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Abstract

We consider a Deligne–Mumford stack X which is the quotient of an affine scheme $\operatorname{Spec} A$ by the action of a finite group G and show that the Balmer spectrum of the tensor triangulated category of perfect complexes on X is homeomorphic to the space of homogeneous prime ideals in the group cohomology ring $H^*(G, A)$.

Contents

1. Introduction

Let $\mathcal{X}$ be an algebraic stack. The category $\operatorname{Perf}(\mathcal{X})$ of perfect complexes of $\mathcal{O}_{\mathcal{X}}$-modules is a tensor triangulated category. One can ask for a classification of all thick tensor ideals in this category, or equivalently for a description of the Balmer spectrum

$$\operatorname{Spc}(\operatorname{Perf}(\mathcal{X})),$$

the space of all prime thick tensor ideals in $\operatorname{Perf}(\mathcal{X})$, as defined in [Bal05]. An answer is known at least in the following cases.

If $\mathcal{X}$ is a quasi-compact, quasi-separated scheme, the thick tensor ideals in $\operatorname{Perf}(\mathcal{X})$ are classified in terms of their support by Thomason [Tho97]. This yields a homeomorphism

$$\operatorname{Spc}(\operatorname{Perf}(\mathcal{X})) \cong |\mathcal{X}|, \tag{1.1}$$

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where $|\mathcal{X}|$ is the underlying topological space of $\mathcal{X}$, by Balmer [Bal19, Theorem 4.1], which relies on [BKS07] when X is not noetherian. More generally, a homeomorphism (1.1) exists when $\mathcal{X}$ is a quasi-compact algebraic stack with separated diagonal and $\mathcal{X}$ is tame, which means that the geometric stabiliser groups of $\mathcal{X}$ are finite and geometrically reductive, by Hall [Hal16].

In a different direction, if $\mathcal{X} = BG$ is the classifying space of a finite group G over a field k, then $\mathrm{Perf}(\mathcal{X})$ is equivalent to $D^b(kG\text{-mod})$, the bounded derived category of finite modules over the group algebra kG. In this case, thick tensor ideals in the stable module category of kG are classified in terms of their cohomological support by Benson, Carlson and Rickard [BCR97] and Benson, Iyengar and Krause [BIK11]; see also [CI15] for a direct proof of a non-stable version. These results yield a homeomorphism

$$\mathrm{Spc}(\mathrm{Perf}(\mathcal{X})) \cong \mathrm{Spec}^h(R_{k,G}) \tag{1.2}$$

where $R_{k,G} = H^*(G, k)$ is the cohomology ring and Spec^h is the space of all homogeneous prime ideals, by Balmer [Bal10, Proposition 8.5]. The homeomorphism (1.2) is complementary to (1.1) because the stack $\mathcal{X} = BG$ is tame if and only if the characteristic of k does not divide the order of G, in which case $\mathrm{Spc}(\mathrm{Perf}(\mathcal{X}))$ is just a point. More generally, if G is a finite group scheme over k, a similar classification of the thick tensor ideals in the stable module category of kG and hence the homeomorphism (1.2) for $\mathcal{X} = BG$ exist by Benson, Iyengar, Krause and Pevtsova [BIKP18].

Our main example

We consider a quotient stack $\mathcal{X} = [\mathrm{Spec}(A)/G]$ where G is a finite group that acts on a commutative ring A. Let $\mathcal{T}_{A,G} = \mathrm{Perf}(\mathcal{X})$. This category is equivalent to the category $D^b(AG)_{A\text{-perf}}$ of bounded complexes of AG-modules which are perfect as complexes of A-modules, viewed as a full subcategory of the derived category $D(AG)$. By [Bal10] there is a natural continuous comparison map

$$\rho_{A,G} : \mathrm{Spc}(\mathcal{T}_{A,G}) \to \mathrm{Spec}^h(R_{A,G})$$

with $R_{A,G} = H^*(G, A)$. If A is a field with the trivial action of G, which we will call the punctual case, then $\rho_{A,G}$ is the homeomorphism (1.2). If G is the trivial group, then $\rho_{A,G}$ is the homeomorphism (1.1) for the affine scheme $\mathcal{X} = \mathrm{Spec}\, A$. The following is our main result.

Theorem 1.3. *The map $\rho_{A,G}$ is a homeomorphism in all cases.*

Even the case $A = \mathbb{Z}$ of integral representations seems to be new.

A stable version

The category $\mathrm{Perf}(AG)$ of perfect complexes of AG-modules is a tensor ideal in $\mathcal{T}_{A,G}$, and hence the Verdier quotient

$$\mathcal{S}_{A,G} = \mathcal{T}_{A,G}/\mathrm{Perf}(AG)$$

is a tensor triangulated category again. As observed in [Bar21], the following variant of Theorem 1.3 is an immediate consequence.

Corollary 1.4. *The map $\rho_{A,G}$ induces a homeomorphism*

$$\mathrm{Spc}(\mathcal{S}_{A,G}) \cong \mathrm{Proj}(R_{A,G}).$$

The category $\mathcal{S}_{A,G}$ can be viewed as the stable category of the Frobenius category $\mathrm{lat}(A, G)$ of all finite A-projective AG-modules, so Corollary 1.4 is in line with the classical results of modular

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representation theory mentioned above. If G acts trivially on A, another stable category of AG-modules with different behavior was introduced in [BIK13], using the Frobenius category of all finitely presented AG-modules with A-split exact sequences.

Recent literature

In the first version of this paper, Theorem 1.3 was proved only when the ring A is regular. Afterwards, related results on stratification of compactly generated triangulated categories of AG-modules in the case where A is regular with trivial G-action appeared in [Bar21, Bar22] and in [BIKP22]. The fact that Theorem 1.3 implies Corollary 1.4 was observed in [Bar21].

Strategy

The proof of Theorem 1.3 is based on a reduction to the punctual case and the case of affine schemes along the following lines. If A is a field with possibly non-trivial action of G, the result follows from the punctual case by a form of Galois descent. In general we can assume that A is essentially of finite type over $\mathbb{Z}$; then it will be sufficient to show that $\rho_{A,G}$ is bijective. There is a commutative diagram of continuous maps

$$\begin{tikzcd} \mathrm{Spc}(\mathcal{T}_{A,G}) \arrow[r, "\rho_{A,G}"] \arrow[d, "\pi_{\mathcal{T}}"] & \mathrm{Spec}^h(R_{A,G}) \arrow[d, "\pi_R"] \\ \mathrm{Spc}(\mathcal{T}_{A^G}) \arrow[r, "\sim", "\rho_{A^G}"'] & \mathrm{Spec}(A^G) \end{tikzcd} \tag{1.5}$$

where A^G is the ring of G-invariants and $\mathcal{T}_{A^G} = \mathrm{Perf}(\mathrm{Spec}(A^G))$ is the associated category of perfect complexes; note that A^G is the degree-zero component of the graded ring $R_{A,G}$. Here ρ_{A^G} is a homeomorphism by the case of affine schemes. Hence, the map $\rho_{A,G}$ is bijective if and only if it restricts to a bijective map between each fibre of $\pi_{\mathcal{T}}$ and the corresponding fibre of π_R. This map between the fibres will be related to the field case of Theorem 1.3 as follows. For a prime ideal $\mathfrak{q} \in \mathrm{Spec}(A^G)$ we consider the reduced fibre of the ring A over $\mathfrak{q}$,

$$A(\mathfrak{q}) = (A \otimes_{A^G} k(\mathfrak{q}))_{\mathrm{red}}.$$

Explicitly this means that $A(\mathfrak{q}) = k(\mathfrak{p}_1) \times \cdots \times k(\mathfrak{p}_r)$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the prime ideals of A over $\mathfrak{q}$. Functoriality gives the following commutative diagram, where the subscript $\mathfrak{q}$ means fibre over $\mathfrak{q}$ in (1.5).

$$\begin{tikzcd} \mathrm{Spc}(\mathcal{T}_{A(\mathfrak{q}),G}) \arrow[r, "\rho_{A(\mathfrak{q}),G}"] \arrow[d, "j_{\mathcal{T}}"'] & \mathrm{Spec}^h(R_{A(\mathfrak{q}),G}) \arrow[d, "j_R"] \\ \mathrm{Spc}(\mathcal{T}_{A,G})_{\mathfrak{q}} \arrow[r, "(\rho_{A,G})_{\mathfrak{q}}"] & \mathrm{Spec}^h(R_{A,G})_{\mathfrak{q}} \end{tikzcd}$$

The field case of Theorem 1.3 implies that $\rho_{A(\mathfrak{q}),G}$ is a homeomorphism. We will show that j_R is a homeomorphism and that $j_{\mathcal{T}}$ is surjective, at least when A is noetherian. It follows that $(\rho_{A,G})_{\mathfrak{q}}$ is bijective as required.

Change of coefficients

The preceding assertions on the maps j_R and $j_{\mathcal{T}}$ relate to the question how the spaces $\mathrm{Spec}^h(R_{A,G})$ and $\mathrm{Spc}(\mathcal{T}_{A,G})$ vary with the ring A. A G-equivariant homomorphism of commutative rings $A \to B$ gives rise to a base change map for the spectra of the cohomology rings,

$$\mathrm{Spec}^h(R_{B,G}) \to \mathrm{Spec}^h(R_{A,G}) \times_{\mathrm{Spec}(A^G)} \mathrm{Spec}(B^G), \tag{1.6}$$

and a base change map for the Balmer spectra,

$$\mathrm{Spc}(\mathcal{T}_{B,G}) \to \mathrm{Spc}(\mathcal{T}_{A,G}) \times_{\mathrm{Spec}(A^G)} \mathrm{Spec}(B^G). \tag{1.7}$$

In the case $B = A(\mathfrak{q})$ these maps can be identified with j_R and $j_{\mathcal{T}}$. The following general result on base change in group cohomology shows in particular that j_R is a homeomorphism.

Theorem 1.8. *The base change map (1.6) is a homeomorphism if B is a localisation of a quotient of A.*

See Corollary 8.27 and the underlying Theorem 8.26, which says that the natural ring homomorphism $(R_{A,G})_{\mathrm{even}} \otimes_{A^G} B^G \to (R_{B,G})_{\mathrm{even}}$ induces a universal homeomorphism of affine schemes if B is a localisation of a quotient of A. The proof of that result proceeds by a reduction to three basic cases: the case where B is a localisation of A, the case $B = A/I$ for a nilpotent ideal I, and the case $B = A/tA$ for an A-regular element $t \in A^G$, under a noetherian assumption.

With hindsight, Theorem 1.3 identifies (1.6) and (1.7). A direct proof of the analogue of Theorem 1.8 for the map (1.7) seems to be difficult, but in the above three basic cases we can at least show that (1.7) is surjective, using a general surjectivity criterion of Balmer [Bal18], saying that a functor between tensor triangulated categories which detects tensor nilpotence induces a surjective map on the Balmer spectra. By formal arguments one deduces that (1.7) is surjective when A is noetherian and $B = A(\mathfrak{q})$; in other words the map $j_{\mathcal{T}}$ is surjective in the noetherian case.

Structure of the paper

In § 2 we review the comparison map ρ and its relation with cohomological support for general tensor triangulated categories. In § 3 we study the category $D^b(AG)_{A\text{-perf}}$ from a purely algebraic point of view. In § 4 we introduce the category $\mathcal{T}_{A,G}$ of perfect complexes on $\mathcal{X}$ and show that it is equivalent to the algebraic category of § 3. In § 5 we study basic properties of the comparison map for these categories. The field case of Theorem 1.3 is established in § 7. Theorem 1.8 is proved in § 8. The corresponding surjectivity results for the map $j_{\mathcal{T}}$ are proved in §§ 9 and 10. Finally, the main result is proved in § 11, and Corollary 1.4 is deduced in § 12.

Notation

For a not necessarily commutative ring R we denote by R-Mod the category of left R-modules and by R-mod the category of finitely generated left R-modules. The latter is abelian if R is left noetherian.

2. Prime spectra and cohomological support

In this section we recall some aspects of Balmer's theory of prime spectra of tensor triangulated categories. Let $\mathcal{T}$ be an essentially small tensor triangulated category with unit object $\mathbb{1}$ and let

$$R = R_{\mathcal{T}} = \mathrm{End}^*_{\mathcal{T}}(\mathbb{1}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \mathbb{1}[n])$$

as a graded ring. This is a graded-commutative ring by the obvious composition of morphisms, or equivalently by the tensor product of morphisms; see [Bal10, Proposition 3.3]. By [Bal10,

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Theorem 5.3] there is a continuous map, denoted there by $\rho^\bullet$ and called the comparison map in [Bal19],

$$\rho = \rho_{\mathcal{T}} : \mathrm{Spc}(\mathcal{T}) \to \mathrm{Spec}^h(R) \tag{2.1}$$

from the space of prime thick tensor ideals in $\mathcal{T}$ to the space of homogeneous prime ideals in R, which is defined by the following property. If $\rho(\mathcal{P}) = \mathfrak{p}$, a homogeneous element $a \in R$ satisfies $a \notin \mathfrak{p}$ if and only if $\mathrm{cone}(a) \in \mathcal{P}$.

If the ring R is noetherian, then ρ is surjective by [Bal10, Theorem 7.3].

2.1 Support

For a graded R-module M we have the support

$$\mathrm{supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}^h(R) \mid M_\mathfrak{p} \neq 0\}$$

where $\mathrm{Spec}^h(R)$ is the set of homogeneous prime ideals of R and $M_\mathfrak{p}$ is the homogeneous localisation of M at $\mathfrak{p}$. For an object X of $\mathcal{T}$ there are two notions of support: the canonical support

$$\mathrm{supp}(X) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid X \notin \mathcal{P}\},$$

which is a closed subset of $\mathrm{Spc}(\mathcal{T})$; and the cohomological support

$$V(X) = \mathrm{supp}_R(\mathrm{End}^*_{\mathcal{T}}(X))$$

as a subset of $\mathrm{Spec}^h(R)$, where $M_X = \mathrm{End}^*_{\mathcal{T}}(X)$ becomes a graded R-module by the graded ring homomorphism $R \to M_X$ defined by $f \mapsto f \otimes \mathrm{id}_X$, whose image is graded-central; the same R-module structure on M_X arises from the tensor product of morphisms; see [Bal10, Proposition 3.3].

The cohomological support $V(X)$ can also be described using localisations of $\mathcal{T}$ as in [Bal10, Construction 3.5]. For $\mathfrak{p} \in \mathrm{Spec}^h(R)$, the localisation $\mathcal{T}_\mathfrak{p}$ has the same objects as $\mathcal{T}$ and homomorphism groups $\mathrm{Hom}_{\mathcal{T}_\mathfrak{p}}(X,Y) = (\mathrm{Hom}^*_{\mathcal{T}}(X,Y)_\mathfrak{p})^0$, where ^{0} means the degree-zero part. Let $X_\mathfrak{p} \in \mathcal{T}_\mathfrak{p}$ be the image of $X \in \mathcal{T}$. Then

$$\mathfrak{p} \in V(X) \iff (M_X)_\mathfrak{p} \neq 0 \iff X_\mathfrak{p} \neq 0. \tag{2.2}$$

The category $\mathcal{T}_\mathfrak{p}$ is equivalent to the Verdier localisation of $\mathcal{T}$ with respect to the thick tensor ideal generated by $\mathrm{cone}(a)$ for all homogeneous elements $a \in R \setminus \mathfrak{p}$ by [Bal10, Theorem 3.6]; in particular, $\mathcal{T}_\mathfrak{p}$ is a tensor triangulated category such that the localisation $\mathcal{T} \to \mathcal{T}_\mathfrak{p}$ is an exact tensor functor.

Lemma 2.3. *For a homogeneous element $a \in R$ we have*

$$V(\mathrm{cone}(a)) = V(a) = \{\mathfrak{p} \in \mathrm{Spec}^h(R) \mid a \in \mathfrak{p}\}.$$

Proof. Using (2.2) and the exactness of $\mathcal{T} \to \mathcal{T}_\mathfrak{p}$, we have $\mathfrak{p} \in V(\mathrm{cone}(a))$ if and only if $\mathrm{cone}(a)_\mathfrak{p} \neq 0$ if and only if a not invertible in $\mathcal{T}_\mathfrak{p}$ if and only if $a \in \mathfrak{p}$. □

One can ask if $(\mathrm{Spec}^h(R), V)$ is a support datum on $\mathcal{T}$ as in [Bal05, Definition 3.1].

