R_1, R_h, R_v\}.$$

Definition 1.5. We say that a self-similar Q-carpet S is symmetric if

$$\mathrm{ISO}(S) = \mathrm{ISO}(Q).$$

A balanced self-similar Q-carpet is not necessarily symmetric. In Figure 1, the five self-similar Q-carpets are all balanced, but only the final two are symmetric.

It is clear that every symmetric self-similar Q-carpet S with $\#\mathcal{C}_1(S) = 4$ is balanced and quasi-round, so the group $\mathrm{QS}(S)$ is finite by Theorem 1.4. We obtain a sufficient condition for such a carpet to be quasisymmetrically rigid. In what follows, we say that a family of contractive similarities is homogeneous if their similarity ratios are equal to each other.

THEOREM 1.6. *Let S be a symmetric self-similar Q-carpet with $\#\mathcal{C}_1(S) = 4$. Suppose that there is a peripheral circle $C \in \mathcal{C}_1(S)$ and a point $z \in C \cap \gamma_0$ such that the family*

$$\Phi_z := \{\phi \in \Phi : z \in \phi(Q)\}$$

is homogeneous and of $\#\Phi_z = 3$. Then, S is quasisymmetrically rigid.

For example, the final two symmetric self-similar Q-carpets in Figure 1 satisfy the assumption condition of Theorem 1.6. Also, there are symmetric self-similar Q-carpets with $\#\mathcal{C}_1(S) = 4$, which do not satisfy this condition; see Figure 2.

Theorem 1.6 improves the result of Zeng and Su [16]. We remark that a similar result can be formulated for symmetric self-similar T-carpets, where T is a closed solid equilateral triangle. For example, one may prove that the first carpet in Figure 1 is quasisymmetrically rigid. The quasisymmetric rigidity question of general balanced self-similar Q-carpets and T-carpets is still open.

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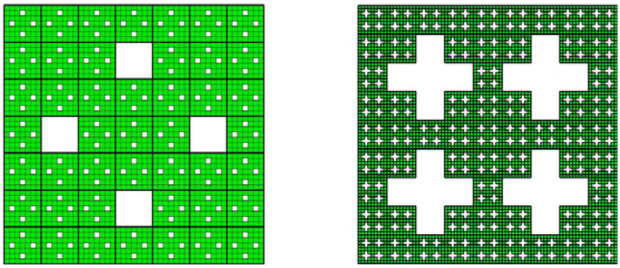

FIGURE 2. Two carpets not satisfying the conditions of Theorem 1.6.

2. *Quasiconformal geometry of carpets*

In this section, we state some results from quasiconformal geometry of carpets that will be used in proving Theorem 1.4.

See Bonk [2]. Let S be a carpet and $\mathcal{C}(S)$ be the family of its peripheral circles. The family $\mathcal{C}(S)$ is said to be uniformly relatively separated if

$$\inf\left\{\frac{\operatorname{dist}(C_1, C_2)}{\operatorname{diam}(C_1)\wedge\operatorname{diam}(C_2)} : C_1, C_2 \in \mathcal{C}(S),\ C_1 \neq C_2\right\} > 0,$$

where $\operatorname{dist}(C_1, C_2) = \min_{z\in C_1, w\in C_2} |z - w|$ is the Euclidean distance between C_1 and C_2 and $a \wedge b = \min\{a, b\}$. We call S a Bonk–Sierpiński carpet if it is of measure zero and its peripheral circles are uniformly relatively separated uniform quasicircles. It is known that this class of carpets are quasisymmetrically invariant.

The following extension theorem is due to Bonk [2].

LEMMA 2.1. *Let S be a quasi-round carpet in $\mathbb{C}$. Then, every quasisymmetric embedding $f : S \to \mathbb{C}$ has a quasiconformal extension $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.*

The following three circle theorem is also due to Bonk [2].

