

C^r -chain closing lemma for certain partially hyperbolic diffeomorphisms

YI SHI[†] and XIAODONG WANG[‡]

[†] *School of Mathematics, Sichuan University, Chengdu 610065, P. R. China*
(e-mail: shiyi@scu.edu.cn)

[‡] *School of Mathematical Sciences, CMA-Shanghai, Shanghai Jiao Tong University, Shanghai 200240, P. R. China*
(e-mail: xdwang1987@sjtu.edu.cn)

(Received 4 November 2022 and accepted in revised form 1 September 2023)

Abstract. For every $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, we prove a C^r -orbit connecting lemma for dynamically coherent and plaque expansive partially hyperbolic diffeomorphisms with one-dimensional orientation preserving center bundle. To be precise, for such a diffeomorphism f , if a point y is chain attainable from x through pseudo-orbits, then for any neighborhood U of x and any neighborhood V of y , there exist true orbits from U to V by arbitrarily C^r -small perturbations. As a consequence, we prove that for C^r -generic diffeomorphisms in this class, periodic points are dense in the chain recurrent set, and chain transitivity implies transitivity.

Key words: partially hyperbolic diffeomorphism, chain closing lemma, periodic orbit, dynamical coherence, chain recurrent set

2020 Mathematics Subject Classification: 37D30 (Primary); 37C20, 37C50 (Secondary)

1. Introduction

Following from Yoccoz [29], the goal of the theory of dynamical systems is to understand most of the dynamics of most systems. Here ‘most systems’ means dense, generic, open and dense dynamical systems. The various perturbation techniques have played crucial roles in the study of most systems, for instance in the exploration of the famous C^1 -stability conjectures. Pugh’s celebrated work: the C^1 -closing lemma [27] which realizes closing a non-wandering orbit to get periodic orbits, initiated the theory of C^1 -perturbations. Mañé’s ergodic closing lemma [22] provides more information on the closing points. Another work by Mañé [23] creates homoclinic points by C^r , $r = 1, 2$ perturbations. These two works [22, 23] are indispensable in solving the C^1 -stability conjecture [22, 24] and the ergodic closing lemma is a milestone in both differentiable dynamical systems and ergodic theory. An improved version of the C^1 -closing lemma is obtained by Hayashi [18] that connects two orbits which visit a same small region. Hayashi’s work is now called the C^1 -connecting lemma, see also works by Wen and Xia [28]. More developments can be found for instance

in [2, 13]. The strongest C^1 -perturbation lemma is the C^1 -chain connecting lemma proved by Bonatti and Crovisier [4], which connects pseudo orbits under small C^1 -perturbations. In particular, works in [4, 8] realize closing a chain recurrent orbit (or a chain recurrent set).

We recall some notions and definitions now. Let M be a C^∞ closed Riemmanian manifold and denote by $\text{Diff}^r(M)$ for any $r \in \mathbb{N} \cup \{\infty\}$ the space of C^r -diffeomorphisms on M endowed with the C^r -topology.

Given $f \in \text{Diff}^r(M)$ and $\delta > 0$, a collection of points

$$\{x_a, x_{a+1}, \dots, x_b\} \quad \text{where } -\infty \leq a < b \leq +\infty$$

is called a δ -pseudo orbit if it holds

$$d(x_{i+1}, f(x_i)) < \delta \quad \text{for every } a \leq i \leq b - 1.$$

Given two points $x, y \in M$, one says that y is *attainable* from x under f , denoted by $x \prec_f y$, if for any neighborhood U of x and any neighborhood V of y , there exist $z \in U$ and $n \geq 1$ such that $f^n(z) \in V$; and y is *chain attainable* from x under f , denoted by $x \dashv_f y$, if for any $\delta > 0$, there exists a δ -pseudo orbit $\{x_0, x_1, \dots, x_n\}$ from x to y , that is, $x_0 = x$ and $x_n = y$. A point $x \in M$ is *non-wandering* if $x \prec_f x$, and the set of non-wandering points of f is denoted by $\Omega(f)$. A point $x \in M$ is *chain recurrent* if $x \dashv_f x$, and the set of chain recurrent points of f is denoted by $\text{CR}(f)$. The set of periodic points is denoted by $\text{Per}(f)$.

Note that compared with non-wandering/chain recurrent points, periodic points have the strongest recurrence property. One would like to ask whether or not the weaker recurrence can be perturbed into stronger ones. Given $f \in \text{Diff}^r(M)$, a point $p \in M$ is called C^r -closable if for any C^r -neighborhood \mathcal{U} of f , there exists $g \in \mathcal{U}$ such that $p \in \text{Per}(g)$. For uniformly hyperbolic systems, Anosov’s shadowing lemma [1] implies that every chain recurrent point is C^r -closable (in fact without perturbation). For a general diffeomorphism f , as has been mentioned above, Pugh’s closing lemma [27] verifies that every $x \in \Omega(f)$ is C^1 -closable and Bonatti–Crovisier’s chain closing lemma [4] verifies every $x \in \text{CR}(f)$ is C^1 -closable.

For dynamics beyond uniform hyperbolicity and when $r \geq 2$, things turn out to be much more complicated and delicate. Gutierrez [15] built an example which showed that the local perturbation method for proving the C^1 -closing lemma does not work in the C^2 -topology. Mañé [23] proved a C^2 -connecting lemma which creates homoclinic points for hyperbolic periodic points with some assumption in the measure sense. Recently, a series progress has been achieved in the C^r -closing lemma for Hamiltonian and conservative surface diffeomorphisms, see [3, 7, 10].

In higher dimensions, Gan and the first author [12] proved the C^r -closing lemma for partially hyperbolic diffeomorphisms with one-dimensional center bundle for every $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Inspired by [12], we explore C^r -perturbation results for a certain class of partially hyperbolic diffeomorphisms.

We say that a diffeomorphism $f \in \text{Diff}^r(M)$ is *partially hyperbolic* if the tangent space TM admits a Df -invariant continuous splitting $TM = E^s \oplus E^c \oplus E^u$ and there exists $k \geq 1$ such that for any $x \in M$, it holds

$$\|Df^k|_{E_x^s}\| < \min\{1, m(Df^k|_{E_x^c})\} \leq \max\{1, \|Df^k|_{E_x^c}\|\} < m(Df^k|_{E_x^u}).$$

Here for a linear operator A , one denotes by $\|A\|$ its norm and by $m(A)$ its conorm, that is, $m(A) = \inf\{\|Av\| : \|v\| = 1\}$.