Lemma 2.4. *The cohomological support $(\mathrm{Spec}^h(R), V)$ is a support datum on $\mathcal{T}$ if and only if for all $X, Y \in \mathcal{T}$ the set $V(X)$ is closed and*

$$V(X \otimes Y) = V(X) \cap V(Y). \tag{2.5}$$

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Proof. The conditions are part of the axioms of a support datum. Since the localisation $\mathcal{T}_p$ is a tensor triangulated category, the remaining axioms of a support datum are easily verified using (2.2). ☐

2.2 The comparison map and support

Definition 2.6. The tensor triangulated category $\mathcal{T}$ will be called End-finite if for each $X \in \mathcal{T}$ the R-module $M_X = \mathrm{End}^*_{\mathcal{T}}(X)$ is noetherian.

Clearly $\mathcal{T}$ is End-finite if and only if the ring $R = \mathrm{End}^*_{\mathcal{T}}(\mathbb{1})$ is noetherian and the R-module $M_X = \mathrm{End}^*_{\mathcal{T}}(X)$ is finite for each $X \in \mathcal{T}$. Moreover, this implies that the R-module $\mathrm{Hom}^*_{\mathcal{T}}(X, Y)$ is finite for all $X, Y \in \mathcal{T}$ since the latter is a direct summand of $\mathrm{End}^*_{\mathcal{T}}(X \oplus Y)$.

If $\mathcal{T}$ is End-finite, $V(X)$ is closed in $\mathrm{Spec}^h(R)$ for each $X \in \mathcal{T}$.

Proposition 2.7. *For each object X of $\mathcal{T}$ we have*

$$\rho(\mathrm{supp}(X)) \subseteq V(X),$$

with equality if $(\mathrm{Spec}^h(R), V)$ is a support datum or $\mathcal{T}$ is End-finite and rigid.

Proof. Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ and $\mathfrak{p} = \rho(\mathcal{P})$. We consider the multiplicative sets $\mathcal{S}$ in the category $\mathcal{T}$ and S in the ring R defined by

$$\begin{aligned} \mathcal{S} &= \{f : X \to Y \text{ in } \mathcal{T} \mid \mathrm{cone}(f) \in \mathcal{P}\}, \\ S &= \mathcal{S} \cap R. \end{aligned}$$

Then S is the set of homogeneous elements in $R \setminus \mathfrak{p}$ by the definition of the map ρ. The corresponding localisations of $\mathcal{T}$ are related by functors

$$\mathcal{T} \longrightarrow S^{-1}\mathcal{T} \longrightarrow \mathcal{S}^{-1}\mathcal{T} = \mathcal{T}/\mathcal{P},$$

where $S^{-1}\mathcal{T} = \mathcal{T}_{\mathfrak{p}}$, and $\mathcal{S}^{-1}\mathcal{T}$ is the Verdier localisation at $\mathcal{P}$. We get

$$(M_X)_{\mathfrak{p}} = 0 \iff X = 0 \text{ in } \mathcal{T}_{\mathfrak{p}} \implies X = 0 \text{ in } \mathcal{T}/\mathcal{P} \iff X \in \mathcal{P},$$

hence $\mathcal{P} \in \mathrm{supp}(X) \implies \mathfrak{p} \in \mathrm{supp}_R(M_X)$. This proves the first assertion.

To prove the second assertion, for a given $\mathfrak{p} \in \mathrm{supp}_R(M_X)$ we have to find a prime ideal $\mathcal{P} \in \mathrm{supp}(X)$ with $\rho(\mathcal{P}) = \mathfrak{p}$. We can replace $\mathcal{T}$ by the localisation $\mathcal{T}_{\mathfrak{p}}$; cf. [Bal10, Theorem 5.4]. Then R is a local graded ring with unique graded maximal ideal $\mathfrak{p}$. Let $\mathcal{M}$ be the multiplicative set of all objects of $\mathcal{T}$ of the form

$$X^{\otimes n} \otimes \mathrm{cone}(a_1) \otimes \cdots \otimes \mathrm{cone}(a_r)$$

with $n \geq 0$ and homogeneous elements $a_1, \ldots, a_r \in \mathfrak{p}$. We claim that $0 \notin \mathcal{M}$. Then by [Bal05, Lemma 2.2] there is a $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ with $\mathcal{M} \cap \mathcal{P} = \emptyset$; in particular, $X \notin \mathcal{P}$ and $\mathrm{cone}(a) \notin \mathcal{P}$ for each homogeneous element $a \in \mathfrak{p}$. The first condition means that $\mathcal{P} \in \mathrm{supp}(X)$; the second condition implies that $\mathfrak{p} \subseteq \rho(\mathcal{P})$ and thus $\mathfrak{p} = \rho(\mathcal{P})$ since $\mathfrak{p}$ was assumed to be maximal. So it remains to verify that $0 \notin \mathcal{M}$.

If $(\mathrm{Spec}^h(R), V)$ is a support datum, the tensor product formula (2.5) implies that $\mathfrak{p} \in V(Z)$ for every $Z \in \mathcal{M}$ because $\mathfrak{p} \in V(X)$ and $\mathfrak{p} \in V(\mathrm{cone}(a))$ for every $a \in \mathfrak{p}$ by Lemma 2.3. Hence, $0 \notin \mathcal{M}$. Assume that $\mathcal{T}$ is End-finite and rigid. We have $X \neq 0$ since $\mathfrak{p} \in V(X)$, hence $X^{\otimes n} \neq 0$ for $n \geq 0$ since $\mathcal{T}$ is rigid. So it suffices to verify that $Y \neq 0$ in $\mathcal{T}$ implies $Y \otimes \mathrm{cone}(a) \neq 0$ for every homogeneous element $a \in \mathfrak{p}$. The triangle $Y \xrightarrow{a} Y \to Y \otimes \mathrm{cone}(a) \to^{+}$ gives an exact sequence

$$\mathrm{End}^*_{\mathcal{T}}(Y) \xrightarrow{a} \mathrm{End}^*_{\mathcal{T}}(Y) \longrightarrow \mathrm{Hom}^*_{\mathcal{T}}(Y, Y \otimes \mathrm{cone}(a)).$$

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Here $\mathrm{End}^*_{\mathcal{T}}(Y)$ is a non-zero finite R-module, so the cokernel of multiplication by a on this module is non-zero by the graded version of Nakayama's lemma. It follows that $Y \otimes \mathrm{cone}(a) \neq 0$. ☐

Corollary 2.8. *If $(\mathrm{Spec}^h(R), V)$ is a support datum or $\mathcal{T}$ is End-finite and rigid, then ρ is bijective if and only if ρ is a homeomorphism.*

Proof. We have to show that if ρ is bijective and $Z \subseteq \mathrm{Spc}(\mathcal{T})$ is closed, then $\rho(Z)$ is closed. The definition of the topology on $\mathrm{Spc}(\mathcal{T})$ implies that Z is an intersection of sets of the form $\mathrm{supp}(X_i)$ with $X_i \in \mathcal{T}$. Proposition 2.7 yields $\rho(\mathrm{supp}(X_i)) = V(X_i)$, which is closed in $\mathrm{Spec}^h(R)$ by the assumption. Since ρ is injective, $\rho(Z)$ is the intersection of the sets $V(X_i)$ and thus closed. ☐

Remark 2.9. One can ask if the conclusion of Corollary 2.8 holds for general tensor triangular categories; see [DS16, Lemma 2.1] and [DS] for a discussion.

Proposition 2.10. *If the category $\mathcal{T}$ is rigid and ρ is a homeomorphism, then $\rho(\mathrm{supp}(X)) = V(X)$ for each $X \in \mathcal{T}$.*

Proof. Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ and $\mathfrak{p} = \rho(\mathcal{P})$. As in the proof of Proposition 2.7, we consider the natural functor $j : \mathcal{T}_\mathfrak{p} \to \mathcal{T}/\mathcal{P}$ from the localisation at $\mathfrak{p}$ to the Verdier quotient by $\mathcal{P}$. We have to show that $X = 0$ in $\mathcal{T}_\mathfrak{p}$ if and only if $X = 0$ in $\mathcal{T}/\mathcal{P}$. In fact we show that j is an equivalence. By [Bal10, Theorem 5.4] there is a cartesian diagram of topological spaces

$$\begin{tikzcd}
\mathrm{Spc}(\mathcal{T}_\mathfrak{p}) \arrow[r, "\rho_{\mathcal{T}_\mathfrak{p}}"] \arrow[d, "\mathrm{Spc}(q)"'] & \mathrm{Spec}^h(R_\mathfrak{p}) \arrow[d] \\
\mathrm{Spc}(\mathcal{T}) \arrow[r, "\rho_{\mathcal{T}}"] & \mathrm{Spec}^h(R)
\end{tikzcd}$$

where $q : \mathcal{T} \to \mathcal{T}_\mathfrak{p}$ is the localisation functor. Since $\rho_\mathcal{T}$ is a homeomorphism, $\rho_{\mathcal{T}_\mathfrak{p}}$ is a homeomorphism. The space $\mathrm{Spec}^h(R_\mathfrak{p})$ has a unique closed point which maps to $\mathfrak{p}$ in $\mathrm{Spec}^h(R)$. Hence, $\mathrm{Spc}(\mathcal{T}_\mathfrak{p})$ has a unique closed point $\mathcal{P}'$ which maps to $\mathcal{P}$ in $\mathrm{Spc}(\mathcal{T})$. Since $\mathcal{T}_\mathfrak{p}$ is rigid, $\mathcal{P}'$ is the zero ideal of $\mathcal{T}_\mathfrak{p}$; see [Bal10, Proposition 4.2]. Explicitly this shows that $\mathcal{P} = q^{-1}(0)$; in particular, q maps $\mathcal{P}$ to zero. Hence, q induces a functor $\mathcal{T}/\mathcal{P} \to \mathcal{T}_\mathfrak{p}$ which is an inverse of j. ☐

2.3 Products and colimits

Let us record how the Balmer spectrum and the comparison map behave under finite products and filtered 2-colimits of $\mathcal{T}$.

Lemma 2.11. *Let $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$ with tensor triangulated categories $\mathcal{T}_i$ and $R_i = \mathrm{End}^*_{\mathcal{T}_i}(\mathbb{1})$. Then there are decompositions into open subspaces*

$$\mathrm{Spc}(\mathcal{T}) = \mathrm{Spc}(\mathcal{T}_1) \sqcup \mathrm{Spc}(\mathcal{T}_2) \quad \textit{and} \quad \mathrm{Spec}^h(R) = \mathrm{Spec}^h(R_1) \sqcup \mathrm{Spec}^h(R_2)$$

such that $\rho_\mathcal{T} = \rho_{\mathcal{T}_1} \sqcup \rho_{\mathcal{T}_2}$.

Proof. This is straightforward; note that $R = R_1 \times R_2$. See [AM69, Chapter 1, Exercise 22] for the corresponding assertion for the spectrum of non-graded rings. ☐

Lemma 2.12. *Let $\mathcal{T} = \varinjlim_i \mathcal{T}_i$ be a filtered 2-colimit of tensor triangulated categories $\mathcal{T}_i$ and $R_i = \mathrm{End}^*_{\mathcal{T}_i}(\mathbb{1})$. Then we have*

$$\mathrm{Spc}(\mathcal{T}) = \varprojlim_i \mathrm{Spc}(\mathcal{T}_i) \quad \textit{and} \quad \mathrm{Spec}^h(R) = \varprojlim_i \mathrm{Spec}^h(R_i)$$

as topological spaces such that $\rho_\mathcal{T} = \varprojlim_i \rho_{\mathcal{T}_i}$.

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Proof. This is straightforward. See [Sta20, Exercise 078L] and [GW20, Proposition 10.53] for the corresponding assertion for the spectrum of non-graded rings. □

3. Modules over the skew group ring

Let A be a commutative ring with an action of a finite group G. We denote by AG the skew group ring for this action, so AG is the free A-module with basis G, and multiplication in AG is defined by $(ag)(bh) = ag(b)gh$ for $a, b \in A$ and $g, h \in G$. Let $D(AG) = D(AG\text{-Mod})$ be the derived category of left AG-modules. The bounded-above derived category $D^-(AG)$ is equivalent to the homotopy category $K^-(AG\text{-Proj})$ of bounded-above complexes of projective AG-modules, which carries the tensor product

$$P \otimes Q = P \otimes_A Q \tag{3.1}$$

with the diagonal action of G. This makes $D^-(AG)$ into a tensor triangulated category. We denote by

$$D^b(AG)_{A\text{-proj}}$$

the full subcategory of $D(AG)$ whose objects are the bounded complexes of AG-modules which are finite projective A-modules, and by

$$D^b(AG)_{A\text{-perf}}$$

the full subcategory of $D(AG)$ whose objects are the complexes of AG-modules which are A-perfect, that is, isomorphic in $D(A)$ to a bounded complex of finite projective A-modules. We will verify that these two categories are equivalent. It is easy to see that they are triangulated subcategories of $D^-(AG)$ and that (at least) $D^b(AG)_{A\text{-perf}}$ is stable under the tensor product.

Lemma 3.2. *For every $P \in D^b(AG)_{A\text{-perf}}$ there is a bounded-above complex of finite projective AG-modules with a quasi-isomorphism $P' \to P$.*

Proof. This is standard; see, for example, [Sta20, Lemma 064Z], where all rings are assumed to be commutative, but that assumption is not used in the proof. Let us sketch a direct argument. Assume $\alpha : Q \to P$ is chosen, where $Q = [Q^m \to \cdots \to Q^n]$ is a finite complex of finite projective AG-modules such that $H^i(\alpha)$ is surjective for $i = m$ and bijective for $i > m$; equivalently, $H^i(\operatorname{cone}(\alpha)) = 0$ for $i \geq m$. Since $\operatorname{cone}(\alpha)$ is A-perfect, it follows that $H^{m-1}(\operatorname{cone}(\alpha))$ is a finite A-module, so we can choose a surjective homomorphism of AG-modules $Q^{m-1} \to H^{m-1}(\operatorname{cone}(\alpha))$ where Q^{m-1} is finite projective. This gives an extension $Q' = [Q^{m-1} \to Q^m \to \cdots \to Q^n]$ of Q and an extension $\alpha' : Q' \to P$ of α such that $H^i(\operatorname{cone}(\alpha')) = 0$ for $i \geq m-1$. By infinite repetition the growing Q gives P'. □

Lemma 3.3. *The inclusion $D^b(AG)_{A\text{-proj}} \to D^b(AG)_{A\text{-perf}}$ is an equivalence of triangulated categories. The resulting tensor product on $D^b(AG)_{A\text{-proj}}$ is given by (3.1).*

Proof. For $P \in D^b(AG)_{A\text{-perf}}$ let $P' \to P$ be a quasi-isomorphism as in Lemma 3.2. For sufficiently small $n \in \mathbb{Z}$, the truncation $P'' = \tau_{\geq n} P'$ is quasi-isomorphic to P as well. Then P'' lies in $D^b(AG)_{A\text{-proj}}$ because P'' is A-perfect and bounded with finite A-projective components except possibly in the left-most degree n. This proves the first assertion.

For the second assertion we note that for quasi-isomorphisms $P' \to P$ and $Q' \to Q$ with $P', Q' \in K^-(AG\text{-Proj})$ and $P, Q \in D^b(AG)_{A\text{-proj}}$ the resulting homomorphism $P' \otimes_A Q' \to P \otimes_A Q$ is again a quasi-isomorphism. □

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Lemma 3.4. *There is an isomorphism of graded rings*

$$\mathrm{End}^*_{D(AG)}(A) \cong H^*(G, A), \tag{3.5}$$

and for two AG-modules M and N such that M is A-projective there is a natural isomorphism of graded modules with respect to (3.5),

$$\mathrm{Hom}^*_{D(AG)}(M, N) \cong H^*(G, \mathrm{Hom}_A(M, N)). \tag{3.6}$$

Again this is standard, at least when G acts trivially on A, but this condition is not essential. We sketch a proof for completeness.