LEMMA 2.2. *Let S be a Bonk–Sierpiński carpet in $\mathbb{C}$. Let C_1, C_2, C_3 be three distinct peripheral circles of S. If f is an orientation-preserving quasisymmetric self-homeomorphism of S such that $f(C_j) = C_j$, $j = 1, 2, 3$, then f is the identity.*

The following rigidity theorems are due to Bonk and Merenkov [4].

LEMMA 2.3. *Let S be a Bonk–Sierpiński carpet in $\mathbb{C}$. Let C_1, C_2 be two distinct peripheral circles of S. Then, the group of all orientation-preserving quasisymmetric self-homeomorphisms of S that fix C_1 and C_2 setwise is a finite cyclic group.*

LEMMA 2.4. *Let S be a Bonk–Sierpiński carpet in $\mathbb{C}$. Let C_1, C_2 be two distinct peripheral circles of S and $p \in S$. If f is an orientation-preserving quasisymmetric self-homeomorphism of S such that $f(C_1) = C_1$, $f(C_2) = C_2$, and $f(p) = p$, then f is the identity.*

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3. *Carpet modulus*

Let ds and m be the Euclidean line element and the Lebesgue measure on $\mathbb{C}$. An embedding $\gamma : (0, 1) \to U$ will be called an open path in U, where U is a subset of $\mathbb{C}$. Let Γ be a family of paths in $\mathbb{C}$. A Borel density $\rho : \mathbb{C} \to [0, \infty]$ is called admissible for Γ if

$$\int_\gamma \rho \, ds \geq 1$$

for each $\gamma \in \Gamma$. The conformal modulus $\mathrm{mod}(\Gamma)$ of Γ is defined by

$$\mathrm{mod}(\Gamma) = \inf_\rho \int_{\mathbb{C}} \rho^2 \, dm,$$

where the infimum is taken over all admissible densities for Γ. We say that an admissible density ρ is extremal for $\mathrm{mod}(\Gamma)$, if it satisfies

$$\mathrm{mod}(\Gamma) = \int_{\mathbb{C}} \rho^2 \, dm.$$

The conformal modulus is invariant under conformal maps and quasi-invariant under quasiconformal maps (cf. [14]). For its various variants and applications, we refer to [2, 8, 9, 12].

Next, we recall the carpet modulus. Let S be a carpet and $\mathcal{C}(S)$ be the family of its peripheral circles. Let Γ be a path family in $\mathbb{C}$. A function $\rho : \mathcal{C}(S) \to [0, +\infty]$ is called admissible for (Γ, S) if there is a subfamily $\Gamma_0 \subseteq \Gamma$ with $\mathrm{mod}(\Gamma_0) = 0$ such that

$$\sum_{C \in \mathcal{C}(S), \gamma \cap C \neq \emptyset} \rho(C) \geq 1$$

for every path $\gamma \in \Gamma \setminus \Gamma_0$. The carpet modulus $\mathrm{mod}_S \Gamma$ of Γ with respect to S is defined by

$$\mathrm{mod}_S \Gamma = \inf_\rho \sum_{C \in \mathcal{C}(S)} \rho(C)^2,$$

where the infimum is taken over all admissible functions for (Γ, S). An admissible function ρ is called extremal for $\mathrm{mod}_S(\Gamma)$ if

$$\mathrm{mod}_S \Gamma = \sum_{C \in \mathcal{C}(S)} \rho(C)^2.$$

By the definition, if $\mathrm{mod}(\Gamma) = 0$, then $\mathrm{mod}_S \Gamma = 0$.

The carpet modulus has been applied as a tool in the study of quasiconformal geometry of carpets in [2, 4]. Among its properties, the following ones are useful in proving Theorem 1.4.

LEMMA 3.1. *Let S be a carpet and Γ_1, Γ_2 be two path families in $\mathbb{C}$. If each path in Γ_1 has a subpath belonging to Γ_2, then*

$$\mathrm{mod}_S \Gamma_1 \leq \mathrm{mod}_S \Gamma_2.$$

The carpet modulus is invariant under quasiconformal maps.