Let $\text{PH}^r(M)$, $r \geq 1$ be the space consisting of all C^r -partially hyperbolic diffeomorphisms of M endowed with the C^r -topology, then $\text{PH}^r(M) \subset \text{Diff}^r(M)$ is an open subset. We say $f \in \text{PH}^r(M)$ is *dynamically coherent* if there exist two f -invariant foliations \mathcal{F}^{cs} and \mathcal{F}^{cu} that are tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$, respectively. It is clear that $\mathcal{F}^c = \mathcal{F}^{cs} \cap \mathcal{F}^{cu}$ is an f -invariant foliation tangent to E^c . For a dynamically coherent partially hyperbolic diffeomorphism f , one always denotes the partially hyperbolic splitting by $TM = E_f^s \oplus E_f^c \oplus E_f^u$ or $TM = E^s \oplus E^c \oplus E^u$ for simplicity when there is no confusion. Also, one denotes the stable/center/unstable foliations by $\mathcal{F}^s/\mathcal{F}^c/\mathcal{F}^u$.

For $f \in \text{PH}^r(M)$ being dynamically coherent and a constant $\varepsilon > 0$, an ε -pseudo orbit $\{x_i\}_{i=a}^b$ is called an ε -center pseudo orbit if $f(x_i) \in \mathcal{F}_\varepsilon^c(x_{i+1})$ for any $a \leq i \leq b - 1$. We say f is ε -plaque expansive if any two bi-infinite ε -center pseudo orbits $\{x_i\}_{i \in \mathbb{Z}}, \{y_i\}_{i \in \mathbb{Z}}$ satisfying

$$d(x_i, y_i) < \varepsilon \quad \text{for every } i \in \mathbb{Z},$$

must satisfy $y_i \in \mathcal{F}_\varepsilon^c(x_i)$ for every $i \in \mathbb{Z}$. Here $\mathcal{F}_\varepsilon^c(x)$ is the disk centered at x in $\mathcal{F}^c(x)$ with radius ε . We say f is *plaque expansive* if it is ε -plaque expansive for some $\varepsilon > 0$.

Notation 1.1. For $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, we denote by $\text{DPH}_1^r(M)$ (respectively $\text{DEPH}_1^r(M)$) the set of all partially hyperbolic diffeomorphisms $f \in \text{PH}^r(M)$ that satisfies properties (a) and (b) (respectively (a), (b), and (c)):

- (a) the center bundle E^c is orientable with $\dim E^c = 1$, and Df preserves the orientation;
- (b) f is dynamically coherent;
- (c) f is plaque expansive.

It has been proved [19, Theorems 7.1 and 7.4] that dynamic coherence plus plaque expansiveness is a robust property for C^1 -partially hyperbolic diffeomorphisms. Thus, $\text{DEPH}_1^r(M)$ forms an open set in $\text{Diff}^r(M)$ for every $r \geq 1$. There are a series of partially hyperbolic diffeomorphisms belonging to $\text{DEPH}_1^r(M)$:

- (1) derived-from-Anosov partially hyperbolic diffeomorphisms on 3-torus \mathbb{T}^3 [16, 26], where Df preserves the center orientation;
- (2) partially hyperbolic diffeomorphisms with circle center fibers on 3-torus \mathbb{T}^3 [16, 26] and 3-nilmanifolds N [17], where Df preserves the center orientation;
- (3) discretized Anosov flows [25], i.e., partially hyperbolic diffeomorphisms which are leaf conjugate to Anosov flows.

In particular, every dynamically coherent and plaque expansive partially hyperbolic diffeomorphism with one-dimensional center admits a double cover with E^c orientable and an iteration of the lifting diffeomorphism belongs to $\text{DEPH}_1^r(M)$.

Remark 1.2. It has been conjectured [19] that every dynamically coherent partially hyperbolic diffeomorphism must be plaque expansive. Actually, all known examples of dynamically coherent partially hyperbolic diffeomorphisms are plaque expansive.

In this paper, we prove that for every $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ and every $f \in \text{DPH}_1^r(M)$, if a point y is chain attainable from x , then any small neighborhoods of x and y can be

connected through true orbits by arbitrarily small C^r -perturbations of f . Let $\mathcal{X}^r(M)$ be the space of C^r -vector fields on M for $r \in \mathbb{N} \cup \{\infty\}$. Given $X \in \mathcal{X}^r(M)$ and $\tau \in \mathbb{R}$, denote by X_τ the time- τ map generated by X . We say X is transverse to a co-dimension one bundle $E^s \oplus E^u$ if $X(x)$ is transverse to $E^s_x \oplus E^u_x$ in $T_x M$ for every $x \in M$.

THEOREM A. *For any $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, assume $f \in \text{DPH}_1^r(M)$ and $X \in \mathcal{X}^r(M)$ is a vector field that is transverse to $E^s \oplus E^u$. For any two points $x, y \in M$ with $x \not\prec_f y$, there exist $p_k \in M$, $\tau_k \in \mathbb{R}$, and $n_k \in \mathbb{N}$ such that:*

- $\tau_k \rightarrow 0$ as $k \rightarrow \infty$;
- $p_k \rightarrow x$ and $(X_{\tau_k} \circ f)^{n_k}(p_k) \rightarrow y$ as $k \rightarrow \infty$.

In particular, if $y = x$, then $(X_{\tau_k} \circ f)^{n_k}(p_k) = p_k$. This means every chain recurrent point of $f \in \text{DPH}_1^r(M)$ is C^r -closable.

Remark 1.3. When $r = 1$, this orbit connecting lemma was proved by Bonatti and Crovisier in [4] for every $f \in \text{Diff}^1(M)$. See [8, 9] for more developments and applications of chain connecting lemmas in C^1 -topology.

Recall that $f \in \text{Diff}^1(M)$ is *transitive* if for any two open sets $U, V \subset M$, there exists $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$. This is equivalent to $x \prec_f y$ for every $x, y \in M$. Similarly, f is *chain transitive* if $x \not\prec_f y$ for every pair of points $x, y \in M$. It is clear that transitivity implies chain transitivity. Concerning that $\text{DEPH}_1^r(M)$ forms an open set in $\text{Diff}^r(M)$ for every $r \geq 1$, we could study generic properties for $\text{DEPH}_1^r(M)$. We show that for C^r -generic $f \in \text{DEPH}_1^r(M)$, transitivity and chain transitivity are equivalent. Recall that a property is *generic* if it holds for every element in a dense G_δ subset.

