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PRODUCT SEMIGROUPS

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In earlier work Borrego, Cohen and DeVun (1971a, b) topological semigroups which were uniquely representable in terms of two subsets were investigated. In this note we extend the definition and prove similar results for a larger class of semigroups.

By a U-semigroup we mean a semigroup iseomorphic to the unit interval $[0,1]$ with the usual multiplication. A semigroup S is said to be the *unique product of U-semigroups* $S_1, S_2, \cdots, S_n$ if each S_i is a U-semigroup and for every $\sigma \in G_n$ (G_n is the symmetric group on n-elements) $S = S_{\sigma(1)}S_{\sigma(2)}\cdots S_{\sigma(n)}$ and for every non-zero element s of S with $s = x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)} = y_{\sigma(1)}y_{\sigma(2)}\cdots y_{\sigma(n)}$ and $x_{\sigma(i)}, y_{\sigma(i)} \in S_{\sigma(i)}$ we have $x_{\sigma(i)} = y_{\sigma(i)}$. Throughout this paper we will assume all semigroups S have the property that $E(S)$ (the set of all idempotents of the semi-group S) and H (the union of all the subgroups of S) consists of $\{0,1\}$ (where 0 is the zero for S and 1 is the identity for S) and S has no zero divisors.

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1. Unique products and reversibility

Recall that a semigroup T is left (right) reversible if any two principal right (left) ideals of T intersect. The purpose of this section is to show when semigroups which are the unique product of U-semigroups are right or left reversible. Later, in section 3, more results on reversibility are obtained.

LEMMA 1.1. *If S is the unique product of U-semigroups $S_1, S_2, \cdots, S_n$, then S_iS_j is a subsemigroup of S for all $i \neq j$, $i, j \in \{1, 2, \cdots, n\}$.*

PROOF. Let $x_i \in S_i$ and $x_j \in S_j$. If $x_jx_i = 0$, then $x_jx_i \in S_iS_j$. If $x_jx_i \neq 0$, then for $\sigma \in G_n$ with $\sigma(1) = i$, $\sigma(2) = j$ we can select $y_{\sigma(k)} \in S_{\sigma(k)}$, $1 \leqq k \leqq n$, such that

$$x_jx_i = y_{\sigma(1)}y_{\sigma(2)}\cdots y_{\sigma(n)}.$$

Now

$$y_{\sigma(1)}y_{\sigma(2)}\cdots y_{\sigma(n-1)} \neq 0,$$

so for $\delta \in G_n$ with $\delta(1) = \sigma(2)$, $\delta(2) = \sigma(1)$, $\delta(k) = \sigma(k)$, $3 \leqq k \leqq n$, we can pick $z_{\delta(l)} \in S_{\delta(l)}$, $1 \leqq l \leqq n$, such that

$$y_{\sigma(1)}y_{\sigma(2)}\cdots y_{\sigma(n-1)} = z_{\delta(1)}z_{\delta(2)}\cdots z_{\delta(n)}.$$

Hence

$$x_jx_i = y_{\sigma(1)}y_{\sigma(2)}\cdots y_{\sigma(n)} = z_{\delta(1)}z_{\delta(2)}\cdots z_{\delta(n)}y_{\sigma(n)}.$$

Thus $z_{\delta(1)} = x_j$, $z_{\delta(2)} = x_i$, $z_{\delta(l)} = 1$, for $3 \leqq l \leqq n-1$ and $z_{\delta(n)}y_{\sigma(n)} = 1$ which implies $y_{\sigma(n)} = 1$. Continuing this process we obtain $y_{\sigma(l)} = 1$ for $3 \leqq l \leqq n$, hence

$$x_jx_i = y_{\sigma(1)}y_{\sigma(2)} \in S_iS_j.$$

This proves S_iS_j is a subsemigroup of S.

COROLLARY 1.2. *Under the hypotheses of* 1.1, *for* $\sigma \in G_n$ *and* $l \in \{1, 2, \cdots, n\}$, $S_{\sigma(1)}S_{\sigma(2)}\cdots S_{\sigma(l)}$ *is a subsemigroup of S and is the unique product of* $S_{\sigma(1)}, S_{\sigma(2)}, \cdots, S_{\sigma(l)}$.