Proof. We have $\mathrm{Hom}^*_{D(AG)}(M,N) = \mathrm{Hom}^*_{D(AG)}(A, \mathrm{Hom}_A(M,N))$, so in (3.6) it suffices to treat the case $M = A$. If $P' \to \mathbb{Z}$ is a $\mathbb{Z}G$-projective resolution, then $P = A \otimes_{\mathbb{Z}} P' \to A$ is an AG-projective resolution, and the complex $\mathrm{Hom}_{AG}(P, N)$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}G}(P', N)$. This gives (3.6) and (3.5) as graded abelian groups. The cup product on $H^*(G, A)$ corresponds to the tensor product in $\mathrm{End}^*_{D(AG)}(A)$, which coincides with the composition product by [Bal10, Proposition 3.3], and hence (3.5) is an isomorphism of graded rings. Similarly (3.6) is an isomorphism of graded modules over these rings. □

3.1 Functoriality

The pairs (G, A) where G is a finite group that acts on a commutative ring A can be made into a category such that a morphism

$$f : (G, A) \to (H, B) \tag{3.7}$$

consists of a group homomorphism $G \leftarrow H$ and an H-equivariant ring homomorphism $f : A \to B$; this is opposite to [Bro82, Chapter III, § 8] and [Wei03, 6.7.6]. A morphism of pairs $f : (G, A) \to (H, B)$ as in (3.7) induces a functor of tensor triangulated categories

$$f_* : D^-(AG) \to D^-(BH)$$

or equivalently $f_* : K^-(AG\text{-Proj}) \to K^-(BH\text{-Proj})$, the latter defined by

$$f_*(P) = P \otimes_A B \tag{3.8}$$

with diagonal H-action using the restriction under $G \leftarrow H$ on the first component. One verifies that the functor f_* restricts to a functor

$$f_* : D^b(AG)_{A\text{-perf}} \to D^b(BH)_{B\text{-perf}}, \tag{3.9}$$

which is given by (3.8) on the objects of $D^b(AG)_{A\text{-proj}}$. Under the isomorphism (3.5), this functor gives a ring homomorphism

$$f_* : H^*(G, A) \to H^*(H, B). \tag{3.10}$$

This is the usual functoriality of group cohomology as a functor of two variables; see [Bro82, Chapter III, § 8] or [Wei03, 6.7.6].

Lemma 3.11. *Let G be a finite group and $A = \varinjlim_i A_i$ a filtered colimit of commutative rings with an action of G. Then*

$$D^b(AG)_{A\text{-perf}} \cong \varinjlim_i D^b(A_i G)_{A_i\text{-perf}}$$

as a filtered 2-colimit of tensor triangulated categories.

Proof. By Lemma 3.3 we can replace $D^b(AG)_{A\text{-perf}}$ by $\mathcal{E}(A) = D^b(AG)_{A\text{-proj}}$. The natural functor $\varinjlim_i \mathcal{E}(A_i) \to \mathcal{E}(A)$ is surjective on isomorphism classes because every complex in $\mathcal{E}(A)$ is determined by finite data of matrices over A subject to finitely many relations. The functor is fully

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faithful because for $X, Y \in \mathcal{E}(A)$, after choosing a quasi-isomorphism $P \to X$ as in Lemma 3.2 we have $\mathrm{Hom}_{\mathcal{E}(A)}(X,Y) = \mathrm{Hom}_{K(AG)}(P,Y)$, which commutes with filtered colimits of A. $\square$

3.2 Finiteness conditions

Let $A^G \subseteq A$ be the ring of G-invariant elements.

DEFINITION 3.12. The pair (G, A) will be called noetherian if the ring A^G is noetherian and A is a finite A^G-module.

LEMMA 3.13. *If A is an algebra of finite type over a noetherian subring B of A^G, then A^G is of finite type over B, and the pair (G, A) is noetherian.*

Proof. Since A is integral over A^G by [Bou75, Chapter V, § 1.9, Proposition 22], [AM69, Proposition 7.8] implies that A^G is a B-algebra of finite type. Hence, A^G is noetherian. Moreover, A is integral and of finite type over A^G, hence finite over A^G. $\square$

COROLLARY 3.14. *If A is a ring of finite type (i.e. a finitely generated $\mathbb{Z}$-algebra), then A^G is of finite type as well, and the pair (G, A) is noetherian.*

Proof. Let B be the image of $\mathbb{Z} \to A$ and apply Lemma 3.13. $\square$

PROPOSITION 3.15. *If the pair (G, A) is noetherian, the triangulated category $D^b(AG)_{A\text{-perf}}$ is End-finite in the sense of Definition 2.6; in particular, the ring $H^*(G, A)$ is noetherian.*

Proof. Let $R = H^*(G, A^G)$. Since A^G is a noetherian ring and A is a finite A^G-module with an A^G-linear action of G, [Eve61, Theorem 6.1 and Corollary 6.2] give that R is a noetherian ring and that $H^*(G, A)$ is a finite R-module, thus a noetherian ring. By Lemma 3.3 it suffices to show that for P, $Q \in D^b(AG)_{A\text{-proj}}$ the graded R-module $M_{P,Q} = \mathrm{Hom}^*_{D(AG)}(P, Q)$ is finite. Since the finite R-modules form a Serre subcategory of the category of all R-modules, one can assume that the complexes P and Q are concentrated in degree zero. In that case, (3.5) gives $M_{P,Q} = H^*(G, \mathrm{Hom}_A(P, Q))$, which is finite over R by [Eve61, Theorem 6.1] again. $\square$

4. Perfect complexes over quotient stacks

As in § 3 let A be a commutative ring with an action of a finite group G. We consider the Deligne–Mumford stack $\mathcal{X} = [\mathrm{Spec}(A)/G]$ and the tensor triangulated category

$$\mathcal{T}_{A,G} = \mathrm{Perf}(\mathcal{X}) \tag{4.1}$$

of perfect complexes in $D(\mathcal{X}_{\mathrm{et}})$, the derived category of $\mathcal{O}_{\mathcal{X}}$-modules on the étale site of $\mathcal{X}$, as well as the graded-commutative ring

$$R_{A,G} = \mathrm{End}^*_{\mathcal{T}_{A,G}}(\mathbb{1}) = \mathrm{End}^*_{D(\mathcal{X}_{\mathrm{et}})}(\mathcal{O}_{\mathcal{X}}). \tag{4.2}$$

We refer to [LM00, (4.6.1)] for the definition of $\mathcal{X}$, to [LM00, § 12] for the étale site of $\mathcal{X}$ and to [Sta20, Section 08G4] for perfect complexes on a ringed site. We note that $\mathrm{Perf}(\mathcal{X})$ is the category of dualisable objects in $D(\mathcal{X}_{\mathrm{et}})$ by [Sta20, Section 0FPP]. In particular, $\mathrm{Perf}(\mathcal{X})$ is a rigid tensor triangulated category.

Remark 4.3. For general algebraic stacks one uses the lisse-étale site and the associated category $D(\mathcal{X}_{\text{lis-et}})$ to define $\mathrm{Perf}(\mathcal{X})$; see for example [HR17]. For Deligne–Mumford stacks this makes no difference because the categories $D_{\mathrm{qc}}(\mathcal{X}_{\mathrm{et}})$ and $D_{\mathrm{qc}}(\mathcal{X}_{\text{lis-et}})$ of complexes of $\mathcal{O}_{\mathcal{X}}$-modules with quasi-coherent cohomology are equivalent by [LM00, Proposition 12.10.1]; cf. also [HR17, §§ 1 and 4].

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Proposition 4.4. *There is an equivalence of tensor triangulated categories*

$$\mathcal{T}_{A,G} \cong D^b(AG)_{A\text{-perf}}, \tag{4.5}$$

and hence an isomorphism of graded rings

$$R_{A,G} \cong H^*(G, A). \tag{4.6}$$

Remark 4.7. We view Proposition 4.4 as a motivation to study the algebraic category $D^b(AG)_{A\text{-perf}}$. In the remainder of the paper, the stack $\mathcal{X}$ will not appear in an essential way, so the reader could skip the rest of this section and take (4.5) as a definition.

Proof of Proposition 4.4. Let $Y = \operatorname{Spec} A$. We denote by $\mathcal{O}_Y[G]$-Mod the category of G-equivariant $\mathcal{O}_Y$-modules on the étale site of Y. Gluing of sheaves for the étale covering $\pi : Y \to \mathcal{X}$ yields an equivalence

$$\mathcal{O}_{\mathcal{X}}\text{-Mod} \cong \mathcal{O}_Y[G]\text{-Mod}, \tag{4.8}$$

using that $Y \times_{\mathcal{X}} Y \cong G \times Y$. There is a pair of adjoint functors

$$AG\text{-Mod} \underset{Q}{\overset{\varphi}{\rightleftarrows}} \mathcal{O}_Y[G]\text{-Mod} \cong \mathcal{O}_{\mathcal{X}}\text{-Mod} \tag{4.9}$$

with φ left adjoint to Q, where $\varphi(M) = \tilde{M}$ is the quasi-coherent $\mathcal{O}_Y$-module associated to M as an A-module, with the action of G on $\tilde{M}$ induced by the given action on M, and $Q(\mathcal{M}) = \Gamma(Y, \mathcal{M})$ as an A-module, carrying the action of G induced by the given action on $\mathcal{M}$. The functor φ is exact and preserves the tensor product defined by $M \otimes_A N$ in AG-Mod and by $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ in $\mathcal{O}_Y[G]$-Mod, in both cases with diagonal G-action. We will show that φ induces the inverse of the desired equivalence (4.5).

Let $D^+_{\mathrm{qc}}(\mathcal{X}_{\mathrm{et}}) \subseteq D^+(\mathcal{X}_{\mathrm{et}})$ be the full subcategory of complexes with quasi-coherent cohomology. The following variant of [SGA6, II, Proposition 3.5] is a special case of [HNR19, Theorem C.1]; we give a direct proof for completeness.

Lemma 4.10. *The functor φ induces an equivalence of triangulated categories $D^+(AG) \cong D^+_{\mathrm{qc}}(\mathcal{X}_{\mathrm{et}})$ with quasi-inverse functor RQ.*

Proof. We begin with two initial remarks.

(1) For $M \in AG$-Mod, the natural map $M \to Q(\varphi(M))$ is an isomorphism, so φ is fully faithful. The image of φ is the category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules in the sense of [Sta20, Definition 03DL] because an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{M}$ is quasi-coherent if and only if $\mathcal{M}_Y = \pi^*(\mathcal{M})$ is a quasi-coherent $\mathcal{O}_Y$-module, which means that $\mathcal{M}_Y \cong \tilde{M}$ for an A-module M by faithfully flat descent.

(2) The functor Q is left exact with derived functors $R^iQ(\mathcal{M}) = H^i(Y_{\mathrm{et}}, \mathcal{M})$ with the induced action of G. Since Y is an affine scheme, for an AG-module M and $i > 0$ we have $H^i(Y_{\mathrm{et}}, \tilde{M}) = 0$, and thus $R^iQ(\varphi(M)) = 0$.

Now the exact functor φ induces an exact functor $\varphi : D^+(AG) \to D^+_{\mathrm{qc}}(\mathcal{X}_{\mathrm{et}})$. We have to show that for complexes $M \in D^+(AG)$ and $\mathcal{M} \in D^+_{\mathrm{qc}}(\mathcal{X}_{\mathrm{et}})$, the natural homomorphisms $\eta_M : M \to RQ(\varphi(M))$ and $\varepsilon_{\mathcal{M}} : \varphi(RQ(\mathcal{M})) \to \mathcal{M}$ are isomorphisms. Both assertions are easily reduced to the case where M and $\mathcal{M}$ are modules concentrated in degree zero. Then η_M is an isomorphism because $M \to Q(\varphi(M))$ is an isomorphism by (1) and $R^iQ(\varphi(M)) = 0$ for $i > 0$ by (2). The module $\mathcal{M}$ is quasi-coherent since it lies in $D^+_{\mathrm{qc}}(\mathcal{X}_{\mathrm{et}})$, thus $\mathcal{M} = \varphi(N)$ for an AG-module N by (1). Hence, $\varepsilon_{\mathcal{M}}$ is an isomorphism since this holds for η_N. $\square$

We continue the proof of Proposition 4.4. In the case $G = 1$, by Lemma 4.10 the functor φ gives an equivalence $D^+(A) \cong D^+_{\mathrm{qc}}(Y_{\mathrm{et}})$. This equivalence restricts to an equivalence $\mathrm{Perf}(A) \cong \mathrm{Perf}(Y_{\mathrm{et}})$ because a complex P of A-modules is perfect if and only if for some faithfully flat ring homomorphism $A \to A'$ the complex $P \otimes_A A'$ is perfect; see [Sta20, Lemma 068T]. For general G it follows that a complex P in $D^+(AG)$ is A-perfect if and only if the complex $\pi^*(\varphi(P))$ in $D^+(Y_{\mathrm{et}})$ is perfect, which means that $\varphi(P)$ in $D^+(\mathcal{X}_{\mathrm{et}})$ is perfect because this is a local condition. Hence, the equivalence of Lemma 4.10 restricts to the desired equivalence $D^b(AG)_{A\text{-perf}} \cong \mathrm{Perf}(\mathcal{X})$. This functor preserves the tensor product because this evidently holds for its restriction to $D^b(AG)_{A\text{-proj}}$; see Lemma 3.3. The isomorphism (4.6) follows by (3.5). □

4.1 Functoriality

A morphism of pairs $f : (G, A) \to (H, B)$ as in (3.7) induces a morphism of algebraic stacks (see, for example, [Sta20, Lemma 046Q])

$$\psi = \mathrm{Spec}(f) : [\mathrm{Spec}(B)/H] \to [\mathrm{Spec}(A)/G],$$

which gives an inverse image functor of tensor triangulated categories

$$f_* = \psi^* : \mathcal{T}_{A,G} \to \mathcal{T}_{B,H}, \tag{4.11}$$

using that the étale topos of Deligne–Mumford stacks is functorial (see, for example, [Zhe15, Construction 2.4], together with [Sta20, Lemma 08H6]). One verifies that under the equivalence of Proposition 4.4, the functor f_* of (4.11) corresponds to the functor f_* of (3.9). We omit further details.

5. The comparison map: basic properties

As earlier, let A be a commutative ring with an action of a finite group G. The comparison map $\rho_{\mathcal{T}}$ of (2.1) for the category $\mathcal{T} = \mathcal{T}_{A,G}$ of (4.1) will be denoted by

$$\rho_{A,G} : \mathrm{Spc}(\mathcal{T}_{A,G}) \to \mathrm{Spec}^h(R_{A,G}). \tag{5.1}$$

5.1 Functoriality

Since the comparison map $\rho_{\mathcal{T}}$ is natural in $\mathcal{T}$, for a morphism of pairs $f : (G, A) \to (H, B)$ as in (3.7) the functor $f_* : \mathcal{T}_{A,G} \to \mathcal{T}_{B,H}$ of (4.11) induces a commutative diagram of topological spaces

$$\begin{tikzcd} \mathrm{Spc}(\mathcal{T}_{B,H}) \arrow[r, "\rho_{B,H}"] \arrow[d, "f^{\mathcal{T}}"'] & \mathrm{Spec}^h(R_{B,H}) \arrow[d, "f^R"] \\ \mathrm{Spc}(\mathcal{T}_{A,G}) \arrow[r, "\rho_{A,G}"] & \mathrm{Spec}^h(R_{A,G}) \end{tikzcd} \tag{5.2}$$

Here $f^{\mathcal{T}}$ is the inverse image map under the functor f_*, and f^R is the inverse image map under the ring homomorphism $f_* : R_{A,G} \to R_{B,H}$ defined by this functor. We recall that under the equivalence (4.5), the functor f_* corresponds to the functor (3.9), and under the isomorphism (4.6), the ring homomorphism f_* corresponds to the homomorphism (3.10).

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5.2 The case of the trivial group

We will write $\mathcal{T}_A = \mathcal{T}_{A,\{e\}} = \mathrm{Perf}(\mathrm{Spec}\, A)$ and $\rho_A = \rho_{A,\{e\}}$. In this case we have $R_{A,\{e\}} = A$ as a graded ring concentrated in degree zero, and the comparison map

$$\rho_A : \mathrm{Spc}(\mathcal{T}_A) \to \mathrm{Spec}(A) \tag{5.3}$$

is a homeomorphism by [Bal10, Proposition 8.1], which is based on [Tho97, Theorem 3.15].