COROLLARY B. *For every $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, there exists a dense G_δ subset $\mathcal{R} \subset \text{DEPH}_1^r(M)$ such that for any $f \in \mathcal{R}$, the following properties hold.*

- (1) $x \not\prec_f y$ if and only if $x \prec_f y$.
- (2) *Periodic points are dense in the chain recurrent set:* $\text{CR}(f) = \Omega(f) = \overline{\text{Per}(f)}$.
- (3) *If f is chain transitive on M , then f is transitive on M .*

Remark 1.4. Transitive dynamics that are beyond uniform hyperbolicity have attracted much interest from dynamists, for instance [5, 6]. There exist chain transitive partially hyperbolic diffeomorphisms with one-dimensional center bundle which are accessible but not transitive [11]. It is worthy to see whether Corollary B holds for C^r -generic partially hyperbolic diffeomorphisms with one-dimensional center bundle.

2. Lifting dynamics and leaf conjugacy

As have been mentioned, our proof is inspired by [12]. This section is devoted to state the necessary preliminaries, some of which are borrowed from [12]. We first give an outline.

- First, we give a Lipschitz center shadowing property for dynamically coherent partially hyperbolic diffeomorphisms proved in [21] which insures that one can always consider center pseudo orbits in our setting.
- Second, we parameterize each center leaf through the real line \mathbb{R} and associate it with an order from \mathbb{R} by the fact that $\dim(E^c) = 1$. Then one could lift the center

dynamics to $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ where each $\mathbb{R}_i = \mathbb{R}$. Moreover, we prove that by replacing some intermediate points of a center pseudo orbit if necessary, one could always assume the center pseudo orbit jumps in an ordered way. This will be done in §2.2.

- Third, we extend the lifting dynamics on $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ to a tubular neighborhood by pulling back a C^∞ -bundle $F^s \oplus F^u$ which is C^0 -close to $E^s \oplus E^u$. Theorem 2.10 guarantees that the center curves for the dynamics in the tubular neighborhood admit a Lipschitz shadowing property under perturbations. This appears in §§2.3 and 2.4.
- Finally, based on the preparations in this section, we will prove a push-forward (or push-backward) perturbation result in §3 which allows us to prove Theorem A by more delicate estimates.

One first recalls the following proposition from [21, Theorem 1] which states a Lipschitz center shadowing property for dynamically coherent partially hyperbolic diffeomorphisms. Roughly speaking, for such a system, any pseudo orbit with small jumps would be shadowed by a center pseudo orbit. See also similar results in [20].

PROPOSITION 2.1. [21] *Assume $f \in \text{PH}^1(M)$ is dynamically coherent. Then there exist $\varepsilon_0 > 0$ and $L > 1$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any ε -pseudo orbit $\{w_a, w_{a+1}, \dots, w_b\}$, $-\infty \leq a < b \leq +\infty$, there exists an $L\varepsilon$ -center pseudo orbit $\{x_a, x_{a+1}, \dots, x_b\}$ satisfying*

$$d(x_i, w_i) < L\varepsilon \quad \text{for any } a \leq i \leq b.$$

Moreover, if $w_a = w_b$ where $-\infty < a < b < +\infty$, then the new center pseudo orbit can be chosen to satisfy $x_a = x_b$.

Proof of the ‘moreover’ part. The statement of Proposition 2.1 before the ‘moreover’ part is [21, Theorem 1]. Thus, we only need to prove the ‘moreover’ part. We first revisit the steps in the proof of [21, Theorem 1]. Note that one could always consider a bi-infinite pseudo orbit $\{w_i\}_{i \in \mathbb{Z}}$ because any finite or one-sided-infinite pseudo orbit can be extended naturally to a bi-infinite one (see for instance §2.2 below).

(1) Fix two constants $\delta_0 > 0$, $L_0 > 1$ such that for any $0 < \delta \leq \delta_0$ and any two points $x, y \in M$ satisfying $d(x, y) < \delta$, then $\mathcal{F}_x^{cu}(L_0\delta)$ intersects $\mathcal{F}_y^s(L_0\delta)$ at a unique point and similarly for $\mathcal{F}_x^{cs}(L_0\delta)$ and $\mathcal{F}_y^u(L_0\delta)$. This local product property is guaranteed by the transversality and continuation of local foliations.

(2) By considering an iterate of f [21, Lemma 1] (or by taking an adapted metric as in §2) and by decreasing δ_0 if necessary, one has

$$d^s(f(x), f(y)) < \lambda d^s(x, y) \quad \text{for all } y \in \mathcal{F}_x^s(\delta_0) \quad \text{for all } x \in M,$$

$$d^u(f^{-1}(x), f^{-1}(y)) < \lambda d^s(x, y) \quad \text{for all } y \in \mathcal{F}_x^u(\delta_0) \quad \text{for all } x \in M,$$

where $d^{s/u}$ denotes the metric on the stable/unstable foliations and $\lambda \in (0, 1)$ is a constant depending only on the contracting/expanding ratio of Df along E^s/E^u .

(3) Then there exist two constants $\varepsilon_0 \in (0, \delta_0)$ and $L_1 > L_0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any bi-infinite ε -pseudo orbit $\{w_i\}_{i \in \mathbb{Z}}$, the map $h_i^s: \mathcal{F}_{w_i}^s(L_1\varepsilon) \rightarrow \mathcal{F}_{w_{i+1}}^s(L_1\varepsilon)$ is well defined where for every $z \in \mathcal{F}_{w_i}^s(L_1\varepsilon)$, the image $h_i^s(z)$ is the unique point of $\mathcal{F}_{f(z)}^{cu}(L_0\delta_0) \cap \mathcal{F}_{w_{i+1}}^s(L_1\varepsilon)$. This corresponds to [21, Lemma 2].

(4) Consider the set $X^s = \prod_{i=-\infty}^{+\infty} \mathcal{F}_{w_i}^s(L_1\varepsilon)$ endowed with the Tikhonov product topology and consider the map $H: X^s \rightarrow X^s$ with (a bit different to [21] as they are working on the tangent bundle while here we consider local leaves on the manifold directly. These two ways are equivalent through the exponential map)

$$H(\{z_i\}) = \{z'_{i+1}\} \quad \text{where } z'_{i+1} = h_i^s(z_i).$$

Applying the Tikhonov–Schauder fixed point theorem, the map H has a (maybe not unique) fixed point. This induces an orbit of center unstable leaves $\{\mathcal{F}_{y_i^s}^{cu}\}_{i \in \mathbb{Z}}$ (that is, $f(\mathcal{F}_{y_i^s}^{cu}) = \mathcal{F}_{y_{i+1}^s}$ for any $i \in \mathbb{Z}$) where $y_{i+1}^s \in \mathcal{F}_{f(y_i^s)}^{cu}(L_1\varepsilon)$ and which satisfies that y_i^s is close to w_i for all $i \in \mathbb{Z}$. Symmetrically, one obtains an orbit of center stable leaves $\{\mathcal{F}_{y_i^u}^{cs}\}_{i \in \mathbb{Z}}$ which satisfies that y_i^u is close to w_i for all $i \in \mathbb{Z}$.