LEMMA 1.3. *If S satisfies the hypotheses of* 1.1, *then* $S - \{0\}$ *is a cancellative subsemigroup of S.*

PROOF. Let $t, u \in S - \{0\}$ and $s \in S_k - \{0\}$, where k is any fixed arbitrary integer between 1 and n. Suppose $ts = us$. Then there exist unique representations of t and u as $t_1t_2\cdots t_{k-1}t_{k+1}\cdots t_nt_k$ and $u_1u_2\cdots u_{k-1}u_{k+1}\cdots u_nu_k$, respectively. From the uniqueness of representation it follows $t_i = u_i$ for $i \neq k$ and $t_ks = u_ks$. Since S_k is a U-semigroup, $t_k = u_k$, and thus $t = u$. Hence any s coming from a fixed S_k can be right cancelled. Now, if s is an arbitrary element of $S - \{0\}$, then let s be represented as $s_1s_2\cdots s_n$. Then, if $ts = us$, we have $ts_1s_2\cdots s_n = us_1s_2\cdots s_n$. Set $\hat{t} = ts_1s_2\cdots s_{n-1}$, $\hat{u} = us_1s_2\cdots s_{n-1}$. Then $\hat{t}s_n = \hat{u}s_n$, hence $\hat{t} = \hat{u}$, by the above argument. Iterating this process yields $t = u$. This completes the proof.

DEFINITION 1.4. *Let A be a subset of an arbitrary semigroup T. The normalizer* $N(A)$ *of A is the set of all x in T with the property that* $xA = Ax$. *If* $N(A) = T$, *then A is said to be normal in T. If S is the unique product of two U-semigroups* S_1 *and* S_2, *then* S_2 *normal in S is equivalent to* $xS_2 = S_2x$ *for all* $x \in S_1$.

LEMMA 1.5. *Let S satisfy the hypotheses of* 1.1. *If* S_j *is normal in* S_iS_j *for some* $i, j \in \{1, 2, \cdots, n\}$, $j \neq i$, *then* S_k *is normal in* S_iS_k *or* S_j *is normal in* S_jS_k *for* $k \in \{1, 2, \cdots, n\}$.

PROOF. It was shown in Borrego, Cohen and DeVun (1971b) that for one U-semigroup W to be normal in the product of two U-semigroups WV we need only show that there exist $v \in V - \{0, 1\}$, $w, w' \in W - \{0, 1\}$ such that $vw = w'v$. Now let $x \in S_i$, $y \in S_j$ and $z \in S_k$ with none equal to 0 or 1. Then $xyz = xz_1y_1 = z_2x_1y_1 = z_2y_2x_1$ for some $x_1 \in S_i$, $y_1, y_2 \in S_j$, and $z_1, z_2 \in S_k$. Also $xyz = y_3xz = y_3z_3x_2$ for some $x_2 \in S_i$, $y_3 \in S_j$, $z_3 \in S_k$. Hence $x_1 = x_2$. Since $x_1y_1 = y_2x_1$ and $xy = y_3x$ (i.e. y_1 has no effect on x_1 and y has no effect on x), the change in x by z_1 must be exactly the same as the change in x by z. However, this can only occur if $z = z_1$ and thus $yz = zy_1$ or z and z_1 have no effect on x and thus $xz = z_3x$. This proves the lemma.

In Borrego, Cohen and DeVun (1971b) it was shown that if S is the unique product of two U-semigroups, then S is right reversible or left reversible. The following example is due to D. Brown and J. Lawson.

EXAMPLE 1.6. Let S be the one point compactification of the product of the semigroups R and L where

$$R = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : 0 < x \leqq 1,\ x + y \leqq 1,\ y \geqq 0 \right\}$$

and

$$L = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x \geqq 1,\ x + y \geqq 1,\ y \leqq 0 \right\},$$

where R and L have the relative topology of the plane and usual matrix multiplication. Now S is the unique product of four U-semigroups but is neither right nor left reversible, since it is the product of a semigroup R which is not left reversible by a semigroup L which is not right reversible see Borrego, Cohen and DeVun (1971b).

THEOREM 1.7. *If S is the unique product of U-semigroups S_1, S_2 and S_3, then $T = S - \{0\}$ is either right reversible or left reversible.*

PROOF. In Borrego, Cohen and DeVun (1971b) it was shown that the subsemigroup $S_iS_j - \{0\}$, $i \neq j$, $i, j \in \{1, 2, 3\}$ must be iseomorphic to one of the following semigroups.