5.3 Restriction to fibres

For a general pair (G, A) there is a morphism of pairs $\pi : (\{e\}, A^G) \to (G, A)$ defined by the inclusion $A^G \to A$ and the unique group homomorphism $\{e\} \leftarrow G$. This gives the following instance of the functoriality diagram (5.2):

$$\begin{tikzcd}
\mathrm{Spc}(\mathcal{T}_{A,G}) \arrow[r, "\rho_{A,G}"] \arrow[d, "\pi^{\mathcal{T}}"'] & \mathrm{Spec}^h(R_{A,G}) \arrow[d, "\pi^R"] \\
\mathrm{Spc}(\mathcal{T}_{A^G}) \arrow[r, "\sim", "\rho_{A^G}"'] & \mathrm{Spec}(A^G)
\end{tikzcd} \tag{5.4}$$

Here ρ_{A^G} is a homeomorphism by [Bal10, Proposition 8.1], as explained in (5.3). We note that π^R is induced by the ring homomorphism $\pi_* : A^G \to R_{A,G}$ given by the inclusion of the degree-zero component, and hence $\pi^R(\mathfrak{p}) = \mathfrak{p}^0$ is the degree-zero part of a homogeneous prime ideal $\mathfrak{p}$ of $R_{A,G}$.

For a given $\mathfrak{q} \in \mathrm{Spec}(A^G)$ with unique inverse image $\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}_{A^G})$ the vertical fibres in (5.4) over these points will be denoted by

$$\mathrm{Spc}(\mathcal{T}_{A,G})_{\mathfrak{q}} = (\pi^{\mathcal{T}})^{-1}(\mathcal{Q}) \quad \text{and} \quad \mathrm{Spec}^h(R_{A,G})_{\mathfrak{q}} = (\pi^R)^{-1}(\mathfrak{q}). \tag{5.5}$$

The map $\rho_{A,G}$ induces a map between these fibres

$$(\rho_{A,G})_{\mathfrak{q}} : \mathrm{Spc}(\mathcal{T}_{A,G})_{\mathfrak{q}} \to \mathrm{Spec}^h(R_{A,G})_{\mathfrak{q}}. \tag{5.6}$$

The following result is evident.

Lemma 5.7. *For a given pair (G, A), the map $\rho_{A,G}$ is bijective if and only if the map $(\rho_{A,G})_{\mathfrak{q}}$ is bijective for each $\mathfrak{q} \in \mathrm{Spec}(A^G)$.*

Remark 5.8. Diagram (5.4) is functorial with respect to (G, A). More precisely, a morphism of pairs $f : (G, A) \to (H, B)$ as in (3.7) gives rise to a commutative cube: [(5.4) for (H, B)] $\to$ [(5.4) for (G, A)]. Since the lower line of (5.4) is always a homeomorphism, the essential information of this cube is captured by the following extension of (5.2),

$$\begin{tikzcd}
\mathrm{Spc}(\mathcal{T}_{B,H}) \arrow[r, "\rho_{B,H}"] \arrow[d, "f^{\mathcal{T}}"'] & \mathrm{Spec}^h(R_{B,H}) \arrow[r, "\pi^R"] \arrow[d, "f^R"] & \mathrm{Spec}(B^H) \arrow[d, "f^0"] \\
\mathrm{Spc}(\mathcal{T}_{A,G}) \arrow[r, "\rho_{A,G}"] & \mathrm{Spec}^h(R_{A,G}) \arrow[r, "\pi^R"] & \mathrm{Spec}(A^G)
\end{tikzcd} \tag{5.9}$$

where f^0 comes from the ring homomorphism $A^G \to B^H$ induced by f.

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5.4 Fibres of the coefficient ring

For a given $\mathfrak{q} \in \operatorname{Spec}(A^G)$ let

$$A(\mathfrak{q}) = (A \otimes_{A^G} k(\mathfrak{q}))_{\mathrm{red}}, \tag{5.10}$$

where the subscript red means maximal reduced quotient, so $\operatorname{Spec}(A(\mathfrak{q}))$ is the reduced fibre over $\mathfrak{q}$ of the morphism $\operatorname{Spec}(A) \to \operatorname{Spec}(A^G)$, and let

$$\psi_{\mathfrak{q}} : A \to A(\mathfrak{q}) \tag{5.11}$$

be the natural homomorphism given by $a \mapsto a \otimes 1$. The action of G on A induces an action on $A(\mathfrak{q})$ via the first factor, and $\psi_{\mathfrak{q}}$ is G-equivariant.

Lemma 5.12. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the prime ideals of A lying over $\mathfrak{q}$. Then*

$$A(\mathfrak{q}) = k(\mathfrak{p}_1) \times \cdots \times k(\mathfrak{p}_r).$$

Proof. The prime ideals of A over $\mathfrak{q}$ form a finite discrete set because they form a single G-orbit in $\operatorname{Spec}(A)$ by [Bou75, Chapter V, § 2.2, Theorem 2(i)], and this set is homeomorphic to $\operatorname{Spec}(A(\mathfrak{q}))$. Hence, the reduced ring $A(\mathfrak{q})$ is the product of its residue fields, and these residue fields coincide with the corresponding residue fields of A since $A(\mathfrak{q})$ is a quotient of a localisation of A. □

Lemma 5.13. *There is a commutative diagram of rings*

$$\begin{tikzcd} A \arrow[rr, "\psi_{\mathfrak{q}}"] & & A(\mathfrak{q}) \\ A^G \arrow[u] \arrow[r, "can"] & k(\mathfrak{q}) \arrow[r] & A(\mathfrak{q})^G \arrow[u] \end{tikzcd}$$

where the vertical homomorphisms are the inclusions. Here $A(\mathfrak{q})^G$ is a field, and $k(\mathfrak{q}) \to A(\mathfrak{q})^G$ is a purely inseparable field extension.

Proof. The composition $A^G \to A \to A(\mathfrak{q})$ factors over $k(\mathfrak{q})$ by the definition of $A(\mathfrak{q})$, and the resulting homomorphism $k(\mathfrak{q}) \to A(\mathfrak{q})$ has image in $A(\mathfrak{q})^G$ since G acts trivially on $k(\mathfrak{q})$. If $\mathfrak{p} \in \operatorname{Spec}(A)$ lies over $\mathfrak{q}$ and if $H \subseteq G$ is the stabiliser of $\mathfrak{p}$, then $A(\mathfrak{q})^G \cong k(\mathfrak{p})^H$, which is a field, and a purely inseparable extension of $k(\mathfrak{q})$ by [Bou75, Chapter V, § 2.2, Theorem 2(ii)]. □

5.5 The fibre diagram

Again let $\mathfrak{q} \in \operatorname{Spec}(A^G)$ be given.

Proposition 5.14. *The functoriality diagram (5.2) for the homomorphism of pairs $\psi_{\mathfrak{q}} : (G, A) \to (G, A(\mathfrak{q}))$ of (5.11) induces a commutative diagram*

$$\begin{tikzcd} \operatorname{Spc}(\mathcal{T}_{A(\mathfrak{q}),G}) \arrow[r, "\rho_{A(\mathfrak{q}),G}"] \arrow[d, "\psi^{\mathcal{T}}_{\mathfrak{q},\mathrm{res}}"'] & \operatorname{Spec}^h(R_{A(\mathfrak{q}),G}) \arrow[d, "\psi^{R}_{\mathfrak{q},\mathrm{res}}"] \\ \operatorname{Spc}(\mathcal{T}_{A,G})_{\mathfrak{q}} \arrow[r, "(\rho_{A,G})_{\mathfrak{q}}"] & \operatorname{Spec}^h(R_{A,G})_{\mathfrak{q}} \end{tikzcd} \tag{5.15}$$

which we call the fibre diagram at $\mathfrak{q}$.

This diagram also appears in the introduction, where the vertical arrows are denoted by $j_{\mathcal{T}}$ and j_R.

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Proof. Let us draw the extended functoriality diagram (5.9) for ψ_q.

$$\begin{tikzcd}
\mathrm{Spc}(\mathcal{T}_{A(\mathfrak{q}),G}) \arrow[r, "\rho_{A(\mathfrak{q}),G}"] \arrow[d, "\psi_{\mathfrak{q}}^{\mathcal{T}}"'] & \mathrm{Spec}^h(R_{A(\mathfrak{q}),G}) \arrow[r, "\pi_R"] \arrow[d, "\psi_{\mathfrak{q}}^{R}"] & \mathrm{Spec}(A(\mathfrak{q})^G) \arrow[d, "\psi_{\mathfrak{q}}^{0}"] \\
\mathrm{Spc}(\mathcal{T}_{A,G}) \arrow[r, "\rho_{A,G}"] & \mathrm{Spec}^h(R_{A,G}) \arrow[r, "\pi_R"] & \mathrm{Spec}(A^G)
\end{tikzcd}$$

We have to verify that the images of the vertical arrows $\psi_{\mathfrak{q}}^{\mathcal{T}}$ and $\psi_{\mathfrak{q}}^{R}$ map to $\mathfrak{q}$ in $\mathrm{Spec}(A^G)$. This holds because the ring homomorphism $A^G \to A(\mathfrak{q})^G$ factors over $k(\mathfrak{q})$ by Lemma 5.13, so the image of $\psi_{\mathfrak{q}}^{0}$ is the singleton $\{\mathfrak{q}\}$. □

LEMMA 5.16. *As earlier, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the prime ideals of A over $\mathfrak{q}$. Moreover, let $L = k(\mathfrak{p}_1)$, and let $H \subseteq G$ be the stabiliser of the element $\mathfrak{p}_1$ of $\mathrm{Spec}(A)$. Then the tensor triangulated category $\mathcal{T}_{A(\mathfrak{q}),G}$ is equivalent to $\mathcal{T}_{L,H}$, and consequently the map $\rho_{A(\mathfrak{q}),G}$ is isomorphic to $\rho_{L,H}$.*

Proof. The group H acts on L by functoriality. We use Lemma 5.12. Since G acts transitively on the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, there is an equivalence

$$A(\mathfrak{q})G\text{-mod} \cong LH\text{-mod}$$

given by $M \mapsto M \otimes_{A(\mathfrak{q})} L$ with diagonal action of H. The resulting equivalence $D^b(A(\mathfrak{q})G\text{-mod}) \cong D^b(LH\text{-mod})$ gives $\mathcal{T}_{A(\mathfrak{q}),G} \cong \mathcal{T}_{L,H}$ by Proposition 4.4, using that every $A(\mathfrak{q})$-module is projective. □

Remark 5.17. On the geometric side, the equivalence $\mathcal{T}_{A(\mathfrak{q}),G} \cong \mathcal{T}_{L,H}$ comes from an isomorphism of stacks $[\mathrm{Spec}(L)/H] \cong [\mathrm{Spec}(A(\mathfrak{q}))/G]$ which is induced by the obvious morphism of pairs $(G, A(\mathfrak{q})) \to (H, L)$.

5.6 Additional comments on the ring $A(\mathfrak{q})$

LEMMA 5.18. *If the ring A^G is local with maximal ideal $\mathfrak{q}$, then $\psi_{\mathfrak{q}}$ is surjective and induces a homeomorphism $\mathrm{Spec}(A(\mathfrak{q})) \cong \mathrm{Max}(A)$.*

Proof. The assumption implies that $A^G \to k(\mathfrak{q})$ is surjective, and hence $\psi_{\mathfrak{q}}$ is surjective. Since $A^G \subseteq A$ is an integral extension, a prime ideal $\mathfrak{p}$ of A is maximal if and only if $\mathfrak{p} \cap A^G$ is a maximal ideal of A^G; see [Bou75, Chapter V, § 2.1, Proposition 1]. Hence, $\mathrm{Max}(A)$ is the set of prime ideals of A lying over $\mathfrak{q}$, which is homeomorphic to $\mathrm{Spec}(A(\mathfrak{q}))$. □

LEMMA 5.19. *Let $f : A \to B$ be a G-equivariant ring homomorphism and let $\tilde{\mathfrak{q}} \in \mathrm{Spec}(B^G)$ with image $\mathfrak{q} \in \mathrm{Spec}(A^G)$ be given. Then f induces an injective ring homomorphism $f' : A(\mathfrak{q}) \to B(\tilde{\mathfrak{q}})$. If B is a localisation of a quotient of A, then f' is bijective.*

Proof. Clearly f induces f'. All assertions follow from Lemma 5.12. Indeed, the G-equivariant map $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ sends the G-orbit over $\tilde{\mathfrak{q}}$ to the G-orbit over $\mathfrak{q}$, and this map between G-orbits is necessarily surjective. Moreover, for $\tilde{\mathfrak{p}} \in \mathrm{Spec}(B)$ with image $\mathfrak{p} \in \mathrm{Spec}(A)$ the homomorphism of residue fields $k(\mathfrak{p}) \to k(\tilde{\mathfrak{p}})$ is injective. Hence, f' is injective. If B is a localisation of a quotient of A, then $\mathrm{Spec}(f)$ is injective, so our map between G-orbits is bijective. Moreover, f induces isomorphisms of the residue fields. Hence, f' is bijective. □

LEMMA 5.20. *Let $A_{\mathfrak{q}} = S^{-1}A$ with $S = A^G \setminus \mathfrak{q}$. The homomorphism $\psi_{\mathfrak{q}}$ factors into G-equivariant homomorphisms*

$$A \to A_{\mathfrak{q}} \xrightarrow{\pi} A(\mathfrak{q})$$

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where π is surjective. The ring $(A_{\mathfrak{q}})^G = (A^G)_{\mathfrak{q}}$ is local with maximal ideal $\mathfrak{q}_{\mathfrak{q}}$. There is an isomorphism $A(\mathfrak{q}) \cong A_{\mathfrak{q}}(\mathfrak{q}_{\mathfrak{q}})$ under which π corresponds to the homomorphism (5.11) for $A_{\mathfrak{q}}$ and $\mathfrak{q}_{\mathfrak{q}}$ in place of A and $\mathfrak{q}$.

Proof. The homomorphism $\psi_{\mathfrak{q}}$ factors over $S^{-1}A$ because $A^G \to k(\mathfrak{q})$ factors over $S^{-1}(A^G)$. We have $(A_{\mathfrak{q}})^G = (A^G)_{\mathfrak{q}}$ since localisation is flat. The rest follows easily; one can also use Lemma 5.19 with $B = S^{-1}A$. $\square$

6. Spectra of graded rings

For a graded-commutative ring R let R_{even} be the subring of R generated by the homogeneous elements of even degree. Then R_{even} is a commutative graded ring, and there is a homeomorphism $\operatorname{Spec}^h(R) \cong \operatorname{Spec}^h(R_{\text{even}})$. In this section we record a number of basic properties of this construction.

Lemma 6.1. *For a homomorphism $S \to T$ of graded-commutative rings let*

$$f : \operatorname{Spec}(T_{\text{even}}) \to \operatorname{Spec}(S_{\text{even}}) \quad \textit{and} \quad f_h : \operatorname{Spec}^h(T) \to \operatorname{Spec}^h(S)$$

be the induced maps. If f is surjective, then so is f_h. If f is a homeomorphism, then so is f_h.