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LEMMA 3.2. [4, Lemma 2.1] *Let S, T be two carpets and Γ be a path family in $\mathbb{C}$. If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map such that $f(S) = T$, then*

$$\mathrm{mod}_T f(\Gamma) = \mathrm{mod}_S \Gamma.$$

Let S be a carpet in $\mathbb{C}$. Let C_1, C_2 be two distinct peripheral circles of S. Let Ω_1 and Ω_2 be respectively the components of $\mathbb{C} \setminus S$ bounded by C_1 and C_2. Let $\Gamma(C_1, C_2; S)$ be the family of paths $\gamma : (0, 1) \to \mathbb{C} \setminus \mathrm{cl}(\Omega_1 \cup \Omega_2)$ with

$$\lim_{t\to 0} \gamma(t) \in C_1 \quad \text{and} \quad \lim_{t\to 1} \gamma(t) \in C_2.$$

LEMMA 3.3. *Let S be a quasi-round carpet in $\mathbb{C}$. Let C_1, C_2 be two distinct peripheral circles of S. If $f \in \mathrm{QS}(S)$, then*

$$\mathrm{mod}_S \Gamma(C_1, C_2; S) = \mathrm{mod}_S \Gamma(f(C_1), f(C_2); S).$$

Proof. By Lemma 2.1, f has a quasiconformal extension $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Note that

$$f(C_1), f(C_2) \in \mathcal{C}(S) \quad \text{and} \quad F(\Gamma(C_1, C_2; S)) = \Gamma(f(C_1), f(C_2); S).$$

We immediately get the desired equality by Lemma 3.2. □

LEMMA 3.4. [4, Proposition 2.4] *Let S be a quasi-round carpet and Γ be a path family in $\mathbb{C}$ with $\mathrm{mod}_S \Gamma < +\infty$. Then, there is a unique extremal admissible function for $\mathrm{mod}_S \Gamma$.*

LEMMA 3.5. [4, Proposition 4.9] *Let S be a Bonk–Sierpiński carpet in $\mathbb{C}$ and let C_1, C_2 be two distinct peripheral circles of S. Then, the carpet modulus $\mathrm{mod}_S \Gamma(C_1, C_2; S)$ is finite and positive. Moreover, if ρ is an extremal admissible function for $\mathrm{mod}_S \Gamma(C_1, C_2; S)$, we have $\rho(C_1) = \rho(C_2) = 0$ and $\rho(C) > 0$ for each $C \in \mathcal{C}(S) \setminus \{C_1, C_2\}$.*

4. *Comparing carpet moduli*

Let S be a self-similar quasi-round carpet with IFS $\{\phi_1, \phi_1, \ldots, \phi_n\}$. To prove Theorem 1.4, we are going to compare the carpet moduli $\mathrm{mod}_S \Gamma(O, M; S)$ and $\mathrm{mod}_S \Gamma(C_1, C_2; S)$, where $M \in \mathcal{C}_1(S)$ and $C_1, C_2 \in \mathcal{C}(S)$.

We use notation from words. Let $A = \{1, 2, \ldots, n\}$ be an alphabet. Let A^k be the set of words of length k and A^* the set of finite words over the alphabet A. By the convention, denote by ε the empty word. For each finite word $w = w_1 w_2 \cdots w_k \in A^k$, let

$$\phi_w := \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_k}$$

denote the composition of maps. If $w = \varepsilon$, ϕ_ε is the identity on $\mathbb{C}$. For simplicity, write E_w for $\phi_w(E)$ if $w \in A^*$ and $E \subset \mathbb{C}$. We say that S_w is a copy of S.