(5) The intersection $\{\mathcal{F}_{y_i^s}^{cs} \cap \mathcal{F}_{y_i^u}^{cu}\}_{i \in \mathbb{Z}}$ forms an orbit of center leaves $\{\mathcal{F}_{y_i^c}^c\}_{i \in \mathbb{Z}}$ with y_i being close to w_i for all $i \in \mathbb{Z}$. Finally, following arguments in [21, pp. 2907–2908], there exists a center pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ that shadows $\{w_i\}_{i \in \mathbb{Z}}$. In this step, the constant ε_0 may be decreasing once again and L_1 is replaced by a larger L .

The ‘moreover’ part is almost included in the proof of [21, Theorem 1]. We give an explanation here. To simplify symbols, we let $0 = a < b = n < +\infty$ and consider a pseudo orbit $\{w_0, w_1, \dots, w_n\}$ with $w_n = w_0$. One first repeats it forward and backward to get a bi-infinite periodic pseudo orbit, that is, let $w_i = w_{i \bmod n}$ for $i \in \mathbb{Z}$. Then Steps (1)–(3) still work. We only need to verify that in Step (4), one could choose a fixed point of H which induces a periodic orbit of center unstable leaves $\{\mathcal{F}_{y_i^s}^{cu}\}_{i \in \mathbb{Z}}$, that is, $y_i^s = y_{i \bmod n}^s$. To do this, we consider the finite composition (rather than considering H)

$$H_n^s = h_{n-1}^s \dots h_1^s \circ h_0^s: \mathcal{F}_{w_0}^s(L_1\varepsilon) \rightarrow \mathcal{F}_{w_n}^s(L_1\varepsilon) = \mathcal{F}_{w_0}^s(L_1\varepsilon),$$

which is well defined by Step (3). Note that each h_i^s is a composition of a local cu -holonomy with f , thus, H_n^s is continuous. However, H_n^s maps $\mathcal{F}_{w_0}^s(L_1\varepsilon)$ into itself where $\mathcal{F}_{w_0}^s(L_1\varepsilon)$ is an s -dimensional disk. Applying the Tikhonov–Schauder fixed point theorem, the map H_n^s has a (maybe not unique) fixed point $y_0^s \in \mathcal{F}_{w_0}^s(L_1\varepsilon)$. This induces an orbit segment of center unstable leaves $\{\mathcal{F}_{y_i^s}^{cu}\}_{i \in [0, n]}$ with $f(\mathcal{F}_{y_i^s}^{cu}) = \mathcal{F}_{y_{i+1}^s}^{cu}$ for $0 \leq i \leq n - 1$ and $y_0^s = y_n^s$. Moreover, one also has that y_i^s is close to w_i for all $i \in [0, n]$ following arguments in [21, pp. 2907–2908]. Repeating $\{\mathcal{F}_{y_i^s}^{cu}\}_{i \in [0, n]}$ forward and backward, that is, let $y_i^s = y_{i \bmod n}^s$ for $i \in \mathbb{Z}$, one gets a periodic orbit of center unstable leaves $\{\mathcal{F}_{y_i^s}^{cu}\}_{i \in \mathbb{Z}}$. Symmetrically, we can get a periodic orbit of center stable leaves $\{\mathcal{F}_{y_i^u}^{cs}\}_{i \in \mathbb{Z}}$. As in Step (5), the intersection $\{\mathcal{F}_{y_i^s}^{cs} \cap \mathcal{F}_{y_i^u}^{cu}\}_{i \in \mathbb{Z}}$ forms a periodic orbit of center leaves $\{\mathcal{F}_{y_i^c}^c\}_{i \in \mathbb{Z}}$ with y_i being close to w_i for all $i \in [0, n]$ and $y_i = y_{i \bmod n}$. We take x_i to be the unique intersection of the local leaf of $\mathcal{F}_{y_i^s}^{cu}$ with the local leaf of $\mathcal{F}_{y_i^u}^{cs}$. Here, $\{x_i\}_{i=0}^n$ is the periodic $L\varepsilon$ -center pseudo orbit we need, see [21, p. 2908] for the estimation of the constant L . (This is the first line of [21, p. 2908] where they use the notation ‘ y_k ’ while here we use ‘ x_i ’.) \square

Remark 2.2. As a consequence of Proposition 2.1, for a dynamically coherent $f \in \text{PH}^1(M)$, for any $x \in \text{CR}(f)$ and any $\varepsilon > 0$, there exists a periodic ε -center pseudo orbit

$$\{x_0, x_1, \dots, x_n = x_0\} \quad \text{satisfying } d(x_0, x) < \varepsilon.$$

2.1. *Preliminary settings.* From now on, unless emphasized, we always assume $f \in \text{DPH}_1^r(M)$, $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. In this subsection, we will fix some elements for $f \in \text{DPH}_1^r(M)$, $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. The first is a metric on M that is adapted to the partially hyperbolic splitting. Second, one associates an order to each (1-dimensional) center bundle. Finally, one fixes two C^∞ -sub-bundles F^s and F^u that are C^0 -close to E^s and E^u , respectively.

Let d be a C^∞ -Riemannian metric on M that is adapted [14] to the splitting $E^s \oplus E^c \oplus E^u$. To be precise, there exist constants $0 < \lambda < 1$ and $0 < \eta_0 < 10^{-3}$ such that for any $z \in M$, it satisfies:

- $\|Df|_{E_z^s}\| < \min\{\lambda, \|Df|_{E_z^c}\|\} \leq \max\{\lambda^{-1}, \|Df|_{E_z^c}\|\} < m(Df|_{E_z^u})$;
- the three sub-bundles are almost orthogonal mutually:

$$\min_{v^\alpha \in E_z^\alpha, \|v^\alpha\|=1} d_{T_z M}(v^\alpha, E_z^\beta \oplus E_z^\gamma) > 1 - \eta_0 \quad \text{where } \{\alpha, \beta, \gamma\} = \{s, c, u\};$$

- for $\alpha = s, c, u$, the projection $\pi_z^\alpha : T_z M \rightarrow E_z^\alpha$ satisfies $\|\pi_z^\alpha(v)\| \leq 2\|v\|$ for any $v \in T_z M$.

Here, $\|\cdot\|$ is the vector norm on TM and $d_{T_z M}(\cdot, \cdot)$ is the distance induced by the Riemannian metric d . Such an adapted Riemannian metric d always exists by the classical arguments in [19].