(a). $\{(x, y)$: x and y are non-negative real numbers$\}$ with the usual addition (the commutative case).

(b). $\left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x \text{ and } y \text{ are real numbers with } x > 0,\ y \geqq 0, \text{ and } x + y \leqq 1 \right\}$ with the usual matrix multiplication (right, not left reversible).

(c). $\left\{ \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} : x \text{ and } y \text{ are real numbers with } x > 0,\ y \leqq 0 \text{ and } x + y \leqq 1 \right\}$ with the usual matrix multiplication (left not right reversible).

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(d). $\left\{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}: x \text{ and } y \text{ are real numbers with } 0 < x \leqq 1,\ y \geqq 0\right\}$ with the usual matrix multiplication (both left and right reversible).

(e). $\left\{\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}: x \text{ and } y \text{ are real numbers with } 0 < x \leqq 1,\ y \geqq 0\right\}$ with the usual matrix multiplication (both left and right reversible).

It should be noted that case (d) and (e) are not commutative, but one factor in each case is normal in the semigroup.

A close examination of the possibilities yields two separate cases. Here we distinguish both.

CASE (1). Assume the subsemigroups $S_1S_2 - \{0\}$, $S_2S_3 - \{0\}$, and $S_1S_3 - \{0\}$ are all left (right) reversible. Let $s, t \in T$ with $s = xyz$, $t = uvw$ and $x, u \in S_1$, $y, v \in S_2$, $z, w \in S_3$ and $x \geqq u$ (where $\geqq$ is the standard cut point order on $[0, 1]$). Let $x_1 \in S_1$ with $xx_1 = u$. Since $S_1S_2 - \{0\}$ and $S_2S_3 - \{0\}$ are left reversible we can select $x_2, x_3 \in S_1$ such that $yx_2 = x_1y_1$ and $zx_3 = x_2z_1$ for some $y_1 \in S_2$ and $z_1 \in S_3$ (see lemma 1, 2, and Borrego, Cohen and DeVun 1971b). Thus

$$uy_1z_1T = xx_1y_1z_1T = xyzx_3T \subset xyzT,$$

and $sT \cap tT \neq \varnothing$ if $uy_1z_1T \cap uvwT \neq \varnothing$. However, $y_1z_1T \cap vwT \neq \varnothing$, since

$$y_1z_1(S_2S_3 - \{0\}) \cap vw(S_2S_3 - \{0\}) \neq \varnothing.$$

So $uy_1z_1T \cap uvwT \neq \varnothing$, and this completes case 1.

CASE (2). Suppose we have the two-one situation, say $S_1S_2 - \{0\}$ and $S_2S_3 - \{0\}$ are of type c) and $S_1S_3 - \{0\}$ is of type b). Let $s, t \in T$ with $s = xyz$, $t = uvw$ and $x, u \in S_1$, $y, v \in S_2$, $z, w \in S_3$ and $x > u$. Let $x_1 \in S_1$, with $xx_1 = u$. From the characterization of $S_1S_2 - \{0\}$ as being of type c), there exist elements $\hat{x} \in S_1$ and $\hat{y} \in S_2$ such that $x_1\hat{y} = y\hat{x}$. Hence $x_1\hat{y}S_2S_3 = y\hat{x}S_2S_3$. From the characterization of $S_1S_3 - \{0\}$ as being of type b), there exists $x_2 \in S_1$ with $x \leqq x_2$ such that $x_2S_3 \cap zS_1 \neq \varnothing$. Again appealing to the characterization of $S_1S_2 - \{0\}$, since $x \leqq x_2$ there exist $y_2, y_3 \in S_2$ such that $\hat{x}y_3 = y_2x_2$. From the results of the preceding sentence, pick $x_4 \in S_1$ and $z_4 \in S_3$ such that $x_2z_4 = zx_4$. Then $y_4zx_4 = y_2x_2z_4 = \hat{x}y_3z_4$. Since $S_2S_3 - \{0\}$ is of type c), for some real number $\alpha > 1$, $y_2z = z^{\alpha}y_5 = zz^{\alpha-1}y_5$, so that $z(z^{\alpha-1}y_5x_4) = \hat{x}y_3z_4$. Let $t = z^{\alpha-1}y_5x_4$. Then