LEMMA 4.1. *Every self-similar quasi-round D-carpet S with IFS $\{\phi_1, \ldots, \phi_n\}$ is a Bonk–Sierpiński carpet and its peripheral circles can be given by*

$$\mathcal{C}(S) = \{O\} \cup \{M_w : w \in A^*, M \in \mathcal{C}_1(S)\}. \tag{1}$$

Proof. By the self-similarity of S,

$$S = D \setminus \bigcup_{w\in A^*} \bigcup_{M\in\mathcal{C}_1(S)} (G(M))_w,$$

where $G(M)$ is the bounded Jordan domain of boundary M. Thus, the peripheral circles of S are given by equation (1).

Let

$$\delta_1 = \min_{M\in\mathcal{C}_1(S)} \frac{\mathrm{dist}(O, M)}{\mathrm{diam}(M)}$$

and

$$\delta_2 = \min_{M,M'\in\mathcal{C}_1(S),M\neq M'} \frac{\mathrm{dist}(M, M')}{\mathrm{diam}(M)\wedge \mathrm{diam}(M')}.$$

Let $\delta = \delta_1 \wedge \delta_2$. Then, $\delta > 0$. Moreover, if C and C' are two distinct peripheral circles of S, we easily get by the self-similarity of S,

$$\frac{\mathrm{dist}(C, C')}{\mathrm{diam}(C)\wedge \mathrm{diam}(C')} \geq \delta,$$

so $\mathcal{C}(S)$ is uniformly relatively separated.

Since S is of Lebesgue measure zero and its peripheral circles are uniform quasicircles, it then follows that S is a Bonk–Sierpiński carpet. The proof is completed. □

Let S be a self-similar quasi-round D-carpet with IFS $\Phi = \{\phi_1, \phi_2, \ldots, \phi_n\}$.

LEMMA 4.2. *Suppose $M \in \mathcal{C}_1(S)$, $w \in A^*$, $w \neq \varepsilon$. Then,*

$$\mathrm{mod}_S\Gamma(O, M_w; S) < \mathrm{mod}_S\Gamma(O, M; S).$$

Proof. By Lemma 4.1, S_w is a Bonk–Sierpiński carpet. Thus, by Lemmas 3.4 and 3.5, the carpet modulus $\mathrm{mod}_{S_w}\Gamma(O_w, M_w; S_w)$ has an extremal admissible function ρ satisfying $\rho(O_w) = 0$. Define $\xi : \mathcal{C}(S) \to [0, +\infty]$ by

$$\xi(C) = \begin{cases} \rho(C) & \text{if } C \in \mathcal{C}(S_w) \setminus \{O_w\}, \\ 0 & \text{if } C \in \mathcal{C}(S) \setminus \mathcal{C}(S_w). \end{cases}$$

Note that every path in $\Gamma(O, M_w; S)$ has a subpath belonging to $\Gamma(O_w, M_w; S_w)$. The admissibility of ρ implies that ξ is admissible for $\Gamma(O, M_w; S)$. However, it is clear that ξ takes zero on infinitely many peripheral circles of S. By Lemma 3.5, ξ cannot be extremal for $\mathrm{mod}_S\Gamma(O, M_w; S)$. It then follows that

$$\mathrm{mod}_S\Gamma(O, M_w; S) < \sum_{C\in\mathcal{C}(S)} \xi(C)^2 = \sum_{C\in\mathcal{C}(S_w)} \rho(C)^2 = \mathrm{mod}_{S_w}\Gamma(O_w, M_w; S_w),$$

which together with the scale invariance of the carpet modulus yields

$$\mathrm{mod}_S\Gamma(O, M_w; S) < \mathrm{mod}_S\Gamma(O, M; S).$$

This completes the proof. □

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LEMMA 4.3. *Let $v, w \in A^*$, $v \neq w$. If int$(D_v) \cap$ int$(D_w) = \emptyset$ or $D_v \supset D_w$, then*

$$\text{mod}_S\Gamma(I_v, M_w; S) < \text{mod}_S\Gamma(O, M; S)$$

for each pair $I, M \in \mathcal{C}_1(S)$.