Denote by $\pi : TM \rightarrow M$ the natural projection from TM to M and by $\pi^\alpha : TM \rightarrow E^\alpha$ the bundle projection map. For $v \in TM$, let $v = v^s + v^c + v^u$ be the direct sum splitting with $v^\alpha = \pi^\alpha(v)$, $\alpha = s, c, u$. To simplify notation, denote $s = \dim(E^s)$ and $u = \dim(E^u)$. For any constant $a > 0$, the a -cone field $\mathcal{C}_a(E^c)$ of E^c is defined as

$$\mathcal{C}_a(E^c) = \{v \in TM : \|v^s + v^u\| \leq a\|v^c\|\}.$$

A C^1 -curve $\gamma : I \rightarrow M$, where I is an interval, is called tangent to the a -cone field of E^c if the tangent line of γ is contained in $\mathcal{C}_a(E^c)$ everywhere.

Let $X \in \mathcal{X}^r(M)$ be a vector field that is transverse to $E^s \oplus E^u$ everywhere. Since $\dim(E^c) = 1$, one has that $E_z^c \cong \mathbb{R}$ for any $z \in M$. Moreover, since E^c is orientable and Df preserves the orientation, one can associate to each E_z^c an order induced by \mathbb{R} such that $Df|_{E^c}$ keeps this order. Without loss of generality, we assume that $X \in \mathcal{X}^r(M)$ is positively transverse to $E^s \oplus E^u$ everywhere. To be precise, it satisfies that $\pi^c(X(z)) > 0$ with respect to the order of E_z^c for every $z \in M$. The other case when X is negatively transverse to $E^s \oplus E^u$ everywhere is symmetric by taking the inverse orientation of E^c .

Note that in general, E^s and E^u are only continuous sub-bundles of TM . One takes two C^∞ -sub-bundles F^s and F^u of TM that are η_0 - C^0 -close to E^s and E^u , respectively. To be precise,

$$\begin{aligned} \dim(F^\alpha) &= \dim(E^\alpha) \quad \text{and} \quad \angle(F^\alpha, E^\alpha) \\ &= \max_{z \in M} \max_{v \in F_z^\alpha, \|v\|=1} d_{T_z M}(v, E_z^\alpha) < \eta_0 \quad \text{for } \alpha = s, u. \end{aligned}$$

Thus, $TM = F^s \oplus E^c \oplus F^u$ is still a direct sum decomposition. In particular, by taking F^s and F^u close enough to E^s and E^u , respectively, one can guarantee that X is also transverse to $F^s \oplus F^u$.

2.2. *Parameterizing and lifting center dynamics.* For each $z \in M$, denote by \mathcal{F}_z^c the center leaf of z . Based on the fact that $\dim(\mathcal{F}_z^c) = 1$, one parameterizes the center leaves $\{\mathcal{F}_z^c\}_{z \in M}$ and lifts the center dynamics on the parameterization. Then we prove that every center pseudo orbit with small jumps could be replaced by one that jumps in an ordered way and that preserves the initial and end points, see Lemma 2.5.

2.2.1. *Parameterization of the center foliation.* Note that \mathcal{F}_z^c either is non-compact and diffeomorphic to \mathbb{R} , or is compact and diffeomorphic to \mathbb{S}^1 . Let $\theta_z: \mathbb{R} \rightarrow \mathcal{F}_z^c$ be a C^1 map such that:

- $\theta_z(0) = z$;
- $\theta'_z(t)$ is the positive unit vector of $E_{\theta_z(t)}^c$ for all $t \in \mathbb{R}$.

To emphasize the dependence of the base point z , we denote $\mathbb{R}_z = \theta_z^{-1}(\mathcal{F}_z^c)$. Note that when \mathcal{F}_z^c is non-compact, then θ_z is a diffeomorphism and is isometric in the sense that $d^c(\theta_z(t_1), \theta_z(t_2)) = |t_2 - t_1|$, where $d^c(\cdot, \cdot)$ is the metric on the center foliations. When \mathcal{F}_z^c is compact, then θ_z is a universal covering map and is locally isometric.

2.2.2. *The constant ε_0 .* Since the length of all (compact) center leaves has a uniform lower bound, one can take a small constant $0 < \varepsilon_0 < 1$ such that for every $z \in M$ and every interval I with length no larger than $10\varepsilon_0$, the restricted map $\theta_z|_I$ is isometric.

Take $\varepsilon \in (0, \varepsilon_0)$. Assume

$$\Gamma = \{x_0, x_1, \dots, x_n\}$$

is an ε -center pseudo orbit. One extends Γ to be bi-infinite. To be precise, let

$$\Gamma_\infty = \{x_i\}_{i=-\infty}^{+\infty} = \{\dots, x_{-1}, x_0, x_1, \dots, x_n, x_{n+1}, \dots\}$$

such that

$$x_i = f^i(x_0) \quad \text{when } i \leq -1 \quad \text{and} \quad x_i = f^{i-n}(x_n) \quad \text{when } i \geq n + 1.$$

To simplify notation, denote $\mathbb{R}_i = \mathbb{R}_{x_i}$ and $\theta_i = \theta_{x_i}: \mathbb{R}_i \rightarrow \mathcal{F}_{x_i}^c$. Since $f(x_i) \in \mathcal{F}_{x_{i+1}}^c$ and each θ_i is locally isometric, then there exists a unique $t_{i+1} \in (-\varepsilon, \varepsilon)$ such that $f(x_i) = \theta_{i+1}(t_{i+1})$. In particular, one has $t_{i+1} = 0$ when $i \leq -1$ or $i \geq n$.

2.2.3. *Lifting dynamics.* Note that $\{\theta_z\}_{z \in M}$ defines a line-fiber over M . The dynamical system $f: M \rightarrow M$ induces natural dynamics $\widehat{f}: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ such that for all $i \in \mathbb{Z}$, the restriction $\widehat{f}|_{\mathbb{R}_i}: \mathbb{R}_i \rightarrow \mathbb{R}_{i+1}$ is a diffeomorphism and for all $t \in \mathbb{R}_i$, it satisfies

$$\theta_{i+1} \circ \widehat{f}(t) = f \circ \theta_i(t) \quad \text{and} \quad |\widehat{f}(0_i)| < \varepsilon.$$

That is to say, the following diagram commutes:

$$\begin{array}{ccc} \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i & \xrightarrow{\widehat{f}} & \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \\ \theta \downarrow & & \downarrow \theta \\ M & \xrightarrow{f} & M \end{array}$$

where $\theta|_{\mathbb{R}_i} = \theta_i$ for $i \in \mathbb{Z}$. Since Df preserves the orientation of E^c , each $\widehat{f}|_{\mathbb{R}_z}$ is strictly increasing. In particular, one has that $\widehat{f}(0_i) = t_{i+1}$ for every $i \in \mathbb{Z}$. Moreover, the map $\theta: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow M$ is a normally hyperbolic leaf immersion following [19, p. 69], see also [12, Proposition 4.4].