$$yzt = y\hat{x}y_3z_4 \in y\hat{x}S_2S_3 = x_1\hat{y}S_2S_3.$$

Therefore

$$xyzt \in xx_1\hat{y}S_2S_3 = u\hat{y}S_2S_3 \subset uS_2S_3.$$

Hence there exist $y_6 \in S_2$ and $z_6 \in S_3$ such that $xyzt = uy_6z_6$. Hence $uy_6z_6(S_2S_3 - \{0\}) \subset xyzT$. Now since $S_2S_3 - \{0\}$ is left reversible, we have that $y_6z_6(S_2S_3 - \{0\}) \cap vw(S_2S_3 - \{0\}) \neq \varnothing$. It follows next that $uy_6z_6(S_2S_3 - \{0\}) \cap uvw(S_2S_3 - \{0\}) \neq \varnothing$. Hence $uvwT \cap xyzT \neq \varnothing$. This completes the proof.

According to Brown and Friedberg (1971) if W is a semigroup defined on a closed subset of Euclidean n-space E^n having non-empty interior in E^n and is left (right) reversible, cancellative, and translation functions are homeomorphisms into, then W is embedded in a Lie group. The semigroup T satisfies this, thus is embedded in a Lie group.

COROLLARY 1.8. *If S satisfies the hypotheses of* 1.7, *then $T = (S - \{0\})$ is embedded in a Lie group.*

2. Uniquely divisible semigroups with two-cells

Recall a semigroup S is *uniquely divisible* if for each $s \in S$ and every positive integer n there exists a unique $z \in S$ such that $z^n = s$. Borrego, Cohen and DeVun (1971a) showed that if S is a uniquely divisible semigroup on the two-cell with $E(S) = \{0, 1\}$, then S is the unique product of two U-semigroups, and in their second paper (1971b) a characterization of these semigroups was given. In this section we extend these results.

DEFINITION 2.1. *Let S be a uniquely divisible semigroup. Then $[x]$ denotes $\{x^r$: r is a positive rational number$\}^*$ (where * denotes topological closure*), see Hildebrant (1967).

THEOREM 2.2. *Let S be a compact uniquely divisible semigroup. If $x, y, z \in S - \{0\}$ with $[x][y] = [y][x]$ and $[x][z] = [z][x]$, then $xy = xz$ implies $y = z$.*

PROOF. The theorem is clear if $x = 1$, $y = 1$, or $z = 1$. So we will assume $x, y, z \in S - \{0, 1\}$. From Borrego, Cohen and DeVun (1971a) we get the semigroups $[x][y]$ and $[x][z]$ are the unique products of two U-semigroups $[x]$, $[y]$ and $[x]$, $[z]$ respectively. Thus we can find $x_n, x'_n, x' \in [x]$, $y_n \in [y]$, and $z'_n, z' \in [z]$ such that $x_ny_n = (xy)^{1/n} = (xz)^{1/n} = x'_nz'_n$ and $xz = z'x'$. Since $(xy)^{1/n} \to 1$, Hudson (1959), we can select a positive integer k such that $(xy)^{1/k} = x_ky_k$ and $x_k \geqq x'$. Let $\bar{x}$ be the unique element of $[x]$ such that $\bar{x}x_k = x'$. Now

$$xyy_k = xzy_k = z'x'y_k = z'\bar{x}x_ky_k = z'\bar{x}x'_kz'_k = \hat{x}\hat{z}$$

for some $\hat{x} \in [x]$ and $\hat{z} \in [z]$. Moreover, since $[xyy_k]$ and $[xz]$ are U-semigroups $(xyy_k)^\alpha = (\hat{x}\hat{z})^\alpha$ for α any positive real number. Since a characterization of the semigroup of $[x][y]$ is known Borrego, Cohen and DeVun (1971b) we know we can pick a positive real number β such that $(xyy_k)^\beta = \bar{x}y$

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for some $\bar{x} \in [x]$ with $\bar{x} > x$. Now $\bar{x}y = (xyy_k)^\beta = (\bar{x}\bar{z})^\beta = x''z''$ for some $x'' \in [x]$ and $z'' \in [z]$. Let $u \in [x]$ with $u\bar{x} = x$. Then

$$xz = xy = u\bar{x}y = ux''z'' \in [x][z].$$

By the uniqueness of representation of the subsemigroup $[x][z]$ we get $z'' = z$ and $ux'' = x = u\bar{x}$, hence $x'' = \bar{x}$. Thus $\bar{x}y = \bar{x}z$ with $\bar{x} > x$. Thus $y = z$.