Proof. It follows by the same argument as that of Lemma 4.2. □

LEMMA 4.4. *Suppose in addition that S is balanced. If $v, w \in A^*$, $v \neq w$, then*

$$\text{mod}_S\Gamma(I_v, M_w; S) < \text{mod}_S\Gamma(O, M; S)$$

for each pair $I, M \in \mathcal{C}_1(S)$.

Proof. Since $\text{int}(D_w) \cap \text{int}(D_v) = \emptyset$ or $D_w \subset D_v$ or $D_v \subset D_w$ by the similarity of S, one has by Lemma 4.3 and the balance of S,

$$\begin{aligned}\text{mod}_S\Gamma(I_v, M_w; S) &< \max\{\text{mod}_S\Gamma(O, I; S),\ \text{mod}_S\Gamma(O, M; S)\} \\ &= \text{mod}_S\Gamma(O, M; S),\end{aligned}$$

as desired. □

5. *Proof of Theorem 1.4*

Let S be a balanced self-similar quasi-round carpet with IFS $\Phi = \{\phi_1, \ldots, \phi_n\}$. We are going to prove Theorem 1.4. Let $g \in \text{QS}(S)$.

CLAIM 5.1. $g(O) \in \{O\} \cup \mathcal{C}_1(S)$.

Proof of Claim 5.1. We argue by contradiction and assume that $g(O) \notin \{O\} \cup \mathcal{C}_1(S)$. Since $g(O)$ is a peripheral circle of the carpet S, by Lemma 4.1, there is a peripheral circle $M \in \mathcal{C}_1(S)$ and a word $w \in A^*$, $w \neq \varepsilon$, such that $g(O) = M_w$. □

Let $J \in \mathcal{C}_1(S)$ be fixed. If $g(J) = O$, one has by Lemmas 3.3 and 4.2,

$$\text{mod}_S\Gamma(O, J; S) = \text{mod}_S\Gamma(M_w, O; S) < \text{mod}_S\Gamma(M, O; S),$$

contradicting the balance of S. If $g(J) = I_v$ for some $I \in \mathcal{C}_1(S)$ and some $v \neq w$, one has by Lemmas 3.3 and 4.4,

$$\text{mod}_S\Gamma(O, J; S) = \text{mod}_S\Gamma(M_w, I_v; S) < \text{mod}_S\Gamma(O, I; S),$$

which is also a contradiction. Thus, $g(J) = I_w$ for some $I \in \mathcal{C}_1(S)$.

It then follows that

$$\{g(O)\} \cup \{g(J) : J \in \mathcal{C}_1(S)\} \subset \{I_w : I \in \mathcal{C}_1(S)\},$$

giving $\#\mathcal{C}_1(S) + 1 \leq \mathcal{C}_1(S)$, which is a contradiction. The proof of Claim 5.1 is completed.

CLAIM 5.2. $g(M) \in \{O\} \cup \mathcal{C}_1(S)$ *for each* $M \in \mathcal{C}_1(S)$.

Proof of Claim 5.2. We argue by contradiction and assume that there exists a peripheral circle $M \in \mathcal{C}_1(S)$ such that $g(M) \notin \{O\} \cup \mathcal{C}_1(S)$. Then, there is a peripheral circle $I \in \mathcal{C}_1(S)$ and a word $w \in A^*$, $w \neq \varepsilon$ such that $g(M) = I_w$. □

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Since Claim 5.1 is proved, one has $g(O) = O$ or $g(O) \in \mathcal{C}_1(S)$. If $g(O) = O$, one has by Lemmas 3.3 and 4.2,

$$\mathrm{mod}_S\Gamma(O, M; S) = \mathrm{mod}_S\Gamma(O, I_w; S) < \mathrm{mod}_S\Gamma(O, I; S),$$

which contradicts the balance of S. If $g(O) \in \mathcal{C}_1(S)$, one has by Lemmas 3.3 and 4.4,

$$\mathrm{mod}_S\Gamma(O, M; S) = \mathrm{mod}_S\Gamma(g(O), I_w; S) < \mathrm{mod}_S\Gamma(O, I; S),$$

which is also a contradiction. This proves Claim 5.2.