Remark 2.3. When $\mathcal{F}_{x_0}^c$ is non-compact, so as $\mathcal{F}_{x_i}^c$ for all $i \in \mathbb{Z}$, then θ_i is a diffeomorphism, and $\widehat{f}|_{\mathbb{R}_i}: \mathbb{R}_i \rightarrow \mathbb{R}_{i+1}$ is defined as $\widehat{f}|_{\mathbb{R}_i} = \theta_{i+1}^{-1} \circ f \circ \theta_i$. When otherwise $\mathcal{F}_{x_0}^c$ is compact, so as $\mathcal{F}_{x_i}^c$ for all $i \in \mathbb{Z}$, then each θ_i can be seen as a universal covering map of \mathbb{S}^1 and $f: \mathcal{F}_{x_i}^c \rightarrow \mathcal{F}_{x_{i+1}}^c$ can be seen as a circle diffeomorphism, thus one can take $\widehat{f}|_{\mathbb{R}_i}: \mathbb{R}_i \rightarrow \mathbb{R}_{i+1}$ to be the unique lifting map of $f|_{\mathcal{F}_{x_i}^c}$ that satisfies $|\widehat{f}(0_i)| < \varepsilon$. Moreover, in both cases, $\widehat{f}(0_i) = 0_{i+1}$ whenever $i \leq -1$ or $i \geq n$.

Remark 2.4. The reason why we extend the center pseudo-orbit Γ to the bi-infinite one Γ_∞ is that, in this way, one obtains an invariant set $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ of the lifting system \widehat{f} . However, one could lift the center dynamics as above over a general bi-infinite pseudo-orbit, but we will concentrate on the finite part Γ in the following.

Recall that $f(x_i) = \theta_{i+1}(t_{i+1})$ which is equivalent to say $\widehat{f}(0_i) = t_{i+1}$, with $|t_{i+1}| < \varepsilon$ for all $i \in \mathbb{Z}$ and in particular $t_{i+1} = 0$ when $i \leq -1$ or $i \geq n$. The following lemma states that replacing some points if necessary, the center pseudo orbit Γ can be chosen to jump in an ordered way.

LEMMA 2.5. *Replacing some of the intermediate points in $\{x_1, x_2, \dots, x_{n-1}\}$ if necessary, one can assume that the ε -center pseudo orbit Γ from x_0 to x_n satisfies that all t_i , $1 \leq i \leq n$ have the same sign. That is to say, either $-\varepsilon < t_i \leq 0$ for all $1 \leq i \leq n$ or $0 \leq t_i < \varepsilon$ for all $1 \leq i \leq n$.*

Proof. Note that $f(\mathcal{F}_{x_i}^c) = \mathcal{F}_{x_{i+1}}^c$. We split the proof into two cases whether the center leaves are compact or not.

Non-compact case. Assume each $\mathcal{F}_{x_i}^c$, $i = 0, 1, \dots, n$ is non-compact and thus diffeomorphic to \mathbb{R} by θ_i . To simplify notation, we define an order ‘ \leq ’ on each $\mathcal{F}_{x_i}^c$ inherited from that of \mathbb{R} . To be precise, for every $w, w' \in \mathcal{F}_{x_i}^c$, one defines $w \leq w'$ (respectively $w < w'$) if and only if $\theta_i^{-1}(w) \leq \theta_i^{-1}(w')$ (respectively $\theta_i^{-1}(w) < \theta_i^{-1}(w')$).

Notice that $f^i(x_0) \in \mathcal{F}_{x_i}^c$ for each $i = 0, 1, \dots, n$. If in the trivial case $f^n(x_0) = x_n$, then one just takes $x_i = f^i(x_0)$ for each $i = 1, 2, \dots, n - 1$ and as a result, every $t_i = 0$. So in the following, without loss of generality, assume that $f^n(x_0) < x_n$ and the other case where $x_n < f^n(x_0)$ is symmetric. One concludes Lemma 2.5 in the non-compact case from the following claim.

CLAIM 2.6. *Replacing some of the intermediate points in $\{x_1, x_2, \dots, x_{n-1}\}$ if necessary, we can choose Γ to satisfy that $f^{i+1}(x_0) \leq f(x_i) \leq x_{i+1}$ for any $0 \leq i \leq n - 1$. This implies that $-\varepsilon < t_i \leq 0$ for all $1 \leq i \leq n$.*

Proof of Claim 2.6. By the fact that Df preserves the orientation of \mathcal{F}^c , the assumption $f^n(x_0) < x_n$ implies $f^i(x_0) < f^{-n+i}(x_n)$ for any $i = 0, 1, \dots, n$.

We associate to the center pseudo orbit $\Gamma = \{x_0, x_1, \dots, x_n\}$ an integer $k = k(\Gamma) \in [0, n]$ defined by the following:

- if $f^n(x_0) \leq f(x_{n-1}) \leq x_n$ holds, then k is the smallest integer in $[0, n - 1]$ such that for any $i \in [k, n - 1]$, it satisfies $f^{i+1}(x_0) \leq f(x_i) \leq x_{i+1}$;
- if otherwise, take $k = n$.

If $k = k(\Gamma) = 0$, then Claim 2.6 satisfies automatically for Γ . Thus, in the following, one assumes that $k = k(\Gamma) \geq 1$.

By the choice of k , one knows that the inequality $f^k(x_0) \leq f(x_{k-1}) \leq x_k$ does not hold. Note that if k is taken as in the first case, then $f^{k+1}(x_0) \leq f(x_k)$ implies $f^k(x_0) \leq x_k$; and if in the second case where $k = n$, then one also has $f^k(x_0) \leq x_k$ by assumption. Thus, considering the orders of $f^k(x_0)$, $f(x_{k-1})$ and x_k , there are the following two cases: either $f(x_{k-1}) < f^k(x_0) \leq x_k$ or $f^k(x_0) \leq x_k < f(x_{k-1})$.

Sub-case (a). If $f(x_{k-1}) < f^k(x_0) \leq x_k$, the fact $d^c(f(x_{k-1}), x_k) < \varepsilon$ implies $d^c(f^k(x_0), x_k) < \varepsilon$. Thus, the new collection

$$\{x_0, f(x_0), \dots, f^{k-1}(x_0), x_k, x_{k+1}, \dots, x_{n-1}, x_n\}$$

is an ε -center pseudo orbit that satisfies Claim 2.6 and one just denotes it by Γ' .