COROLLARY 2.3. *If S is a compact uniquely divisible semigroup and for all $x, y \in S$, $[x][y] = [y][x]$, then $T = S - \{0\}$ is cancellative.*

The following example shows that there are compact uniquely divisible semigroups S with $E(S) = \{0, 1\} = H$ and $T = S - \{0\}$ cancellative, but $[x][y] \neq [y][x]$ for some $x, y \in S$. Another example having this property has been given by Hinman (1971).

EXAMPLE 2.4. Let S be the one point compactification of the product of the semigroups R and U where $R = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : 0 < a \leqq 1,\ a + b \leqq 1,\ b \geqq 0\right\}$ with the relative topology of the plane and usual matrix multiplication and $U = (0, 1]$ with the usual topology and multiplication. Then S is a compact uniquely divisible semigroup with $E(S) = \{0, 1\} = H$ and $T = S - \{0\}$ cancellative, but $[x][y] \neq [y][x]$ when

$$x = \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\right) \quad \text{and} \quad y = \left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, 1\right).$$

LEMMA 2.5. *If S satisfies the conditions of 2.3, then if $x, y, w \in S - \{0, 1\}$ and $[x][y] - \{0\}$ is left reversible and not right reversible, then $[x][w] - \{0\}$ is left reversible.*

PROOF. Suppose $[x][w] - \{0\}$ is right reversible and not left reversible. Then for every $x_1 \in [x] - \{0, 1\}$, $w_1 \in w - \{0, 1\}$ there exist $x_2 \in [x] - \{0, 1\}$, $w_2 \in [w] - \{0, 1\}$ with $w_2 > w_1$, $x_1 > x_2$, and $x_1 w_1 = w_2 x_2$. However, for each $x_1 \in [x] - \{0, 1\}$, $y_1 \in [y] - \{0, 1\}$ there exist $x_3 \in [x] - \{0, 1\}$, $y_2 \in [y] - \{0, 1\}$ with $y_1 > y_2$, $x_3 > x_1$ and $x_1 y_1 = y_2 x_3$. From the hypothesis $[x][yw] = [yw][x]$, so $x(yw) = (y'w')\bar{x}$ for some $(y'w') \in [yw]$ and $\bar{x} \in [x]$. But it is impossible for $y'w' \in [yw]$ with $y > y'$ and $w' > w$. Thus $[x][w] - \{0\}$ is left reversible.

The above argument shows $w > w'$. Thus it is impossible to have elements $p, q \in S$ with the property that $[p][q]$ is right reversible and not left reversible.

THEOREM 2.6. *If S satisfies the hypothesis of 2.3, then $T = S - \{0\}$ is right reversible or left reversible.*

Proof. From 2.5 we may assume the semigroup $[x][y]-\{0\}$ is left reversible for all $x, y \in T$. Let $w, z \in T$. Then $w([w][z]-\{0\}) \cap z([w][z]-\{0\}) \neq \varnothing$. Thus $wT \cap zT \neq \varnothing$. So T is left reversible.

Thus one can see that if $T = S - \{0\}$ and S satisfies the hypotheses of 2.6 and appropriate Euclidian conditions, then T is embedded in a Lie group.

3. Reversibility in unique products

In this section we shall show that if S is the unique product of U-semigroups $S_1, S_2, \cdots, S_n$ where $S_iS_j - \{0\}$ is left (right) reversible, then $T = S - \{0\}$ is left (right) reversible and S is uniquely divisible. The key to this problem lies in the solution for $n = 3$. By direct computations we obtained the following results for $n = 3$. The calculations concerning (2) can be found in DeVun (to appear).

Let S be the unique product of U-semigroups S_1, S_2 and S_3.

(1). If S_2 is normal in S_1S_2 and S_2 is not normal in S_2S_3, then we must have S_1 normal in S_1S_2 (S_1S_2 is commutative), and S_3 normal in S_1S_3.

(2). If $S_1S_2 - \{0\}$, $S_2S_3 - \{0\}$ and $S_1S_3 - \{0\}$ are all left (right) reversible and not right (left) reversible, then $[s_1s_2][s_3] = [s_3][s_1s_2]$ for $s_1 \in S_1$, $s_2 \in S_2$, and $s_3 \in S_3$.