By Claims 5.1 and 5.2, we get

$$\{g(O)\} \cup \{g(M) : M \in \mathcal{C}_1(S)\} = \{O\} \cup \mathcal{C}_1(S). \tag{2}$$

Next, we show that the group QS(S) is finite. Let G be the group of all orientation-preserving quasisymmetric self-homeomorphisms of S. It suffices to show that G is finite.

Case 1. $\#\mathcal{C}_1(S) = 1$, say $\mathcal{C}_1(S) = \{M\}$. In this case, S has only one peripheral circle M of first generation. Let

$$G_0 = \{g \in G : g(O) = O,\ g(M) = M\}$$

and

$$P = \{g \in G : g(O) = M,\ g(M) = O\}.$$

Since equation (2) is proved, one has $G = G_0 \cup P$. By Lemma 2.3, G_0 is a finite cyclic group. However, if $P \neq \emptyset$, one has $P = gG_0$ for an arbitrarily given $g \in P$. Thus, either $\#P = 0$ or $\#P = \#G_0$. It follows that G is finite.

Case 2. $\#\mathcal{C}_1(S) > 1$. In this case, let $M, M' \in \mathcal{C}_1(S)$, $M \neq M'$, be given. By equation (2), each $g \in G$ is a solution of an equation system of the form

$$g(O) = C_1,\ g(M) = C_2,\ g(M') = C_3,$$

where C_1, C_2, C_3 are three distinct peripheral circles in $\{O\} \cup \mathcal{C}_1(S)$. Since there are only finitely many such equation systems and each of them has at most one solution in G by Lemma 2.2 (three circle theorem), we get that G is finite. The proof of Theorem 1.4 is completed.

By the proof of Theorem 1.4, we have the following result.

COROLLARY 5.3. *Let S be a balanced self-similar quasi-round carpet and $g \in QS(S)$. Then,*

$$\{g(O)\} \cup \{g(M) : M \in \mathcal{C}_1(S)\} = \{O\} \cup \mathcal{C}_1(S).$$

6. *Proof of Theorem 1.6*

Denote by γ_0 and γ_1 the diagonals of the unite square Q passing through the vertices 0 and 1, respectively. Denote by γ_h and γ_v the horizontal and vertical median lines of Q. Denote by R_0, R_1, R_h, and R_v the reflections in γ_0, γ_1, γ_h, and γ_v. Denote by $r_{\pi/2}$, r_π, and $r_{3\pi/2}$ the rotations of angles $\pi/2$, π, and $3\pi/2$ around the center of Q.

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Let S be a symmetric self-similar Q-carpet with IFS Φ. Suppose that S satisfies the condition of Theorem 1.6. Let $M \in \mathcal{C}_1(S)$ be a peripheral circle such that $M \cap \gamma_0 \neq \emptyset$. Since S is symmetric and $\mathcal{C}_1(S) = 4$, we may write

$$\mathcal{C}_1(S) = \{M, C_1, C_2, C_3\},$$

where C_1, C_2, C_3 satisfy

$$C_1 = r_{\pi/2}(M), \quad C_2 = r_\pi(M), \quad \text{and} \quad C_3 = r_{3\pi/2}(M).$$

It is clear that

$$R_0(M) = M, \quad R_0(C_2) = C_2, \quad C_1 = R_v(M), \quad \text{and} \quad C_3 = R_v(C_2).$$

By symmetry, $M \cap \gamma_0$ has exactly two points, so we may write

$$M \cap \gamma_0 = \{u, v\}.$$

By the condition of Theorem 1.6, we may assume without loss of generality that

$$\Phi_u := \{\phi \in \Phi : u \in \phi(Q)\}$$

is homogeneous and of $\#\Phi_u = 3$.