Sub-case (b). Now we assume $f^k(x_0) \leq x_k < f(x_{k-1})$. By the orientation-preserving property, we have $f^{-1}(x_k) < x_{k-1}$ and $x_0 \leq f^{-k}(x_k)$. Thus, there exists a minimal $j \in [1, k - 1]$ such that $x_i \leq f^{-k+i}(x_k)$ for any $i \in [0, j - 1]$ while $f^{-k+j}(x_k) < x_j$. In particular, when $i = j - 1$, the inequality $x_{j-1} \leq f^{-k+j-1}(x_k)$ implies that $f(x_{j-1}) \leq f^{-k+j}(x_k) (< x_j)$. Thus,

$$d^c(f(x_{j-1}), f^{-k+j}(x_k)) \leq d^c(f(x_{j-1}), x_j) < \varepsilon.$$

Consider the new collection

$$\{x_0, x_1, \dots, x_{j-1}, f^{-k+j}(x_k), f^{-k+j+1}(x_k), \dots, f^{-1}(x_k), x_k, x_{k+1}, \dots, x_{n-1}, x_n\}.$$

This is an ε -center pseudo orbit and on $\mathcal{F}_{x_k}^c$, it satisfies that

$$f^k(x_0) < f(f^{-1}(x_k)) = x_k \leq x_k.$$

So we denote the new collection by Γ'' in Sub-case (b).

Note that in Sub-case (b), the newly defined ε -center pseudo orbit Γ'' satisfies that $k(\Gamma'') \leq k(\Gamma) - 1 < k(\Gamma)$. Moreover, the two endpoints x_0 and x_n are always kept. Since the length of the pseudo orbit is finite, this process ends in finite steps. The final ε -center pseudo orbit, denoted also by Γ' as in Sub-case (a), satisfies Claim 2.6. □

Compact case. Assume each $\mathcal{F}_{x_i}^c, i = 0, 1, \dots, n$ is compact and thus $\theta_i: \mathbb{R}_i \rightarrow \mathcal{F}_{x_i}^c$ is a universal covering map. By the locally isometric property of θ_i , one can see that the piece $\widehat{\Gamma} = \{0_0, 0_1, 0_2, \dots, 0_{n-1}, 0_n\}$ is an ε -pseudo orbit of the lifting dynamics \widehat{f} , where $0_i \in \mathbb{R}_i$ is the zero point on \mathbb{R}_i for any $i = 0, 1, \dots, n$. Note that $\widehat{f}(0_i) = t_{i+1}$. Applying the same arguments as in the non-compact case, one can assume that all $t_i, 1 \leq n$ have the same sign. This completes the proof of Lemma 2.5. □

Remark 2.7. Since each $\widehat{f}|_{\mathbb{R}_i}$ is strictly increasing, thus, when Γ is taken to satisfy that $-\varepsilon < t_i \leq 0$ for all $1 \leq i \leq n$ as in Lemma 2.5, then $\widehat{f}^i(0_0) \leq 0_i$ for all $1 \leq i \leq n$. The other case is symmetric.

One fixes the constants $0 < \lambda < 1$ and $0 < \eta_0 < 1/10^{-3}$ from §2.1 and $0 < \varepsilon_0 < 1$ in §2.2.

2.3. *Lifting bundle.* Recall that we have fixed two C^∞ -sub-bundles F^s and F^u that are C^0 -close to E^s and E^u , respectively. In this section, for a bi-infinite center pseudo orbit Γ , we pull back the bundle $F^s \oplus F^u$ over M to $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ through the C^1 -immersion map $\theta: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow M$. Then we obtain a tubular neighborhood of $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ on which one could define an exponential map to M and a bundle map which is partially hyperbolic. The construction follows essentially from [12, §4.1] where the ideas originates from [19]. We sketch the arguments and conclusions here. Also some of the notation is borrowed from [12].

Assume $\varepsilon \in (0, \varepsilon_0)$ and let $\Gamma = \{x_0, x_1, \dots, x_n\}$ be an ε -center pseudo orbit from §2.2 that satisfies Lemma 2.5. Let $\Gamma_\infty = \{x_i\}_{i=-\infty}^{+\infty}$ be the extended bi-infinite center pseudo orbit such that $x_i = f^i(x_0)$ when $i \leq -1$ and $x_i = f^{i-n}(x_n)$ when $i \geq n + 1$ as in §2.2. One pulls back the C^∞ -bundle $F^s \oplus F^u$ over M by the C^1 -leaf immersion $\theta: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow M$ to get a fiber bundle $\theta^*(F^s \oplus F^u)$ over $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$. Thus, one has the following commuting diagram:

$$\begin{array}{ccc}
 \theta^*(F^s \oplus F^u) & \xrightarrow{\theta^*} & F^s \oplus F^u \\
 \pi \downarrow & & \downarrow \pi \\
 \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i & \xrightarrow{\theta} & M
 \end{array}$$

For each $t \in \mathbb{R}_i$ where $i \in \mathbb{Z}$, one pulls back the Riemannian metric on $F^s_{\theta_i(t)} \oplus F^u_{\theta_i(t)}$ and thus the fiber

$$\theta^*(F^s \oplus F^u)_t := \theta^*(F^s_{\theta_i(t)} \oplus F^u_{\theta_i(t)})$$

over $t \in \mathbb{R}_i$ is a linear space equipped with an inner product structure. In this way, one defines a metric $\|\cdot\|_t$ on $\theta^*(F^s \oplus F^u)_t$. For any constant $\delta > 0$, any $t \in \mathbb{R}_i$, and any interval $I \subset \mathbb{R}_i$, one denotes

$$\theta^*(F^s \oplus F^u)_t(\delta) = \{v \in \theta^*(F^s \oplus F^u)_t : \|v\|_t \leq \delta\},$$

$$\theta^*(F^s \oplus F^u)_I(\delta) = \bigcup_{t \in I} \theta^*(F^s \oplus F^u)_t(\delta),$$

and

$$\theta^*(F^s \oplus F^u)(\delta) = \bigcup_{t \in \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i} \theta^*(F^s \oplus F^u)_t(\delta).$$

In particular, when $\delta = 0$, the 0-section $\theta^*(F^s \oplus F^u)(0)$ is reduced to $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$.