Theorem 3.1. *If S is the unique product of U-semigroups $S_1, S_2, \cdots, S_n$ and $S_iS_j - \{0\}$ left (right) reversible for $i \neq j$, $i, j \in \{1, 2, \cdots, n\}$, then so is $T = S - \{0\}$.*

Proof. By assumption the result holds for $k = 2$. Assume the result holds for $k = n - 1$. We will show the result holds for $k = n$. Let $s, t \in T$ with $s = s_1s_2 \cdots s_n$, $t = t_1t_2 \cdots t_n$ with $s_i, t_i \in S_i$ for $i \in \{1, 2, \cdots, n\}$ and $t_1 \leqq s_1$. For each $w_i \in S_i$ and $z_j \in S_j$ there exist $w'_i \in S_i$ and $z'_j \in S_j$ such that $w_iz_j = z'_jw'_i$. Since $z_j \to 0$ implies $z'_j \to 0$, by the continuity of multiplication we can select $z_1 \in S_1$ such that $sz_1 = t_1u'_2u'_3 \cdots u'_n$ with $u'_i \in S_i$. Hence

$$sT \cap tT \supset su_1T \cap tT = t_1u'_2u'_3 \cdots u'_nT \cap t_1t_2 \cdots t_nT \neq \varnothing$$

if $u'_2u'_3 \cdots u'_nT \cap t_2t_3 \cdots t_nT \neq \varnothing$. However,

$$u'_2u'_3 \cdots u'_nT \cap t_2t_3 \cdots t_nT \supset u'_2u''_3 \cdots u'_n(S_2S_3 \cdots S_n - \{0\})$$
$$\cap\, t_2t_3 \cdots t_n(S_2S_3 \cdots S_n - \{0\}) \neq \varnothing.$$

So our results hold.

Note that under the conditions of 3.1, T is embedded in a Lie group.

Theorem 3.2. *If S satisfies the conditions of* 3.1, *then S is uniquely divisible.*

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Proof. In Borrego, Cohen and DeVun (1971b) the result was shown for $k = 2$. Assume the result holds for $(n-1)$ products. Now S is clearly divisible since the map $s \to s^n$ sends the boundary of S onto the boundary of S. To show S is uniquely divisible we distinguish three cases.

Case 1. Suppose $S_iS_j - \{0\}$ is left reversible and not right reversible for each pair $i \neq j$, $i, j \in \{1, 2, \cdots, n\}$. Let $s, t \in S - \{0\}$ with $s^m = t^m$ for some positive integer m, and $s = s_1s_2 \cdots s_n$, $t = t_1t_2 \cdots t_n$ with $s_i, t_i \in S_i$, $i \in \{1, 2, \cdots, n\}$. From the discussion preceding 3.1 we get

$$s^m = s_1(s_2s_3 \cdots s_n) \cdots s_1(s_2s_3 \cdots s_n) = s_1^p(s_2s_3 \cdots s_n)^q,$$

and

$$t^m = t_1(t_2t_3 \cdots t_n) \cdots t_1(t_2t_3 \cdots t_n) = t_1^r(t_2t_3 \cdots t_n)^u.$$

Thus $s_1^p = t_1^r$ and $(s_2s_3 \cdots s_n)^q = (t_2t_3 \cdots t_n)^u$. Thus $s_1 = t_1^{r/p}$ and $s_2s_3 \cdots s_n = (t_2t_3 \cdots t_n)^{u/q}$. So $s, t \in [s_1][s_2s_3 \cdots s_n] = [t_1][t_2t_3 \cdots t_n]$ with $s^m = t^m$. But $[s_1][s_2s_3 \cdots s_n]$ is uniquely divisible Borrego, Cohen and DeVun (1971b), and we get $s = t$.