LEMMA 6.1. *If $f \in QS(S)$, then $f(u) \neq 0$.*

Proof. Let ψ be the contractive similarity in Φ with $0 \in \psi(Q)$ and let c be its ratio. By the symmetry of S, if p is a vertex of Q and $\phi \in \Phi$ satisfying $p \in \phi(Q)$, then the ratio of ϕ is equal to c. Let

$$W(S, 0) := \{\infty\} \cup \bigcup_{n=0}^{\infty} c^{-n} S$$

be a weak tangent of S at 0.

Let

$$N(u) = \bigcup_{\phi \in \Phi_u} \phi(S).$$

Then, $N(u)$ is a relative neighborhood of u in S. Noticing that Φ_u is homogeneous and that u is a vertex of $\phi(Q)$ for each $\phi \in \Phi_u$, one has by the self-similarity and the symmetry of S,

$$c^{-n}(N(u) - u) \subset c^{-n-1}(N(u) - u).$$

Let

$$W(S, u) := \{\infty\} \cup \bigcup_{n=0}^{\infty} c^{-n}(N(u) - u)$$

be a weak tangent of S at u. By the homogeneity of Φ_u, we see that $W(S, u)$ is similar to

$$W(S, 0) \cup e^{-i\pi/2} W(S, 0) \cup e^{i\pi/2} W(S, 0).$$

Arguing as that of [4, Lemma 7.3], we see that there is no quasisymmetric homeomorphism $g : W(S, u) \to W(S, 0)$ such that $g(0) = 0$ and $g(\infty) = \infty$.

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Now, let $f \in \mathrm{QS}(S)$. If $f(u) = 0$, then, by the argument of [4, Lemma 7.2], there is a quasisymmetric homeomorphism $g : W(S, u) \to W(S, 0)$ such that $g(0) = 0$ and $g(\infty) = \infty$, which is a contradiction. Thus, $f(u) \neq 0$. □

By Lemma 6.1 and the symmetry of S, if $f \in \mathrm{QS}(S)$, then $f(u)$ cannot be any vertex of Q.

LEMMA 6.2. *If $f \in QS(S)$ is orientation-preserving such that $f(O) = O$ and $f(M) = M$, then f is the identity on S.*

Proof. Suppose that $f \in \mathrm{QS}(S)$ satisfies the condition of Lemma 6.2. Then, by Corollary 5.3, we have

$$f(C_1) \in \{C_1, C_2, C_3\}.$$

If $f(C_1) = C_1$, one has $f = \mathrm{id}$ by the three circle theorem.
If $f(C_1) = C_2$, one has

$$f(C_2) \in \{C_1, C_3\}.$$

In the case $f(C_2) = C_1$, one has $f(C_3) = C_3$, so $f = \mathrm{id}$ by the three circle theorem, which is a contradiction. In the case $f(C_2) = C_3$, the equalities $f(O) = O$, $f(C_1) = C_2$, and $f(C_2) = C_3$ imply

$$f(O) = r_{\pi/2}(O), \quad f(C_1) = r_{\pi/2}(C_1), \quad \text{and} \quad f(C_2) = r_{\pi/2}(C_2),$$

so $f = r_{\pi/2}$ by the three circle theorem, which contradicts $f(M) = M$. Thus, the case $f(C_1) = C_2$ is impossible.
Similarly, the case $f(C_1) = C_3$ is impossible. This completes the proof. □

By Lemma 6.2 and the symmetry of S, if $f \in \mathrm{QS}(S)$ is orientation-preserving such that $f(O) = O$ and $f(C) = C$ for some $C \in \mathcal{C}_1(S)$, then $f = \mathrm{id}$.