Proof. One can replace S and T by S_{even} and T_{even}. Since $\operatorname{Spec}^h(S) \subseteq \operatorname{Spec}(S)$ and $\operatorname{Spec}^h(T) \subseteq \operatorname{Spec}(T)$ carry the subspace topology, it suffices to show that f_h is surjective if f is surjective. For $\mathfrak{p} \in \operatorname{Spec}^h(S)$ we consider the residue field $k(\mathfrak{p}) = \operatorname{Frac}(S/\mathfrak{p})$ and the graded residue field $k((\mathfrak{p})) = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$, where $S_{\mathfrak{p}}$ is the graded localisation of S at $\mathfrak{p}$. There are ring homomorphisms $S \to k((\mathfrak{p})) \to k(\mathfrak{p})$. Since f is surjective, the ring $T \otimes_S k(\mathfrak{p})$ is non-zero, hence the graded ring $T \otimes_S k((\mathfrak{p}))$ is non-zero. Any graded prime ideal of $T \otimes_S k((\mathfrak{p}))$ gives a graded prime ideal of T that maps to $\mathfrak{p}$. $\square$

Lemma 6.2. *Let R be a graded-commutative ring with an action of a finite group Γ by automorphisms of graded rings. Then the inclusion $R^\Gamma \to R$ induces a homeomorphism $\operatorname{Spec}^h(R)/\Gamma \cong \operatorname{Spec}^h(R^\Gamma)$.*

Proof. The lemma includes its non-graded version because a commutative ring can be considered as a graded ring concentrated in degree zero. The non-graded version is well known and appears, for example, in [SGA1, Exp. V, Proposition 1.1]. In more detail, $R^\Gamma \to R$ is integral and gives a bijective continuous map $\operatorname{Spec}(R)/\Gamma \to \operatorname{Spec}(R^\Gamma)$ by [Bou75, Chapter V, § 2, Theorem 2]; this map is also closed by [GW20, Proposition 5.12]. In the graded case it follows that the natural map $\operatorname{Spec}^h(R)/\Gamma \to \operatorname{Spec}^h(R^\Gamma)$ is the inclusion of a subspace, and the map is surjective by Lemma 6.1, hence a homeomorphism. $\square$

Lemma 6.3. *A graded commutative ring $R = \bigoplus_{n\geq 0} R_n$ is noetherian if and only if the ring R_0 is noetherian and R is an R_0-algebra of finite type. If this holds, for any $d > 0$ the ring $R^{(d)} = \bigoplus_n R_{nd}$ is noetherian as well, and R is a finite $R^{(d)}$-module.*

Proof. See, for example, [Mat80, Theorem 13.1] for the first assertion. The second assertion is reduced to the case $R = R_0[T_1, \ldots, T_r]$ where each T_i is homogeneous of some positive degree. Then $R' = R_0[T_1^d, \ldots, T_r^d] \subseteq R^{(d)} \subseteq R$ where R' is noetherian and R is finite over R', and the second assertion follows. $\square$

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7. The comparison map: the field case

Let L be a field with an action of a finite group G. By Proposition 4.4, the tensor triangulated category $\mathcal{T}_{L,G}$ is equivalent to $D^b(LG\text{-mod})$. The aim of this section is to prove the following result.

Theorem 7.1. *If L is a field, the map $\rho_{L,G} : \mathrm{Spc}(\mathcal{T}_{L,G}) \to \mathrm{Spec}^h(R_{L,G})$ of (5.1) is a homeomorphism.*

If G acts trivially on L, this is the content of [Bal10, Proposition 8.5], which is eventually based on the fact that thick tensor ideals in $D^b(LG\text{-mod})$ are classified by their cohomological support in $\mathrm{Spec}^h(R_{L,G})$, or equivalently that thick tensor ideals in the stable module category of G over L are classified by their cohomological support in $\mathrm{Proj}(R_{L,G})$. The stable version is proved in [BCR97, Theorem 3.4] when L is algebraically closed, and in [BIK11, Theorem 11.4] for arbitrary fields L; a direct proof of the unstable version appears in [CI15].

The general case will be reduced to the case of trivial action. Let

$$H = \mathrm{Ker}(G \to \mathrm{Aut}(L))$$

so that G/H acts faithfully on L. The morphism of pairs

$$f : (G, L) \to (H, L)$$

defined by the identity of L and the inclusion $G \leftarrow H$ gives the following instance of the functoriality diagram (5.2):

$$\begin{array}{ccc} \mathrm{Spc}(\mathcal{T}_{L,H}) & \xrightarrow{\rho_{L,H}} & \mathrm{Spec}^h(R_{L,H}) \\ {\scriptstyle f^{\mathcal{T}}}\big\downarrow & & \big\downarrow{\scriptstyle f^{R}} \\ \mathrm{Spc}(\mathcal{T}_{L,G}) & \xrightarrow{\rho_{L,G}} & \mathrm{Spec}^h(R_{L,G}) \end{array} \tag{7.2}$$

Here the vertical arrows are induced by the functor $f_* : \mathcal{T}_{L,G} \to \mathcal{T}_{L,H}$ which corresponds to the restriction functor

$$\mathrm{res}^G_H : D^b(LG\text{-mod}) \to D^b(LH\text{-mod})$$

under the equivalence of Proposition 4.4.

Lemma 7.3. *The group $\bar{G} = G/H$ acts on all spaces in (7.2) with trivial action on the lower line such that all maps in (7.2) are $\bar{G}$-equivariant.*

Proof. The actions are induced by the conjugation action of G on the ring LG and on the subring LH; note that H is a normal subgroup of G.

In more detail, for $g \in G$ and an LH-module X we form the LH-module X^g which is X with the action of $z \in LH$ by gzg^{-1}. This defines a right action of G on the triangulated category $\mathcal{T}_{L,H} \cong D^b(LH\text{-mod})$, here called the conjugation action. The conjugation action admits the following alternative description, which shows that the action of G on $\mathcal{T}_{L,H}$ preserves the tensor structure. Each $g \in G$ gives a homomorphism of pairs $(H, L) \to (H, L)$ defined by $L \to L$, $a \mapsto g^{-1}(a)$ and $H \leftarrow H$, $ghg^{-1} \leftarrow\!\shortmid h$. The resulting endomorphism of $\mathcal{T}_{L,H}$ by functoriality with respect to (H, L) is isomorphic to the endomorphism $X \mapsto X^g$ because there is an isomorphism $X^g \to X \otimes_{L,g^{-1}} L$, $y \mapsto y \otimes 1$. Hence, the action of G on the pair (H, L) induces by functoriality the conjugation action of G on $\mathcal{T}_{L,H}$.

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The conjugation action of G on $\mathcal{T}_{L,H}$ induces compatible left actions of G on the source and target of $\rho_{L,H}$. Similarly, G acts on the tensor triangulated category $\mathcal{T}_{L,G}$, which induces compatible actions of G on the source and target of $\rho_{L,G}$ such that all maps in (7.2) are equivariant. It remains to verify that H acts trivially on the source and target of $\rho_{L,H}$ and G acts trivially on the source and target of $\rho_{L,G}$.

For an LG-module X and $g \in G$, multiplication by g is an isomorphism $X \cong X^g$. It follows that G acts trivially on $\mathrm{Spc}(\mathcal{T}_{L,G})$. The alternative description of the conjugation action implies that the resulting action of G on $R_{L,G} \cong H^*(G,L)$ corresponds to the conjugation action in group cohomology as defined in [Bro82, Chapter III, § 8], which is trivial by [Bro82, Chapter III, Proposition (8.1)]. Similarly, H acts trivially on $\mathrm{Spc}(\mathcal{T}_{L,H})$ and on $R_{L,H}$. □

Lemma 7.4. *The map f^R of (7.2) induces a homeomorphism*

$$\mathrm{Spec}^h(R_{L,H})/\bar{G} \xrightarrow{\bar{f}^R} \mathrm{Spec}^h(R_{L,G}).$$

Proof. Let $K = L^G$. Then L/K is a finite Galois extension with Galois group $\bar{G}$. We consider the sequence of left exact functors

$$LG\text{-Mod} \xrightarrow{(-)^H} L\bar{G}\text{-Mod} \xrightarrow{(-)^{\bar{G}}} K\text{-Mod} \tag{7.5}$$

whose composition is the functor of G-invariants. By Galois descent [GW20, Theorem 14.85], the functor $(-)^{\bar{G}}$ in (7.5) is an equivalence, in particular, exact. Hence (7.5) yields an isomorphism of δ-functors

$$H^i(G,M) \xrightarrow{\sim} H^i(H,M)^{\bar{G}}$$

for $M \in LG$-Mod, which coincides with the restriction map in group cohomology because this holds for $i = 0$, as is easily verified. For $M = L$ it follows that the homomorphism $f_* : R_{L,G} \to R_{L,H}$ identifies $R_{L,G}$ with the ring of $\bar{G}$-invariants in $R_{L,H}$. Then Lemma 6.2 finishes the proof. □

Lemma 7.6. *The map $f^{\mathcal{T}}$ of (7.2) is surjective.*

Proof. The restriction functor $f_* : LG\text{-mod} \to LH\text{-mod}$ has an exact right adjoint $f^!$ defined by $f^!(M) = \mathrm{Hom}_{LH}(LG, M)$, which is an LG-module using the right LG-module structure of LG. This induces an exact right adjoint $f^!$ of the tensor triangulated functor $f_* : D^b(LG\text{-mod}) \to D^b(LH\text{-mod})$. By [Bal18, Theorem 1.7] it follows that the image of the map $f^{\mathcal{T}} = \mathrm{Spc}(f_*)$ is equal to the support of $f^!(\mathbb{1}) = \mathrm{Hom}_{LH}(LG, L)$, viewed as an object of $D^b(LG\text{-mod})$. One verifies that H acts trivially on $f^!(\mathbb{1})$, so this is an $L\bar{G}$-module. An $L\bar{G}$-module is determined by its L-dimension by Galois descent [GW20, Theorem 14.85]. Hence, $f^!(\mathbb{1}) \cong L^{[G:H]}$ and thus $\mathrm{supp}(f^!(\mathbb{1})) = \mathrm{supp}(\mathbb{1}) = \mathrm{Spc}(\mathcal{T}_{L,G})$. □

Proof of Theorem 7.1. By Lemma 7.3, the commutative diagram (7.2) induces a commutative diagram of topological spaces

$$\begin{tikzcd} \mathrm{Spc}(\mathcal{T}_{L,H})/\bar{G} \arrow[r, "\tilde{\rho}_{L,H}"] \arrow[d, "\bar{f}^{\mathcal{T}}"'] & \mathrm{Spec}^h(R_{L,H})/\bar{G} \arrow[d, "\bar{f}^R"] \\ \mathrm{Spc}(\mathcal{T}_{L,G}) \arrow[r, "\rho_{L,G}"] & \mathrm{Spec}^h(R_{L,G}) \end{tikzcd} \tag{7.7}$$

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Here $\tilde{\rho}_{L,H}$ is a homeomorphism because $\rho_{L,H}$ is a homeomorphism by [Bal10, Proposition 8.5], $\tilde{f}^R$ is a homeomorphism by Lemma 7.4, and $\tilde{f}^{\mathcal{T}}$ is surjective by Lemma 7.6. It follows that all arrows in (7.7) are homeomorphisms. □

Remark 7.8. In the situation of Theorem 7.1 one could ask for a relation between the stacks $[\mathrm{Spec}(L)/G]$ and $[\mathrm{Spec}(K)/H]$ or between the corresponding module categories LG-mod and KH-mod. In general, these stacks are not isomorphic, and the categories are not equivalent. For example, let $L = \mathbb{C}$ and $G = \mathbb{Z}/4\mathbb{Z}$ such that a generator of G acts on L by complex conjugation, so $K = \mathbb{R}$ and $H = 2\mathbb{Z}/4\mathbb{Z}$. Then KH-mod has two isomorphism classes of one-dimensional representations while LG-mod has only one such class, so these categories are not equivalent as tensor categories because the number of invertible objects up to isomorphism is different.

8. Change of coefficients in group cohomology

We fix a finite group G. For a commutative ring A with an action of G we ask how the graded-commutative ring $R_{A,G} = H^*(G, A)$ and its homogeneous prime spectrum depend on A.

8.1 Base change homomorphisms

A G-equivariant ring homomorphism

$$u : A \to B$$

induces a homomorphism of graded-commutative rings

$$u' : R_{A,G} \to R_{B,G}$$

whose degree-zero component is the homomorphism $A^G \to B^G$ defined by u. This gives a homomorphism of graded-commutative rings

$$u'' : R_{A,G} \otimes_{A^G} B^G \to R_{B,G}$$

and, by restriction to the subrings generated by the elements of even degree, a homomorphism of commutative graded rings

$$u''' : (R_{A,G})_{\mathrm{even}} \otimes_{A^G} B^G \to (R_{B,G})_{\mathrm{even}}. \tag{8.1}$$

A homomorphism of commutative rings $S \to T$ will be called a universal homeomorphism if it induces a universal homeomorphism $\mathrm{Spec}\, T \to \mathrm{Spec}\, S$ of schemes, which means that for every ring homomorphism $S \to S'$ the natural map $\mathrm{Spec}(T \otimes_S S') \to \mathrm{Spec}(S')$ is a homeomorphism.

Definition 8.2. We denote by Coh-uh(G) the class of all G-equivariant homomorphisms of commutative rings $u : A \to B$ such that the base change homomorphism u''' in (8.1) is a universal homeomorphism.

Remark 8.3. The class Coh-uh(G) does not contain all G-equivariant ring homomorphisms. For example, if L/K is a finite Galois extension with Galois group G, then L is an induced G-module by the normal basis theorem, the inclusion map $K \to L$ is G-equivariant, and the associated homomorphism $R_{K,G} \to R_{L,G}$ is the augmentation $H^*(G, K) \to K$.

Proposition 8.4. *The class* Coh-uh(G) *is stable under composition.*

Proof. If $A \xrightarrow{u} B \xrightarrow{v} C$ is a sequence of G-equivariant ring homomorphisms, then $(v \circ u)''' = v''' \circ (u''' \otimes_{B^G} C^G)$. □

We will show that the class Coh-uh(G) contains all G-equivariant surjections and localisations. First we study the effect of such homomorphisms on the rings of G-invariants, that is, on the degree-zero part of $R_{A,G}$.

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Proposition 8.5. *Let A be a commutative ring with an action of G and let $B = A/I$ for a G-stable ideal I of A. Then the natural ring homomorphism $A^G/I^G \to B^G$ is a universal homeomorphism.*

Proof. Since A is integral over A^G by [Bou75, Chapter V, § 1.9, Proposition 22], the injective ring homomorphism $A^G/I^G \to B$ is integral, so $A^G/I^G \to B^G$ is integral and injective as well, and the induced morphism

$$f : \mathrm{Spec}(B^G) \to \mathrm{Spec}(A^G/I^G)$$

is integral and surjective by [Sta20, Lemma 00GQ]. We have

$$\mathrm{Spec}(A^G) = \mathrm{Spec}(A)/G \quad \text{and} \quad \mathrm{Spec}(B^G) = \mathrm{Spec}(B)/G$$

as topological spaces; see Lemma 6.2. Since $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is injective it follows that $\mathrm{Spec}(B^G) \to \mathrm{Spec}(A^G)$ is injective, hence f is injective and thus bijective. Since f is integral and bijective, by [GD67, 18.12.11] it remains to show that the residue field extensions induced by f are purely inseparable.

For given $\tilde{\mathfrak{q}} \in \mathrm{Spec}(B^G)$ with image $\mathfrak{q} \in \mathrm{Spec}(A^G)$ the homomorphism $A \to B$ induces an isomorphism $A(\mathfrak{q}) \to B(\tilde{\mathfrak{q}})$ by Lemma 5.19. The sequence of G-equivariant ring homomorphisms

$$A \to B \to B(\tilde{\mathfrak{q}}) \cong A(\mathfrak{q})$$

gives ring homomorphisms $A^G \to B^G \to B(\tilde{\mathfrak{q}})^G \cong A(\mathfrak{q})^G$, which induces field extensions $k(\mathfrak{q}) \to k(\tilde{\mathfrak{q}}) \to B(\tilde{\mathfrak{q}})^G \cong A(\mathfrak{q})^G$ by Lemma 5.13 applied to B, and the total extension $k(\mathfrak{q}) \to A(\mathfrak{q})^G$ is purely inseparable by Lemma 5.13 applied to A. Hence, $k(\mathfrak{q}) \to k(\tilde{\mathfrak{q}})$ is purely inseparable as well. $\square$

Lemma 8.6. *Let $A \to B$ be a G-equivariant homomorphism of rings such that B is a localisation of A, and let $S \subseteq A$ be the set of elements which become invertible in B. Then $B^G = (S^G)^{-1}A^G$ and $B = A \otimes_{A^G} B^G$.*

Proof. If $s \in S$ then $\prod_{g\in G} g(s) \in S^G$. Hence, $B = S^{-1}A = A \otimes_{A^G} (S^G)^{-1}A^G$. Since localisation is exact it follows that $B^G = (S^G)^{-1}A^G$. $\square$

8.2 Calculation by resolutions

We will use the following description of the graded ring $R_{A,G} = H^*(G, A)$. Let $P \to A$ be a resolution of A by a complex P of finite projective AG-modules; for example, one can take a resolution $P' \to \mathbb{Z}$ by finite projective $\mathbb{Z}G$-modules and set $P = A \otimes_{\mathbb{Z}} P'$. Let

$$E = \mathrm{End}_{AG}(P) \tag{8.7}$$

as a differential graded algebra. Then

$$R_{A,G} = H^*(E) \tag{8.8}$$

as a graded ring. Moreover, let

$$N = \mathrm{Hom}_{AG}(P, A) \tag{8.9}$$

as a right differential graded (dg) E-module. The homomorphism $P \to A$ induces a quasi-isomorphism of right dg E-modules $E \to N$ and thus

$$R_{A,G} = H^*(N) \tag{8.10}$$

as a graded right $R_{A,G}$-module, in particular as a graded abelian group.