Case 2. Suppose there exists $i, j \in \{1, 2, \cdots, n\}$ $i \neq j$ with S_j is normal in S_iS_j but S_i does not commute with S_j. From the discussion preceding 3.1 we have for each $k \in \{1, 2, \cdots, n\}$, S_j is normal in S_jS_k. Thus for $k \in \{1, 2, \cdots, n\}$ with $s_k \in S_k$ and $s_j \in S_j$ there exists $s'_j \in S_j$ such that $s_ks_j = s'_js_k$. Now let $s, t \in S - \{0\}$ with $s^m = t^m$. Set $s = s_1s_2 \cdots s_n$ and $t = t_1t_2 \cdots t_n$ with $s_i, t_i \in S_i$, $i \in \{1, 2, \cdots, n\}$. Then

$$\begin{aligned} s^m &= (s_1s_2 \cdots s_{n-1}s_js_{j+1} \cdots s_n) \cdots (s_1s_2 \cdots s_{j-1}s_js_{j+1} \cdots s_n) \\ &= s_j^\alpha(s_1s_2 \cdots s_{j-1}s_{j+1} \cdots s_n)^m \end{aligned}$$

and

$$\begin{aligned} t^m &= (t_1t_2 \cdots t_{j-1}t_jt_{j+1} \cdots t_n) \cdots (t_1t_2 \cdots t_{j-1}t_jt_{j+1} \cdots t_n) \\ &= t_j^\beta(t_1t_2 \cdots t_{j-1}t_{j+1} \cdots t_n)^m \end{aligned}$$

for appropriate real numbers α and β. Since

$$(s_1s_2 \cdots s_{j-1}s_{j+1} \cdots s_n)^m = (t_1t_2 \cdots t_{j-1}t_{j+1} \cdots t_n)^m$$

we get

$$s_1s_2 \cdots s_{j-1}s_{j+1} \cdots s_n = t_1t_2 \cdots t_{j-1}t_{j+1} \cdots t_n.$$

However, this implies $s_j = t_j$. So $s = t$.

Case 3. Suppose there exists $i, j \in \{1, 2, \cdots, n\}$, $i \neq j$ with S_i commuting with S_j. Then for each $k \in \{1, 2, \cdots, n\}$ we must have S_i commute with S_k or S_j commutes with S_k. Let $S_{i1}, S_{i2}, \cdots, S_{ik}$ commute with S_i alone, $S_{j1}, S_{j2} \cdots S_{jl}$ commute with S_j alone, and $S_{q1}, S_{q2}, \cdots, S_{qr}$ commute with both S_i and S_j.

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Using the results of (1) and lemma 1.5 several times one sees that for all $t \in \{1, 2, \cdots, r\}$ S_{qt} commutes with S_{iu} and S_{jv} for all $u \in \{1, 2, \cdots, k\}$ and $v \in \{1, 2, \cdots, l\}$. Since S_{qt} commutes with $S_i(S_j)$, and $S_i(S_j)$ does not commute $S_{jv}(S_{iu})$, S_{qt} must commute with $S_{jv}(S_{iu})$. Also S_{iu} commutes with S_{jv} for all $u \in \{1, 2, \cdots, k\}$ and $v \in \{1, 2, \cdots, l\}$. This can be seen since S_{iu} commutes with S_i, S_{jv} does not commute with S_i, so S_{iu} must commute with S_{jv}. Now let $s, t \in S - \{0\}$ with $s^m = t^m$ and

$$s = s_i(s_{j1}, s_{j2}, \cdots, s_{jk})s_j(s_{i1}s_{i2} \cdots s_{il})(s_{q1}s_{q2} \cdots s_{ql})$$

and

$$t = t_i(t_{j1}t_{j2} \cdots t_{jk})t_j(t_{i1}t_{i2} \cdots t_{il})(t_{q1}t_{q2} \cdots t_{ql}),$$

$s_i, t_i \in S_i$; $s_{jv}, t_{jv} \in S_{jv}$; $s_j, t_j \in S_j$; $s_{iu}, t_{iu} \in S_{iu}$; $s_{qt}, t_{qt} \in S_{qt}$. Then

$$\begin{aligned} s^m &= (s_i(s_{j1}s_{j2} \cdots s_{jk}))^m (s_j(s_{i1}s_{i2} \cdots s_{il}))^m (s_{q1}s_{q2} \cdots s_{qv})^m \\ &= (t_i(t_{j1}t_{j2} \cdots t_{jk}))^m (t_j(t_{i1}t_{i2} \cdots t_{il}))^m (t_{q1}t_{q2} \cdots t_{qv})^m. \end{aligned}$$

Since each of the components is uniquely divisible $s = t$. This completes the proof.

Finally, it can be seen in Brown and Friedburg (1968) that S may be a commutative uniquely divisible semigroup without being a unique product. Thus the converse of 3.2 is false.

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