LEMMA 6.3. *There is no $f \in QS(S)$ satisfying*

$$f(O) = M \quad \textit{and} \quad f(M) = O. \tag{3}$$

Proof. We argue by contradiction and assume that there is a map $f \in \mathrm{QS}(S)$ satisfying equation (3). Then, $f^{-1} \circ R_0 \circ f \circ R_0 \in \mathrm{QS}(S)$ is orientation-preserving and satisfies

$$f^{-1} \circ R_0 \circ f \circ R_0(O) = O \quad \text{and} \quad f^{-1} \circ R_0 \circ f \circ R_0(M) = M.$$

By Lemma 6.2, we get $f^{-1} \circ R_0 \circ f \circ R_0 = \mathrm{id}$, so $f \circ R_0 = R_0 \circ f$, so $f(u) = R_0(f(u))$, so $f(u) \in O \cap \gamma_0$, and so $f(u) = 0$ or $1 + i$, which contradicts Lemma 6.1. The proof is completed. □

LEMMA 6.4. *There is no $f \in QS(S)$ satisfying $f(O) = M$.*

Proof. We argue by contradiction and assume that there is a map $f \in \mathrm{QS}(S)$ such that $f(O) = M$. Since Corollary 5.3 and Lemma 6.3 are proved, we only need consider

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the case $f(M) \in \{C_1, C_2, C_3\}$. By the symmetry of S, we may assume without loss of generality that $f(M) = C_1$. Then, by Corollary 5.3, there are three possible cases:

$$f(C_1) = O, \ \ f(C_2) = O, \ \ f(C_3) = O.$$

Case 1. $f(C_1) = O$. In this case, one has

$$f \circ R_v(M) = f(C_1) = O \quad \text{and} \quad f \circ R_v(O) = f(O) = M,$$

which contradicts Lemma 6.3.

Case 2. $f(C_2) = O$. In this case, one has

$$f \circ R_1(M) = O \quad \text{and} \quad f \circ R_1(O) = M,$$

which contradicts Lemma 6.3.

Case 3. $f(C_3) = O$. In this case, one has

$$f \circ R_h(M) = O \quad \text{and} \quad f \circ R_h(O) = M,$$

which contradicts Lemma 6.3.

Thus, $f(O) = M$ is impossible. This completes the proof. □

LEMMA 6.5. *QS*$(S) =$ *ISO*(S).

Proof. It suffices to show $\mathrm{QS}(S) \subseteq \mathrm{ISO}(S)$. Let $f \in \mathrm{QS}(S)$ be orientation-preserving. By Lemma 6.4, $f(O) \neq M$. Since S is symmetric, we further have

$$f(O) \notin \{M, C_1, C_2, C_3\}.$$

It then follows from Corollary 5.3 that $f(O) = O$.

If $f(M) = M$, one has $f = \mathrm{id}$ by Lemma 6.2, so $f \in \mathrm{ISO}(S)$.

If $f(M) = C_1$, one has

$$f \circ r_{3\pi/2}(C_1) = C_1 \quad \text{and} \quad f \circ r_{3\pi/2}(O) = O,$$

so $f \circ r_{3\pi/2} = \mathrm{id}$ by Lemma 6.2, so $f \in \mathrm{ISO}(S)$.

If $f(M) = C_2$, one has

$$f \circ r_{\pi}(C_2) = C_2 \quad \text{and} \quad f \circ r_{\pi}(O) = O,$$

so $f \circ r_{\pi} = \mathrm{id}$ by Lemma 6.2, so $f \in \mathrm{ISO}(S)$.

If $f(M) = C_3$, one has

$$f \circ r_{\pi/2}(C_3) = C_3 \quad \text{and} \quad f \circ r_{\pi/2}(O) = O,$$

so $f \circ r_{\pi/2} = \mathrm{id}$ by Lemma 6.2, so $f \in \mathrm{ISO}(S)$.

Therefore, $f \in \mathrm{ISO}(S)$ if $f \in \mathrm{QS}(S)$ is orientation-preserving. Now, let $f \in \mathrm{QS}(S)$ be orientation-reversing. Then, $f \circ R_0 \in \mathrm{QS}(S)$ is orientation-preserving, so $f \circ R_0 \in \mathrm{ISO}(S)$, and so $f \in \mathrm{ISO}(S)$. □

Now the proof of Theorem 1.6 is completed.

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