Lemma 8.11. *For a flat ring homomorphism $A^G \to B_0$ let $B = A \otimes_{A^G} B_0$, where G acts on the first factor. Then $B^G = B_0$, and the homomorphism of graded rings $R_{A,G} \otimes_{A^G} B^G \to R_{B,G}$ is an isomorphism. In particular, the homomorphism $A \to B$ lies in the class* Coh-uh(G).

Proof. Flatness implies that $B^G = B_0$. For $N = \mathrm{Hom}_{AG}(P, A)$ as in (8.9) there is an isomorphism

$$N \otimes_{A^G} B^G \cong \mathrm{Hom}_{BG}(P \otimes_A B, B) \tag{8.12}$$

because P consists of finite projective AG-modules. Since $A^G \to B^G$ is flat, the cohomology of the left-hand side of (8.12) is $R_{A,G} \otimes_{A^G} B^G$. Since $P \otimes_A B \to B$ is a resolution by finite projective BG-modules, the cohomology of the right-hand side of (8.12) is $R_{B,G}$. □

Proposition 8.13. *The class* Coh-uh(G) *of Definition 8.2 contains all G-equivariant localisation homomorphisms.*

Proof. By Lemma 8.6 we have $B = A \otimes_{A^G} B_0$ where B_0 is a localisation of A^G, so the assertion follows from Lemma 8.11. □

Proposition 8.14. *Let $\bar{A} = A/I$ for a G-stable nilpotent ideal I. Then the natural ring homomorphism*

$$\pi_{\mathrm{even}} : (R_{A,G})_{\mathrm{even}} \to (R_{\bar{A},G})_{\mathrm{even}}$$

is a universal homeomorphism, and the projection $u : A \to \bar{A}$ lies in the class Coh-uh(G) *of Definition 8.2.*

Proof. For a prime p let p^r be the maximal p-power dividing $|G|$ and let $m_p = |G|/p^r$. Then $\mathrm{Spec}(\mathbb{Z})$ is covered by the open sets $\mathrm{Spec}(\mathbb{Z}[1/m_p])$ for varying p, and we can replace A by $A[1/m_p]$ for a fixed prime p, using Lemma 8.11. Then $p^r H^i(G, M) = 0$ for any AG-module M and $i > 0$.

To prove that π_{even} is a universal homeomorphism, by an induction using the sequence of G-equivariant ring homomorphisms $A \to A/I^2 \to A/I$ we can assume that I has square zero. Then the exact sequence of AG-modules

$$0 \to I \to A \to \bar{A} \to 0$$

induces a long exact sequence in group cohomology

$$H^*(G, I) \xrightarrow{j} R_{A,G} \xrightarrow{\pi} R_{\bar{A},G} \xrightarrow{\delta} H^*(G, I). \tag{8.15}$$

Here π is a homomorphism of graded rings that restricts to the homomorphism π_{even} of the proposition, $H^*(G, I)$ is a graded left and right $R_{\bar{A},G}$-module since I is an $\bar{A}$-module, and j is $R_{A,G}$-linear. Moreover, δ is a graded derivation by Lemma 8.17 below. If $a \in R_{\bar{A},G}$ is homogeneous of even degree, we obtain

$$\delta(a^{p^r}) = \sum_{i=1}^{p^r} a^{i-1}\delta(a)a^{p^r-i} = p^r\delta(a)a^{p^r-1} = 0, \tag{8.16}$$

hence a^{p^r} lies in the image of π; moreover, $p^r a$ lies in the image of π since $p^r\delta(a) = 0$. The image j has square zero since this holds for I. It follows that π_{even} is a universal homeomorphism by [Sta20, Lemma 0BRA].

To prove that u lies in Coh-uh(G) we can drop the assumption that I has square zero. The degree-zero component of π_{even} factors into the homomorphisms $A^G \to A^G/I^G \to \bar{A}^G$, which are both universal homeomorphisms by Proposition 8.5 and since I^G is nilpotent. Now π_{even} is the composition

$$(R_{A,G})_{\text{even}} \to (R_{A,G})_{\text{even}} \otimes_{A^G} \bar{A}^G \xrightarrow{u'''} (R_{\bar{A},G})_{\text{even}}$$

where the first arrow is a universal homeomorphism since this holds for $A^G \to \bar{A}^G$, and u''' is the homomorphism associated to $u : A \to \bar{A}$ as in (8.1). Hence, u''' is a universal homeomorphism since this holds for π_{even}. □

Lemma 8.17. *The homomorphism δ of (8.15) is a graded derivation, that is, $\delta(ab) = \delta(a)b + (-1)^{|a|}a\delta(b)$ for homogeneous elements $a, b \in R_{\bar{A},G}$.*

This is a variant of the well-known fact that the Bockstein homomorphism is a graded derivation; see, for example, [Hat02, Example 3E.1]. We include a proof for completeness.

Proof. We use that $R_{A,G} = H^*(E)$ with $E = \operatorname{End}_{AG}(P)$ as in (8.7). Let $\bar{P} = P/IP$. Then $\bar{P} \to \bar{A}$ is a resolution of $\bar{A}$ by finite projective $\bar{A}G$-modules, and $IP \to I$ is a resolution of I by $\bar{A}G$-modules. Let $\bar{E} = \operatorname{End}_{\bar{A}G}(\bar{P})$ and $J = \operatorname{Hom}_{AG}(P, IP) = \operatorname{Hom}_{\bar{A}G}(\bar{P}, IP)$. The obvious exact sequence $0 \to IP \to P \to \bar{P} \to 0$ induces an exact sequence

$$0 \to J \to E \xrightarrow{\tilde{\pi}} \bar{E} \to 0 \tag{8.18}$$

where $\tilde{\pi}$ is a homomorphism of dg algebras, so J is a two-sided dg ideal of E, and J becomes a dg $\bar{E}$-bimodule since J has square zero. The cohomology sequence of (8.18) can be identified with (8.15). For $a \in R_{\bar{A},G}$ let $\tilde{a} \in E$ be an inverse image under $\tilde{\pi}$ of a representative of a in $\bar{E}$. Then $\delta(a) = [d(\tilde{a})]$ where $[\,]$ is the cohomology class of a cycle in J, and the lemma follows from the relation $d(\tilde{a}\tilde{b}) = d(\tilde{a})\tilde{b} + (-1)^{|a|}\tilde{a}d(\tilde{b})$. □

We recall that the pair (G, A) is called noetherian if the ring A^G is noetherian and if A is finite over A^G; see Definition 3.12.

Proposition 8.19. *Let $\bar{A} = A/tA$ where $t \in A^G$ is an A-regular element and assume that (G, A) is noetherian. Then the natural ring homomorphism*

$$\bar{\pi}_{\text{even}} : (R_{A,G}/tR_{A,G})_{\text{even}} \to (R_{\bar{A},G})_{\text{even}}$$

is a universal homeomorphism, and the projection $u : A \to \bar{A}$ lies in the class Coh-uh(G) *of Definition 8.2.*

Proof. We begin with two initial remarks.

First, we can replace t by a positive power t^m using Proposition 8.14 for $A/t^m \to A/t$; note that the projection $(R_{A,G}/t^m)_{\text{even}} \to (R_{A,G}/t)_{\text{even}}$ is a universal homeomorphism, and Coh-uh(G) is stable under composition by Proposition 8.4. The value of m will be determined later.

Second, as in the proof of Proposition 8.14, after replacing A by a localisation we can assume that for a fixed prime p we have $p^r H^i(G, M) = 0$ for any AG-module M and $i > 0$.

The exact sequence of AG-modules,

$$0 \to A \xrightarrow{t} A \to \bar{A} \to 0,$$

induces a long exact sequence in group cohomology,

$$R_{A,G} \xrightarrow{t} R_{A,G} \xrightarrow{\pi} R_{\bar{A},G} \xrightarrow{\delta} R_{A,G} \xrightarrow{t} R_{A,G}, \tag{8.20}$$

and thus an injective homomorphism of graded rings,

$$\bar{\pi} : R_{A,G}/tR_{A,G} \to R_{\bar{A},G},$$

that restricts to the homomorphism $\bar{\pi}_{\text{even}}$ of the proposition. By Lemma 8.21 below, after replacing t by t^m for some positive m, for each homogeneous element $a \in R_{\bar{A},G}$ of even degree, a^{p^r} lies

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in the image of π. Moreover, $p^r a$ lies in the image of π since $p^r\delta(a) = 0$. It follows that $\bar{\pi}_{\text{even}}$ is a universal homeomorphism by [Sta20, Lemma 0BRA].

The degree-zero component of $\bar{\pi}_{\text{even}}$ is given by $A^G/tA^G \to \bar{A}^G$, which is a universal homeomorphism by Proposition 8.5; note that $tA^G = (tA)^G$ since t is A-regular. Now $\bar{\pi}_{\text{even}}$ is the composition

$$(R_{A,G})_{\text{even}} \otimes_{A^G} A^G/tA^G \to (R_{A,G})_{\text{even}} \otimes_{A^G} \bar{A}^G \xrightarrow{u'''} (R_{\bar{A},G})_{\text{even}}$$

where the first arrow is a universal homeomorphism since this holds for $A^G/tA^G \to \bar{A}^G$, and u''' is the homomorphism associated to $u : A \to \bar{A}$ as in (8.1). Hence, u''' is a universal homeomorphism since this holds for $\bar{\pi}_{\text{even}}$. □

LEMMA 8.21. *In the situation of Proposition 8.19, after replacing t by t^m for some fixed positive integer m, for every homogeneous element $a \in R_{\bar{A},G}$ of even degree we have $\delta(a^{p^r}) = 0$ in $R_{A,G}$.*

Proof. This is similar to the calculation in (8.16), but with some complications since we do not have a direct analogue of Lemma 8.17. Since the ring $R_{A,G}$ is noetherian by Proposition 3.15, the ideals $J_i = \operatorname{Ker}(t^i : R_{A,G} \to R_{A,G})$ stabilise for large i. After replacing t by t^m for some $m \geq 1$ we can assume that $J_1 = J_2$. We use again that $R_{A,G} = H^*(E)$ with $E = \operatorname{End}_{AG}(P)$ as in (8.8). Let $\bar{P} = P/tP$. Then $\bar{P} \to \bar{A}$ is a resolution of $\bar{A}$ by finite projective $\bar{A}G$-modules. Let $\bar{E} = \operatorname{End}_{\bar{A}G}(\bar{P})$. There is an exact sequence

$$0 \to E \xrightarrow{t} E \xrightarrow{\tilde{\pi}} \bar{E} \to 0,$$

whose cohomology sequence can be identified with (8.20). If $\tilde{a} \in E$ is an inverse image of a representative of a in $\bar{E}$, then $\delta(a) = [d(\tilde{a})/t]$ where [ ] denotes the class of a cycle in E. Since $R_{\bar{A},G}$ is a graded-commutative ring, for homogeneous elements $x, y, z \in E$ with $d(x), d(y), d(z) \in tE$ we have

$$xyz - (-1)^{|x|\cdot|y|}yxz \in d(E) + tE. \tag{8.22}$$

Modulo $d(E) + tE$ we obtain

$$\frac{d(\tilde{a}^{p^r})}{t} = \sum_{i=1}^{p^r} \tilde{a}^{i-1}\frac{d(\tilde{a})}{t}\tilde{a}^{p^r-i} \equiv p^r\frac{d(\tilde{a})}{t}\tilde{a}^{p^r-1} \equiv 0,$$

so $\delta(a^{p^r}) = [d(x) + ty] = [ty]$ for certain $x, y \in E$. Necessarily, $d(x) + ty$ is a cycle, so $0 = d(ty) = td(y)$. Hence, $d(y) = 0$ since t is E-regular, and thus $\delta(a^{p^r}) = tc$ with $c = [y] \in R_{A,G}$. We have $t^2c = t\delta(a^{p^r}) = 0$ since $t \circ \delta = 0$ in (8.20). Since $J_1 = J_2$ it follows that $\delta(a^{p^r}) = tc = 0$. □

The following combination of Propositions 8.14 and 8.19 will be used for a noetherian induction.

PROPOSITION 8.23. *Assume that the pair (G, A) is noetherian and the ring A^G is local with maximal ideal $\mathfrak{q}$. If the homomorphism $\psi_{\mathfrak{q}} : A \to A(\mathfrak{q})$ of (5.11) is not an isomorphism, then there is a non-zero G-invariant ideal I of A with $I \subseteq \operatorname{Ker}(\psi_{\mathfrak{q}})$ such that the projection $A \to A/I$ lies in the class* Coh-uh(G) *of Definition 8.2.*

Proof. If A is not reduced, let $I \subseteq A$ be a non-zero G-stable nilpotent ideal, for example the nil-radical of A. Then $I \subseteq \operatorname{Ker}(\psi_{\mathfrak{q}})$ since $A(\mathfrak{q})$ is reduced, and the projection $A \to A/I$ lies in Coh-uh(G) by Proposition 8.14. So we can assume that A is reduced.

By Lemma 5.18, the homomorphism ψ_q is surjective and induces a homeomorphism $\operatorname{Spec}(A(\mathfrak{q})) \cong \operatorname{Max}(A)$. If some maximal ideal of A is also a minimal prime ideal, this holds for every maximal ideal of A because $\mathrm{Max}(A)$ is a single G-orbit in $\mathrm{Spec}(A)$, hence $\mathrm{Spec}(A) = \mathrm{Max}(A)$, and $\psi_{\mathfrak{q}}$ induces a bijective map $\mathrm{Spec}(A(\mathfrak{q})) \to \mathrm{Spec}(A)$. Since A is reduced and $\psi_{\mathfrak{q}}$ surjective, this implies that $\psi_{\mathfrak{q}}$ is bijective, which was excluded. Hence, every minimal prime ideal $\mathfrak{p}$ of A is non-maximal and therefore satisfies $\mathfrak{p} \cap A^G \neq \mathfrak{q}$.

By prime avoidance in A^G we find an element $t \in A^G$ with $t \in \mathfrak{q}$ such that t is not contained in any minimal prime ideal of A; the minimal prime ideals of A form a finite set since A is noetherian. Since A is reduced, t is A-regular, so $A \to A/tA$ lies in Coh-uh(G) by Proposition 8.19. $\square$

PROPOSITION 8.24. *If the pair (G, A) is noetherian, for each $\mathfrak{q} \in \mathrm{Spec}(A^G)$ the homomorphism $\psi_{\mathfrak{q}} : A \to A(\mathfrak{q})$ of (5.11) lies in the class* Coh-uh(G) *of Definition 8.2.*

Proof. Using the factorisation $A \to A_{\mathfrak{q}} \to A(\mathfrak{q})$ of $\psi_{\mathfrak{q}}$ of Lemma 5.20, Propositions 8.4 and 8.13 allow us to replace A by $A_{\mathfrak{q}}$. Then A^G is local with maximal ideal $\mathfrak{q}$. The case $A = A(\mathfrak{q})$ is clear, so let $\psi_{\mathfrak{q}}$ not be an isomorphism. Proposition 8.23 gives a factorisation of $\psi_{\mathfrak{q}}$ into G-equivariant homomorphisms

$$A \xrightarrow{\pi} A' = A/I \xrightarrow{\varphi} A(\mathfrak{q}) \tag{8.25}$$

where I is non-zero and π lies in Coh-uh(G).

Here A'^G is local with maximal ideal $\mathfrak{q}'$ lying over $\mathfrak{q}$ because the natural map $\mathrm{Spec}(A'^G) \to \mathrm{Spec}(A^G/I^G)$ is a homeomorphism by Proposition 8.5. The resulting homomorphism $A(\mathfrak{q}) \to A'(\mathfrak{q}')$ is an isomorphism by Lemma 5.19, so φ can be identified with the homomorphism (5.11) for A' and $\mathfrak{q}'$ in place of A and $\mathfrak{q}$. One verifies that the pair (G, A') is noetherian using the chain $A^G \to A'^G \subseteq A'$, where A' is finite over A^G and A^G is noetherian.

