Canad. Math. Bull. Vol. **14** (4), 1971

ON A TOPOLOGY GENERATED BY MEASURABLE COVERS

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1. **Introduction.** In [2] we showed how, for a certain class of outer measures on a metric space, a measurable cover $\bar{A}$ could be constructed for each subset A of the space. The function $A \to \bar{A}$ is a closure operator, and in this note some of the properties of the resulting topology are investigated. In particular, we obtain a sufficient condition for the space to be connected.

2. **Preliminaries.** The situation is as in [2]: (X, ρ) is a metric space, C a sequential covering class of closed sets, τ a gauge on C, and ϕ is the outer measure defined by C, τ. The strong upper density of $A \subseteq X$ at $x \in X$ is defined to be

$$D(A, x) = \lim_{\epsilon \to 0^+} \sup \frac{\phi(A \cap I)}{\tau(I)}$$

where the supremum is taken as I ranges over all those sets in C which contain x and whose diameter is less than ε.

We assume that:

(i) the regularity conditions of [2] hold. That is, to every set I from C there corresponds a set I' from C such that:

(a) $I' \supseteq \{p : \rho(p, I) \leq \alpha \cdot d(I)\}$ where α is a finite number greater than 1 and independent of I and $\rho(p, I)$ is the distance from p to I.
(b) $\tau(I') \leq \beta \cdot \tau(I)$ where β is a finite number independent of I.
(c) for every $\varepsilon > 0$ there is a $\delta > 0$ such that $d(I) < \delta$ implies $d(I') < \varepsilon$.

(ii) $\phi(X)$ is finite or, more generally, spheres of finite radius have finite outer measure.

3. **The measure topology.** It is shown in [2] that, for each $A \subseteq X$, the set

$$\bar{A} = A \cup \{x \mid D(A, x) > 0\}$$

is a measurable cover for A. We now note that the function $A \to \bar{A}$ is a topological closure operator. The required four properties are easily verified: to show that

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

it is only necessary to note that

$$D(A \cup B, x) \leq D(A, x) + D(B, x),$$

Received by the editors May 11, 1970 and, in revised form, October 30, 1970.

https://doi.org/10.4153/CMB-1971-089-3 Published online by Cambridge University Press

and the proof that $\bar{\bar{A}}=\bar{A}$ follows from [2, Theorem 10].

We will call the topology generated by this closure operator the *measure topology*. It is similar to that discussed in [5] and the same properties hold. In particular, a set is nowhere dense if and only if it is null, and a set is Borel if and only if it is measurable.

Let f be a real-valued function with domain X. In addition to the properties analogous to those obtained in [5], we note here that if f is bounded and measurable, then it is dominated by an upper semi-continuous function (the topologies referred to are the measure topology on X and the usual topology on the reals) which is equal to f almost everywhere: the cover function obtained in [3] is such a function.

It is interesting to note that proof (i) of [1] is identical to the usual proof of the analogous result for the Lebesgue integral if the latter is expressed in terms of the measure topology.

4. **Sufficient conditions for the connectedness of X.** It is shown in [4, Theorem 3] that if X is euclidean n-space and ϕ is n-dimensional Lebesgue outer measure, then X with the resulting measure topology is connected. This is generalized in [6, Theorem 5.2] to euclidean n-space with outer measures which have a positive parameter of regularity. We now generalize this result, using a modification of the methods of [4], [6].

THEOREM. *The space X, topologized by the measure topology, is connected if, in addition to the conditions assumed in* §2, *the following also hold:*

(i) (X, ρ) is connected and complete;

(ii) each set in C is connected in the metric topology;

(iii) for each $x \in X$ and each positive number δ, there is a set $I \in C$ whose diameter is δ and whose interior (with respect to the metric topology) contains x;

(iv) for some positive number k and all $I \in C$,

$$\phi(I) \geq k \cdot \tau(I).$$

Proof. Suppose that X is not connected in the measure topology. Then there is a set $S \subseteq X$ such that neither S nor its complement $\tilde{S}$ are empty and such that, if $D(S, x) > 0$ then $x \in S$ and similarly for $\tilde{S}$.

For each $\varepsilon > 0$ we consider the sets

$$C(\varepsilon) = \{x \mid x \in i(I), I \in C, 2\varepsilon/(\alpha+1) < d(I) < \varepsilon \Rightarrow \phi(I \cap S) \geq 2 \cdot \phi(I \cap \tilde{S})\}$$

and $D(\varepsilon)$, where $D(\varepsilon)$ is obtained by interchanging S, $\tilde{S}$ in the definition of $C(\varepsilon)$. Here $i(I)$ is the interior of I with respect to the metric topology, $d(I)$ is the diameter of I, and α is the number given in the regularity conditions which we assumed in §2.