Hence, the hypotheses of Proposition 8.23 are satisfied by A' and $\mathfrak{q}'$ in place of A and $\mathfrak{q}$, and we can apply Proposition 8.23 repeatedly as long as the new rings A' differ from $A(\mathfrak{q})$. Since A is noetherian, the process necessarily stops and we arrive at a finite chain of G-equivariant ring homomorphisms $A \to A' \to A'' \to \cdots \to A^{(n)} = A(\mathfrak{q})$ where all arrows lie in the class Coh-uh(G). Then $\psi_{\mathfrak{q}}$ lies in Coh-uh(G) by Proposition 8.4. $\square$

THEOREM 8.26. *The class* Coh-uh(G) *of Definition 8.2 contains all G-equivariant ring homomorphisms $u : A \to B$ such that B is a localisation of a quotient of A.*

Proof. The homomorphism $u : A \to B$ factors into G-equivariant homomorphisms $A \xrightarrow{w} A' \xrightarrow{v} B$ where w is surjective and v is an injective localisation. By Propositions 8.4 and 8.13 we can assume that u is surjective.

Let us first assume that the pair (G, A) is noetherian in the sense of Definition 3.12. Then $R_{B,G}$ is a finite module over the noetherian ring $R_{A,G}$ by Proposition 3.15. Moreover, $R_{A,G}$ is a finite module over the noetherian ring $(R_{A,G})_{\mathrm{even}}$ by Lemma 6.3. Hence, $(R_{B,G})_{\mathrm{even}}$ is finite over $(R_{A,G})_{\mathrm{even}}$. So the ring homomorphism u''' of (8.1) is finite, thus universally closed, and it suffices to show that u''' is universally bijective. This holds if and only if for each $\mathfrak{q} \in \mathrm{Spec}(B^G)$ the base change of u''' under the natural ring homomorphism $B^G \to k(\mathfrak{q})$ is universally bijective, which is verified as follows.

Let $\mathfrak{q}' \in \mathrm{Spec}(A^G)$ be the image of $\mathfrak{q}$ and let $C = B(\mathfrak{q})$. Since $A(\mathfrak{q}') \cong B(\mathfrak{q})$ by Lemma 5.19, in the sequence of G-equivariant homomorphisms $A \xrightarrow{u} B \to C$ the homomorphisms $A \to C$ and $B \to C$ lie in Coh-uh(G) by Proposition 8.24. By two-out-of-three for universal homeomorphisms it follows that the base change of u''' under the ring homomorphism $B^G \to C^G$ is a universal homeomorphism; see the proof of Proposition 8.4. By Lemma 5.13, $B^G \to C^G$ factors as

$B^G \to k(\mathfrak{q}) \to C^G$, where $k(\mathfrak{q}) \to C^G$ is a purely inseparable field extension and hence a universal homeomorphism. It follows that the base change of u''' under $B^G \to k(\mathfrak{q})$ is a universal homeomorphism as well. This finishes the proof if the pair (G, A) is noetherian.

The general case follows by a limit argument. We have $A = \varinjlim_i A_i$ as a filtered direct limit where A_i runs through all finitely generated G-invariant subrings of A, and $B = \varinjlim_i B_i$ where B_i is the image of A_i in B. Let $u_i : A_i \to B_i$ be the restriction of u. There is the following commutative diagram.

$$\begin{array}{ccc} \varinjlim_i((R_{A_i,G})_{\mathrm{even}} \otimes_{A_i^G} B_i^G) & \xrightarrow{\varinjlim u_i'''} & \varinjlim_i(R_{B_i,G})_{\mathrm{even}} \\ \downarrow & & \downarrow \\ (R_{A,G})_{\mathrm{even}} \otimes_{A^G} B^G & \xrightarrow{u'''} & (R_{B,G})_{\mathrm{even}} \end{array}$$

Each pair (G, A_i) is noetherian by Corollary 3.14, so u_i lies in Coh-uh(G) by the first part of the proof, that is, u_i''' is a universal homeomorphism. It follows that $\varinjlim u_i'''$ is a universal homeomorphism since Spec transforms a filtered colimit of rings into a limit of topological spaces; see [GW20, Proposition 10.53]. The vertical arrows of the diagram are isomorphisms since the cohomology of a finite group preserves filtered colimits of the coefficients. Hence, u''' is a universal homeomorphism as desired. □

COROLLARY 8.27. *Let $u : A \to B$ be a G-equivariant ring homomorphism where B is a localisation of a quotient of A. Then there is the following cartesian diagram of topological spaces with immersions as vertical arrows.*

$$\begin{array}{ccc} \mathrm{Spec}^h(R_{B,G}) & \xrightarrow{\pi^R} & \mathrm{Spec}(B^G) \\ {\scriptstyle f^R}\downarrow & & \downarrow{\scriptstyle f^0} \\ \mathrm{Spec}^h(R_{A,G}) & \xrightarrow{\pi^R} & \mathrm{Spec}(A^G) \end{array} \tag{8.28}$$

Proof. The diagram is the right-hand square of (5.9) in the case $H = G$ and $f = u$. The homomorphism u factors into a G-equivariant surjection and a G-equivariant localisation, and it suffices to treat these cases separately.

If B is a G-equivariant localisation of A, then $B = S^{-1}A$ for a multiplicative set $S \subseteq A^G$ by Lemma 8.6, so we have $R_{B,G} = S^{-1}R_{A,G}$ by Lemma 8.11, and all assertions follow. If $B = A/I$ for a G-invariant ideal I, then $A^G/I^G \to B^G$ is a universal homeomorphism by Proposition 8.5. Hence, in the chain of homomorphisms

$$R_{A,G} \otimes_{A^G} A^G/I^G \to R_{A,G} \otimes_{A^G} B^G \to R_{B,G}$$

the first arrow is a universal homeomorphism, while the second arrow is a universal homeomorphism by Theorem 8.26. Therefore, using Lemma 6.1, the upper line of (8.28) can be replaced by

$$\mathrm{Spec}^h(R_{A,G}/I^G R_{A,G}) \to \mathrm{Spec}(A^G/I^G),$$

and again all assertions follow. □

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Corollary 8.29. *For an arbitrary pair (G, A) and $\mathfrak{q} \in \mathrm{Spec}(A^G)$ the map*

$$\psi^R_{\mathfrak{q},\mathrm{res}} : \mathrm{Spec}^h(R_{A(\mathfrak{q}),G}) \to \mathrm{Spec}^h(R_{A,G})_{\mathfrak{q}}$$

in the fibre diagram (5.15) is a homeomorphism.

Proof. This follows from Corollary 8.27 with $B = A(\mathfrak{q})$ because the continuous map $\mathrm{Spec}(A(\mathfrak{q})^G) \to \mathrm{Spec}(A^G)$ can be identified with the inclusion $\{\mathfrak{q}\} \to \mathrm{Spec}(A^G)$ by Lemma 5.13. □

Remark 8.30. If the pair (G, A) is noetherian, to deduce Corollary 8.29 it is sufficient to use Proposition 8.24 instead of Theorem 8.26.

9. Tensor nilpotence

Let $F : \mathcal{K} \to \mathcal{L}$ be a tensor triangulated functor between essentially small tensor triangulated categories. Following [Bal18], we say that F detects tensor nilpotence of morphisms if every morphism $f : X \to Y$ in $\mathcal{K}$ with $F(f) = 0$ satisfies $f^{\otimes n} = 0$ for some $n \geq 1$.

If F detects tensor nilpotence of morphisms and $\mathcal{K}$ is rigid, then the map $\mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ is surjective by [Bal18, Theorem 1.3].

Proposition 9.1. *Let G be a finite group and let $f : A \to A'$ be a G-equivariant homomorphism of commutative rings. The resulting functor*

$$f_* : \mathcal{T}_{A,G} \to \mathcal{T}_{A',G}$$

detects tensor nilpotence of morphisms in the following cases:

(1) *$A' = A/N$ for a G-invariant nilpotent ideal N;*
(2) *$A' = A_b \times A/b$ for an A-regular element $b \in A^G$.*

Proof. We use the equivalence $\mathcal{T}_{A,G} \cong D^b(AG)_{A\text{-proj}}$ given by Proposition 4.4 together with Lemma 3.3. Let $f : X \to Y$ be a morphism in $D^b(AG)_{A\text{-proj}}$ such that $f' : X' \to Y'$ is zero in $D^b(A'G)$, where $X' = X \otimes_A A'$ etc. We have to show that $f^{\otimes n} = 0$ for some $n \geq 1$.

We choose a quasi-isomorphism $u : P \to X$ where P is a bounded-above complex of finite projective AG-modules; see Lemma 3.2. Then f is represented by a homomorphism of complexes $g : P \to Y$ which is unique up to homotopy, namely $g = fu$ in $D(AG)$. The base change $g' : P' \to Y'$ is homotopic to zero because $g' = f'u'$ in $D(A'G)$ where $f' = 0$.

In case (1) we write $g' = dh' + h'd$ for a homomorphism of graded $A'G$-modules $h' : P' \to Y'[-1]$. Since P consists of projective AG-modules, h' lifts to a homomorphism of graded AG-modules $h : P \to Y[-1]$. We can replace g by $g - (dh + hd)$ and thus assume that g' is zero, which means that g factors as $P \to NY \to Y$. Then $g^{\otimes r}$ factors as $P^{\otimes r} \to (NY)^{\otimes r} \to N^r Y^{\otimes r} \to Y^{\otimes r}$. If $N^r = 0$ it follows that $g^{\otimes r} = 0$ as a homomorphism of complexes and thus $f^{\otimes r} = 0$ in $D(AG)$. This refines (1) because the exponent r is explicit (the nilpotence order of N).

In case (2) let $X_b = X \otimes_A A_b$ etc. Since the components of P are finite projective AG-modules we have

$$\begin{aligned}\mathrm{Hom}_{D(A_bG)}(X_b, Y_b) &= \mathrm{Hom}_{K(A_bG)}(P_b, Y_b)\\ &= \mathrm{Hom}_{K(AG)}(P, Y)_b = \mathrm{Hom}_{D(AG)}(X, Y)_b.\end{aligned}$$

The assumption $f' = 0$ in $D(AG)$ implies that $f_b : X_b \to Y_b$ is zero in $D(A_bG)$, and hence $b^r f = 0$ in $D(AG)$ for some $r \geq 1$. The obvious exact triangle $X \xrightarrow{b^r} X \xrightarrow{\pi} X/b^r \to^+$ in $D(AG)$ then shows

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that f factors as

$$X \xrightarrow{\pi} X/b^r \xrightarrow{\tilde{f}} Y.$$

Now there is the following commutative diagram in $D(AG)$ where $(X/b^r)^{\otimes m}$ denotes the iterated tensor product over A/b^r, and similarly for Y, and where $f_r : X/b^r \to Y/b^r$ is the reduction of f.

$$\begin{tikzcd}
X^{\otimes r} \otimes_A X \arrow[r, "f^{\otimes r} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \pi"'] & Y^{\otimes r} \otimes_A X \arrow[r, "\mathrm{id} \otimes f"] \arrow[d, "\mathrm{id} \otimes \pi"'] & Y^{\otimes r} \otimes_A Y \\
X^{\otimes r} \otimes_A X/b^r \arrow[r, "f^{\otimes r} \otimes \mathrm{id}"] \arrow[d, "\simeq"'] & Y^{\otimes r} \otimes_A X/b^r \arrow[ur, "\mathrm{id} \otimes \tilde{f}"'] \arrow[d, "\simeq"'] & \\
(X/b^r)^{\otimes r} \otimes_{A/b^r} X/b^r \arrow[r, "f_r^{\otimes r} \otimes \mathrm{id}"] & (Y/b^r)^{\otimes r} \otimes_{A/b^r} X/b^r &
\end{tikzcd}$$

Indeed, the upper square and triangle are evident, and the lower square follows from the corresponding square in $K(AG)$ with P in place of X and g in place of f. The assumption $f' = 0$ implies that $f_1 : X/b \to Y/b$ is zero in $D((A/b)G)$. By the refined version of (1) applied to the G-equivariant homomorphism $A/b^r \to A/b$ it follows that $f_r^{\otimes r}$ is zero in $D((A/b^r)G)$ and hence in $D(AG)$. It follows that $f^{\otimes(r+1)} = 0$ in $D(AG)$. □

10. Change of coefficients for Spc

As in § 8, we fix a finite group G. For a G-equivariant ring homomorphism $f : A \to B$ we consider the outer square of the extended functoriality diagram (5.9) for $G = H$,

$$\begin{tikzcd}
\mathrm{Spc}(\mathcal{T}_{B,G}) \arrow[r, "\bar{\rho}_{B,G}"] \arrow[d, "f^{\mathcal{T}}"'] & \mathrm{Spec}(B^G) \arrow[d, "f^0"] \\
\mathrm{Spc}(\mathcal{T}_{A,G}) \arrow[r, "\bar{\rho}_{A,G}"] & \mathrm{Spec}(A^G)
\end{tikzcd} \tag{10.1}$$

and the corresponding fibre product of topological spaces,

$$(f^0)^* \, \mathrm{Spc}(\mathcal{T}_{A,G}) = \mathrm{Spc}(\mathcal{T}_{A,G}) \times_{\mathrm{Spec}(A^G)} \mathrm{Spec}(B^G).$$

Diagram (10.1) induces a continuous map,

$$f^{\mathcal{T}}_{\mathrm{res}} : \mathrm{Spc}(\mathcal{T}_{B,G}) \to (f^0)^* \, \mathrm{Spc}(\mathcal{T}_{A,G}), \tag{10.2}$$

which we call the base change map associated to f.

DEFINITION 10.3. We denote by Spc-surj(G) the class of all G-equivariant homomorphisms of commutative rings $f : A \to B$ such that the base change map $f^{\mathcal{T}}_{\mathrm{res}}$ is surjective.

One should compare this with Definition 8.2.

Remark 10.4. For the homomorphism $f = \psi_{\mathfrak{q}} : A \to A(\mathfrak{q})$ of (5.11) the base change map $f^{\mathcal{T}}_{\mathrm{res}}$ can be identified with the map $\psi^{\mathcal{T}}_{\mathfrak{q},\mathrm{res}}$ in the fibre diagram (5.15) because the natural map $\mathrm{Spec}(A(\mathfrak{q})^G) \to \mathrm{Spec}(A^G)$ can be identified with the inclusion $\{\mathfrak{q}\} \to \mathrm{Spec}(A^G)$ by Lemma 5.13.