Clearly, both $C(\varepsilon)$ and $D(\varepsilon)$ are closed in the metric topology. They are disjoint, by assumption (iii). If $x \in S$, then $D(\tilde{S}, x) = 0$ and so, by (iv), $x \in C(\varepsilon)$ for all sufficiently small ε. Similarly if $x \in \tilde{S}$, and so we can choose a number $\varepsilon_1 > 0$ so that $C(\varepsilon_1)$ and $D(\varepsilon_1)$ are not empty.

Since X is connected there is a point c_1 not in $C(\varepsilon_1) \cup D(\varepsilon_1)$, so there are sets $I_1, J_1 \in C$, each containing c_1 in its interior, and such that

$$2\varepsilon_1/(\alpha+I) < d(I_1), \qquad d(J_1) < \varepsilon_1,$$
$$\phi(I_1 \cap S) < 2\cdot\phi(I_1 \cap \tilde{S}),$$
$$\phi(J_1 \cap \tilde{S}) < 2\cdot\phi(J_1 \cap S).$$

Consider $I_1 \cup J_1$; it is closed, connected, and contains points from S and from $\tilde{S}$. Let $\varepsilon_2 > 0$ be chosen so that

$$(I_1 \cup J_1) \cap C(\varepsilon_2), \qquad (I_1 \cup J_1) \cap D(\varepsilon_2)$$

are not empty, and also choose ε_2 to be less than the number δ given in regularity condition (iii) of [2] where ε is the smallest of $(\alpha-1)\cdot d(I_1)/2$, $(\alpha-1)\cdot d(J_1)/2$, $\alpha\cdot d(I_1)$, $\alpha\cdot d(J_1)$.

As before, there is a point $c_2 \in I_1 \cup J_1$ which is not in $C(\varepsilon_2) \cup D(\varepsilon_2)$ and so there are sets $I_1, J_2 \in C$ each containing c_2 in its interior and such that

$$2\varepsilon_2/(\alpha+1) < d(I_2), \qquad d(J_2) < \varepsilon_2,$$
$$\phi(I_2 \cap S) < 2\cdot\phi(I_2 \cap \tilde{S}),$$
$$\phi(J_2 \cap \tilde{S}) < 2\cdot\phi(J_2 \cap S).$$

Each set $I \in C$ is associated with a particular set $I' \in C$, by the regularity conditions. We now show that

$$I_2' \cap J_2' \subseteq I_1' \cap J_1'.$$

Either $c_2 \in I_1$ or $c_2 \in J_1$. Let us suppose that $c_2 \in I_1$; the other case is similar. Let $x \in I_2' \cap J_2'$. Since $c_1 \in J_1$,

$$\rho(x, J_1) \le \rho(x, c_1) \le \rho(x, c_2) + \rho(c_2, c_1).$$

But $x, c_2 \in I_2'$, so that

$$\rho(x, c_2) \le d(I_2') < (\alpha-1)\cdot d(J_1)/2$$

and also, since $c_1, c_2 \in I_1$.

$$\rho(c_1, c_2) \le d(I_1) < (\alpha+1)\cdot d(J_1)/2$$

so that

$$\rho(x, J_1) < \alpha\cdot d(J_1)$$

and thus $x \in J_1'$. Now we must show that $x \in I_1'$. This follows from the inequalities

$$\rho(x, I_1) \le \rho(x, c_2) \le d(I_2') < \alpha\cdot d(I_1).$$

https://doi.org/10.4153/CMB-1971-089-3 Published online by Cambridge University Press

Thus we have the required inclusion.

Continuing in this way, we obtain a sequence of points $\{c_n\}$ and two sequences of sets from C, $\{I_n\}$ and $\{J_n\}$, such that, for all n,

$$c_n \in I_n \cap J_n;$$

$$I'_{n+1} \cap J'_{n+1} \subseteq I'_n \cap J'_n;$$

$$\phi(I_n \cap S) < 2\cdot\phi(I_n \cap \tilde{S});$$

$$\phi(J_n \cap \tilde{S}) < 2\cdot\phi(J_n \cap S).$$

Furthermore, we can choose the I_n so that $d(I_1)+d(I_2)+\cdots$ converges; this implies that the sequence $\{c_n\}$ is Cauchy and so has a limit c. Since

$$c_n \in I'_m \cap J'_m$$

for all $n \geq m$, it follows that

$$c \in I'_m \cap J'_m$$

for all m. Suppose that $c \in S$. Then

$$\lim_{m\to\infty} \frac{\phi(I'_m \cap S)}{\tau(I'_m)} = 0.$$

But

$$0 < k \leq \frac{\phi(I_m)}{\tau(I_m)} < 3\cdot\frac{\phi(I_m \cap \tilde{S})}{\tau(I_m)} < 3\beta\cdot\frac{\phi(I'_m \cap \tilde{S})}{\tau(I'_m)},$$

which yields a contradiction, and there is a similar contradiction if $c \in \tilde{S}$. This completes the proof.

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