PROPOSITION 10.5. *The class* Spc-surj(G) *is stable under composition.*

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Proof. If $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence of G-equivariant ring homomorphisms, then $(g \circ f)^{\mathcal{T}}_{\mathrm{res}}$ factors as

$$\mathrm{Spc}(\mathcal{T}_{C,G}) \xrightarrow{g^{\mathcal{T}}_{\mathrm{res}}} (g^0)^* \,\mathrm{Spc}(\mathcal{T}_{B,G}) \xrightarrow{(g^0)^* f^{\mathcal{T}}_{\mathrm{res}}} (g^0)^*(f^0)^* \,\mathrm{Spc}(\mathcal{T}_{A,G}). \qquad \square$$

Proposition 10.6. *The class* $\mathrm{Spc\text{-}surj}(G)$ *contains all* G*-equivariant localisation homomorphisms.*

Proof. Let $f : A \to B$ be a G-equivariant localisation, so $B = S^{-1}A$ for a multiplicative set $S \subseteq A^G$ by Lemma 8.6. By [Bal10, Corollary 3.10], the Verdier localisation of $\mathcal{T}_{A,G}$ at the thick tensor ideal generated by $\mathrm{cone}(s)$ for $s \in S$ is a tensor triangulated category $S^{-1}\mathcal{T}_{A,G}$ with $\mathrm{End}(\mathbb{1}) = S^{-1}A^G = B^G$. Since the functor $f_* : \mathcal{T}_{A,G} \to \mathcal{T}_{B,G}$ maps S to isomorphisms, it factors into tensor triangulated functors

$$\mathcal{T}_{A,G} \xrightarrow{j} S^{-1}\mathcal{T}_{A,G} \xrightarrow{\varphi} \mathcal{T}_{B,G},$$

and accordingly the transpose of diagram (10.1) factors as follows.

$$\begin{tikzcd}
\mathrm{Spc}(\mathcal{T}_{B,G}) \arrow[r, "\mathrm{Spc}(\varphi)"] \arrow[d, "\bar\rho_{B,G}"'] & \mathrm{Spec}(S^{-1}\mathcal{T}_{A,G}) \arrow[r, "\mathrm{Spc}(j)"] \arrow[d, "\bar\rho"] & \mathrm{Spc}(\mathcal{T}_{A,G}) \arrow[d, "\bar\rho_{A,G}"] \\
\mathrm{Spec}(B^G) \arrow[r, "\sim"] & \mathrm{Spec}(S^{-1}A^G) \arrow[r] & \mathrm{Spec}(A^G)
\end{tikzcd} \tag{10.7}$$

Here the right-hand square is cartesian by [Bal10, Theorem 5.4]. The functor φ is fully faithful because homomorphisms in $D^b(AG)_{A\text{-proj}}$ are homomorphisms in $K^-(AG\text{-proj})$ by Lemma 3.2, and these commute with localisation at S by finiteness. Hence, $\mathrm{Spc}(\varphi)$ is surjective by [Bal18, Cor. 1.8]. Together it follows that $f^{\mathcal{T}}_{\mathrm{res}}$ is surjective. $\square$

Proposition 10.8. *The class* $\mathrm{Spc\text{-}surj}(G)$ *contains all surjective* G*-equivariant homomorphisms* $f : A \to B$ *with nilpotent kernel.*

Proof. Let N be the kernel of $A \to B$. The map f^0 in (10.1) factors into $\mathrm{Spec}(B^G) \to \mathrm{Spec}(A^G/N^G) \to \mathrm{Spec}(A^G)$, where both maps are homeomorphisms by Proposition 8.5 and because N^G is a nilpotent ideal. So f^0 is a homeomorphism. The functor $f_* : \mathcal{T}_{A,G} \to \mathcal{T}_{B,G}$ detects tensor nilpotence by Proposition 9.1(1), so the map $f^{\mathcal{T}} = \mathrm{Spc}(f_*)$ is surjective by [Bal18, Theorem 1.1]. This map factors as

$$\mathrm{Spc}(\mathcal{T}_{B,G}) \xrightarrow{f^{\mathcal{T}}_{\mathrm{res}}} (f^0)^* \,\mathrm{Spc}(\mathcal{T}_{A,G}) \xrightarrow{\pi} \mathrm{Spc}(\mathcal{T}_{A,G}) \tag{10.9}$$

where π is a base change of f^0 and thus bijective. Hence $f^{\mathcal{T}}_{\mathrm{res}}$ is surjective. $\square$

Proposition 10.10. *For each* A*-regular element* $b \in A^G$*, the projection homomorphism* $g : A \to A/b$ *lies in the class* $\mathrm{Spc\text{-}surj}(G)$*.*

Proof. Let $B = A_b \times A/b$. We consider the diagram (10.1) for the natural G-equivariant homomorphism $f : A \to B$. Since $\mathcal{T}_{B,G} \cong \mathcal{T}_{A_b,G} \times \mathcal{T}_{A/b,G}$ as tensor triangulated categories and $B^G = (A_b)^G \times (A/b)^G$ as rings, there are compatible disjoint decompositions into open and closed subspaces

$$\mathrm{Spc}(\mathcal{T}_{B,G}) \cong \mathrm{Spc}(\mathcal{T}_{A_b,G}) \sqcup \mathrm{Spc}(\mathcal{T}_{A/b,G}), \tag{10.11}$$

$$\mathrm{Spec}(B^G) \cong \mathrm{Spec}((A_b)^G) \sqcup \mathrm{Spec}((A/b)^G), \tag{10.12}$$

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using Lemma 2.11 for (10.11). There is a similar disjoint decomposition into an open and a closed subspace

$$\operatorname{Spec}(A^G) \cong \operatorname{Spec}((A^G)_b) \sqcup \operatorname{Spec}(A^G/b). \tag{10.13}$$

Here $(A^G)_b = (A_b)^G$ and $A^G/b = A^G/(bA)^G$ since b is A-regular. Hence, the decompositions (10.12) and (10.13) together with Proposition 8.5 applied to $A \to A/b$ show that the map $f^0 : \operatorname{Spec}(B^G) \to \operatorname{Spec}(A^G)$ is bijective.

Now the situation is similar to Proposition 10.8. The functor $f_* : \mathcal{T}_{A,G} \to \mathcal{T}_{B,G}$ detects tensor nilpotence of morphisms by Proposition 9.1(2), so the map $f^{\mathcal{T}} = \operatorname{Spc}(f_*)$ is surjective by [Bal18, Theorem 1.1]. Again this map factors as (10.9) where π is bijective since f^0 is bijective, so $f^{\mathcal{T}}_{\mathrm{res}}$ is surjective. The decompositions (10.11) and (10.12) yield that

$$g^{\mathcal{T}}_{\mathrm{res}} : \operatorname{Spc}(\mathcal{T}_{A/b,G}) \to (g^0)^* \operatorname{Spc}(\mathcal{T}_{A,G})$$

is a retract of $f^{\mathcal{T}}_{\mathrm{res}}$, so $g^{\mathcal{T}}_{\mathrm{res}}$ is surjective as well. $\square$

Proposition 10.14. *Assume that the ring A is noetherian and the ring A^G is local with maximal ideal $\mathfrak{q}$. If the homomorphism $\psi_{\mathfrak{q}} : A \to A(\mathfrak{q})$ of (5.11) is not an isomorphism, then there is a non-zero G-invariant ideal I of A with $I \subseteq \operatorname{Ker}(\psi_{\mathfrak{q}})$ such that the projection $A \to A/I$ lies in the class $\operatorname{Spc-surj}(G)$ of Definition 10.3.*

Proof. This is parallel to Proposition 8.23, using Propositions 10.8 and 10.10 instead of Propositions 8.14 and 8.19. $\square$

Proposition 10.15. *If the ring A is noetherian, then for each $\mathfrak{q} \in \operatorname{Spec}(A^G)$ the homomorphism $\psi_{\mathfrak{q}} : A \to A(\mathfrak{q})$ of (5.11) lies in the class $\operatorname{Spc-surj}(G)$; in other words, the map $\psi^{\mathcal{T}}_{\mathfrak{q},\mathrm{res}}$ in the fibre diagram (5.15) is surjective.*

Proof. This is parallel to Proposition 8.24, using Propositions 10.5, 10.6, and 10.14 instead of Propositions 8.4, 8.13, and 8.23. See Remark 10.4 for the assertion 'in other words'. $\square$

11. The comparison map: conclusion

Theorem 11.1. *For every pair (G, A) the map*

$$\rho_{A,G} : \operatorname{Spc}(\mathcal{T}_{A,G}) \to \operatorname{Spec}^h(R_{A,G})$$

of (5.1) is a homeomorphism.

Proof. We recall that $\mathcal{T}_{A,G}$ is equivalent to $D^b(AG)_{A\text{-perf}}$ by Proposition 4.4. Since A is the filtered colimit of its finitely generated G-invariant subrings, by Lemmas 3.11 and 2.12 we can assume that A is of finite type; in particular, the pair (G, A) is noetherian in the sense of Definition 3.12 by Corollary 3.14. Then the rigid tensor category $\mathcal{T}_{A,G}$ is End-finite by Proposition 3.15, so by Corollary 2.8 the map $\rho_{A,G}$ is a homeomorphism if and only if it is bijective. By Lemma 5.7 this holds if and only if for each $\mathfrak{q} \in \operatorname{Spec}(A^G)$ the map $(\rho_{A,G})_{\mathfrak{q}}$ of (5.6) is bijective. This map is the lower arrow in the fibre diagram (5.15), in which the upper arrow $\rho_{A(\mathfrak{q}),G}$ is a homeomorphism by Lemma 5.16 and Theorem 7.1, the right arrow $\psi^{R}_{\mathfrak{q},\mathrm{res}}$ is a homeomorphism by Corollary 8.29, and the left arrow $\psi^{\mathcal{T}}_{\mathfrak{q},\mathrm{res}}$ is surjective by Proposition 10.15. It follows that $(\rho_{A,G})_{\mathfrak{q}}$ is bijective as required. $\square$

Remark 11.2. In the first version of this paper, Theorem 11.1 was proved only when the ring A is regular by a similar route, including also results on the functoriality of cohomological support. In view of Proposition 2.10 these results are now obsolete.

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12. A stable variant

Let us record a variant of Theorem 11.1 with Proj in place of Spec^h, which is a rather formal consequence. This was first observed in [Bar21, § 3.4] when A is regular with trivial action of G. We begin with an elementary remark.

Remark 12.1. The restriction of scalars $\mathrm{res} : AG\text{-Mod} \to A\text{-Mod}$ has a left adjoint $\mathrm{ind} : A\text{-Mod} \to AG\text{-Mod}$ defined by $\mathrm{ind}(M) = AG \otimes_A M$, using the right A-module structure of AG for the tensor product. Let $P_0 = AG$ as a left AG-module. For an AG-module Q there is a natural isomorphism

$$\mathrm{ind}\,\mathrm{res}\, Q \cong P_0 \otimes_A Q \tag{12.2}$$

where the tensor product is formed as in (3.1), that is, G acts diagonally. Indeed, the A-linear map $\mathrm{res}\, Q \to P_0 \otimes_A Q$, $x \mapsto 1 \otimes x$ gives by adjunction an AG-linear map $\mathrm{ind}\,\mathrm{res}\, Q \to P_0 \otimes_A Q$, which is an isomorphism.

Lemma 12.3. *The category* $\mathrm{Perf}(AG)$ *of perfect complexes of AG-modules is a thick tensor ideal in* $D^b(AG)_{A\text{-perf}}$.

Proof. Clearly $\mathrm{Perf}(AG)$ is a thick subcategory of $D^b(AG)$. By Lemma 3.3 it suffices to show that for a finite AG-modules P, Q where P is projective and Q is A-projective, $P \otimes_A Q$ is AG-projective. We can assume that $P = AG$. Then $P \otimes_A Q \cong \mathrm{ind}\,\mathrm{res}\, Q$ by (12.2), which is AG-projective. □

Using Proposition 4.4, we identify $\mathcal{T}_{A,G}$ and $D^b(AG)_{A\text{-perf}}$. Then $\mathrm{Perf}(AG)$ is a tensor ideal in $\mathcal{T}_{A,G}$ by Lemma 12.3, so the Verdier quotient

$$\mathcal{S}_{A,G} = \mathcal{T}_{A,G}/\,\mathrm{Perf}(AG)$$

is a tensor triangulated category.

Corollary 12.4. *For every pair (G, A) the homeomorphism $\rho_{A,G}$ of Theorem 11.1 restricts to a homeomorphism*

$$\bar{\rho}_{A,G} : \mathrm{Spc}(\mathcal{S}_{A,G}) \cong \mathrm{Proj}(R_{A,G}).$$

Proof. Compare [Bar21, Corollary 3.32]. The ring homomorphism $q : R_{A,G} \to A^G$ defined by projection to degree zero gives a closed immersion of topological spaces $\mathrm{Spec}(A^G) \to \mathrm{Spec}^h(R_{A,G})$, whose complement is $\mathrm{Proj}(R_{A,G})$; moreover, the natural functor $\mathcal{T}_{A,G} \to \mathcal{S}_{A,G}$ induces a homeomorphism

$$\mathrm{Spc}(\mathcal{S}_{A,G}) \cong \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}_{A,G}) \mid \mathrm{Perf}(AG) \subseteq \mathcal{P}\}$$

by [Bal05, Proposition 3.11]. Hence, it suffices to show that $\rho_{A,G}$ induces by restriction a bijective map $\bar{\rho}_{A,G}$ as indicated. But some $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}_{A,G})$ satisfies $\mathrm{Perf}(AG) \subseteq \mathcal{P}$ if and only if $AG \in \mathcal{P}$ if and only if $\mathcal{P} \notin \mathrm{supp}(AG)$ if and only if $\rho(\mathcal{P}) \notin V(AG)$ by Proposition 2.10. Now $\mathrm{End}^*(AG) = AG$ in degree zero, which is an $R_{A,G}$-module via q, and it follows that $V(AG) = \mathrm{Spec}(A^G)$. Hence, $\rho(\mathcal{P}) \notin V(AG)$ if and only if $\rho(\mathcal{P}) \in \mathrm{Proj}(R_{A,G})$. □

Remark 12.5. The Verdier quotient $\mathcal{S}_{A,G}$ can be viewed as the stable category associated to the Frobenius category $\mathrm{lat}(A, G)$ of A-projective finite AG-modules (here lat is short for lattice). More precisely, $\mathrm{lat}(A, G)$ is an exact subcategory of the abelian category AG-Mod, and with this exact structure $\mathrm{lat}(A, G)$ is a Frobenius category where the projective objects are the finite projective AG-modules; note that exact sequences in $\mathrm{lat}(A, G)$ are automatically A-split, and $\mathrm{lat}(A, G)$ has a duality involution defined by $M^\vee = \mathrm{Hom}_A(M, A)$ with G-action by conjugation.

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The stable category $\underline{\mathrm{lat}}(A,G)$ is a triangulated category by [Hap88, Theorem 2.6], and it is a tensor triangulated category since the projective objects of $\mathrm{lat}(A,G)$ form a tensor ideal. By an obvious extension of [Ric89, Theorem 2.1] there is a tensor triangulated equivalence

$$\underline{\mathrm{lat}}(A,G) \cong \mathcal{S}_{A,G}. \tag{12.6}$$

In more detail, the composition $\mathrm{lat}(A,G) \to D^b(AG)_{A\text{-perf}} \to \mathcal{S}_{A,G}$ induces the functor (12.6), and an inverse functor is given by stabilised syzygies as follows. The category $D^b(AG)_{A\text{-perf}}$ is equivalent to the homotopy category $\mathcal{K} = K^-(AG\text{-proj})_{A\text{-perf}}$ of upper bounded complexes of finite projective AG-modules which are A-perfect. Let T denote the suspension functor of $\underline{\mathrm{lat}}(AG)$. Each $X \in \mathcal{K}$ has bounded cohomology. If X has trivial cohomology in degree at most n, then the AG-module $Z^n(X) = \ker(d : X^n \to X^{n+1})$ is A-projective and the object $T^{-n}Z^n(X)$ of $\underline{\mathrm{lat}}(A,G)$ is independent of n. This construction gives a functor $\mathcal{K} \to \underline{\mathrm{lat}}(A,G)$, which induces an inverse of (12.6) as is easily verified. If G acts trivially on A, the equivalence (12.6) is also proved in [Bar21, Proposition 3.26] using homotopy theoretic arguments.

Remark 12.7. If G acts trivially on A, a different stable category $\mathrm{stmod}(AG)$ is considered in [BIK13]. The category of all AG-modules with the exact structure given by the A-split exact sequences is a Frobenius category, the associated stable category is denoted by $\mathrm{StMod}(AG)$, and $\mathrm{stmod}(AG)$ is the full subcategory of $\mathrm{StMod}(AG)$ whose objects are the finitely presented AG-modules. There is an obvious functor $\mathcal{S}_{A,G} \cong \underline{\mathrm{lat}}(A,G) \to \mathrm{stmod}(AG)$, but these categories behave quite differently with respect to tensor ideals. For example, for $A = \mathbb{Z}$ and a prime p dividing the order of G, by [BIK13, § 7] there is an infinite ascending sequence $D_1 \subsetneq D_2 \subsetneq \cdots$ of radical tensor ideals in $\mathrm{stmod}(AG)$, where D_n consists of all finite AG-modules isomorphic to modules annihilated by p^n. This sequence is not visible in $\underline{\mathrm{lat}}(A,G)$ because the inverse image of D_n in $\underline{\mathrm{lat}}(A,G)$ is zero for all n.

Remark 12.8. We also note the following geometric description of $\mathcal{S}_{A,G}$. The natural morphism $h : \mathrm{Spec}\, A \to [\mathrm{Spec}(A)/G]$ gives a direct image functor $h_* : \mathcal{T}_A \to \mathcal{T}_{A,G}$, which corresponds to the functor

$$\mathrm{Perf}(A) \to D^b(AG), \quad Q \mapsto AG \otimes_A Q.$$

It follows that $\mathrm{Perf}(AG)$ coincides with $\langle h_*\mathcal{T}_A\rangle$, the thick subcategory of $\mathcal{T}_{A,G}$ generated by the image of h_*, and hence $\mathcal{S}_{A,G} \cong \mathcal{T}_{A,G}/\langle h_*\mathcal{T}_A\rangle$.

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