For $t \in \mathbb{R}_i$ and $v \in \theta^*(F^s \oplus F^u)_t$, let

$$\Phi(v) = \exp_{\theta_i(t)}(\theta_*(v)).$$

In this way, one defines the exponential map $\Phi: \theta^*(F^s \oplus F^u) \rightarrow M$. Then by [12, Proposition 4.5], there exists a constant $\delta_0 > 0$ that depends only on f, F^s, F^u such that for any $0 < \delta \leq \delta_0$ and for any $t \in \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$, the set $\Phi(\theta^*(F^s \oplus F^u)_{(t-\delta, t+\delta)}(\delta))$ is open in M . Moreover, the exponential map Φ is a diffeomorphism from $\theta^*(F^s \oplus F^u)_{(t-\delta, t+\delta)}(\delta)$ to its image. As a consequence, there exists a constant $C_0 > 1$ that depends on f such that for any $t \in \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ and any $v \in \theta^*(F^s \oplus F^u)_t(\delta_0/C_0)$, there exists a unique vector

$$\theta^* f(v) := \Phi^{-1} \circ f \circ \Phi(v) \in \theta^*(F^s \oplus F^u)_t(\delta_0/2),$$

where Φ^{-1} is defined from $\Phi(\theta^*(F^s \oplus F^u)_{(t-\delta_0, t+\delta_0)}(\delta_0))$ to $\theta^*(F^s \oplus F^u)_{(t-\delta_0, t+\delta_0)}(\delta_0)$. That is to say, one has the following commuting diagram:

$$\begin{array}{ccc} \theta^*(F^s \oplus F^u)(\delta_0/C_0) & \xrightarrow{\theta^* f} & \theta^*(F^s \oplus F^u)(\delta_0/2) \\ \Phi \downarrow & & \downarrow \Phi \\ M & \xrightarrow{f} & M \end{array}$$

Moreover, the lifting map $\theta^* f$ is a diffeomorphism from $\theta^*(F^s \oplus F^u)(\delta_0/C_0)$ to its image. However, one pulls back the Riemannian metric on M through Φ to obtain a local metric $\tilde{d}(\cdot, \cdot)$ on $\theta^*(F^s \oplus F^u)(\delta_0)$. To be precise, for any two $v_1, v_2 \in \theta^*(F^s \oplus F^u)_{(t-\delta_0, t+\delta_0)}(\delta_0)$ for some $t \in \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$, one has $\tilde{d}(v_1, v_2) = d(\Phi(v_1), \Phi(v_2))$. See similar statements in [12, pp. 32].

Remark 2.8. When restricted to the 0-section $\theta^*(F^s \oplus F^u)(0)$ (which is $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ as pointed out above), the map

$$\theta^* f|_{\theta^*(F^s \oplus F^u)(0)}: \theta^*(F^s \oplus F^u)(0) \rightarrow \theta^*(F^s \oplus F^u)(0)$$

is reduced to $\hat{f}: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$. In this sense, the map $\theta^* f$ is in fact an extension of \hat{f} to a tubular neighborhood of $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$. Thus, in the following, we identify the 0-section $\theta^*(F^s \oplus F^u)(0)$ to $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ and associate the order on $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ to $\theta^*(F^s \oplus F^u)(0)$. One has

$$\theta^* f|_{\theta^*(F^s \oplus F^u)(0)} = \hat{f}: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i.$$

Remark 2.9. We point out that [12, Proposition 4.5] deals with periodic center leaves while here we consider general ones. However, the existence of δ_0 and C_0 only depends on the uniform hyperbolicity of the two extreme bundles E^s/E^u (or equivalently on the constant $0 < \lambda < 1$) and the bounded norm of f .

The partially hyperbolic splitting $TM = E^s \oplus E^c \oplus E^u$ induces a partially hyperbolic structure for the bundle map $\theta^* f: \theta^*(F^s \oplus F^u)(\delta_0/C_0) \rightarrow \theta^*(F^s \oplus F^u)(\delta_0/2)$ over the

diffeomorphism $\widehat{f}: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ with partially hyperbolic splitting

$$\begin{aligned} T \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \theta^*(F^s \oplus F^u) &= \theta^*(E^s)|_{\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i} \oplus T \left(\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \right) \oplus \theta^*(E^u)|_{\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i} \\ &= (\theta^*(E^s) \oplus \theta^*(E^c) \oplus \theta^*(E^u))|_{\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i}. \end{aligned}$$

The main ingredient is that $\theta: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow M$ is a normally hyperbolic immersion following [19, p. 69]. The detailed proofs can be found in [12, Proposition 4.6].

One fixes the constants $\delta_0 > 0$ and $C_0 > 1$ from §2.3.

2.4. *Lipschitz shadowing of invariant manifolds and leaf conjugacy.* Recall that $X \in \mathcal{X}^r(M)$ is a C^r vector field which is positively transverse to $E^s \oplus E^u$ (and also transverse to $F^s \oplus F^u$) everywhere and X_τ is the time- τ map generated by X . In this section, we consider the lifting bundle map of the composition $X_\tau \circ f$ (which can be seen as C^r -small perturbations of f provided τ is small). This allows us to have a Lipschitz shadowing property of invariant manifolds (Theorem 2.10) and a leaf conjugacy map (§2.4.1).

The transverse property implies that X has no singularity. Denote $f_\tau = X_\tau \circ f: M \rightarrow M$ for every $\tau \in \mathbb{R}$. One lifts X to be a vector field \widetilde{X} on $\theta^*(F^s \oplus F^u)(\delta_0)$ as follows:

$$\widetilde{X}(v) = D\Phi^{-1}(X(\Phi(v))) \quad \text{for all } v \in \theta^*(F^s \oplus F^u)(\delta_0).$$

The lifting vector field \widetilde{X} is C^0 -continuous on $\theta^*(F^s \oplus F^u)(\delta_0)$, while for τ small, the time τ -map \widetilde{X}_τ generated by \widetilde{X} is a C^1 -diffeomorphism from $\theta^*(F^s \oplus F^u)(\delta_0/2)$ to its image in $\theta^*(F^s \oplus F^u)(\delta_0)$ and satisfies $\widetilde{X}_\tau = \Phi^{-1} \circ X_\tau \circ \Phi$. One has the following theorem which is stated in [12, Lemma 4.9, Theorems 4.2 and 4.3], and whose proofs are originated from [19, Theorem 6.8] and based on [19, Theorem 6.1] which shows the existence and properties of local invariant manifolds. Recall that we have fixed the constant $0 < \eta_0 < 10^{-3}$ from §2.1.

THEOREM 2.10. *There exist two constants $\tau_0 > 0$ and $L > 1$ such that for any $\tau \in [-\tau_0, \tau_0]$, the following properties are satisfied.*

(1) *The family of C^1 -diffeomorphisms*

$$\widetilde{f}_\tau = \widetilde{X}_\tau \circ \theta^* f: \theta^*(F^s \oplus F^u)(\delta_0/C_0) \rightarrow \theta^*(F^s \oplus F^u)(\delta_0)$$

is well defined and $\widetilde{f}_\tau \rightarrow \theta^ f$ in the C^1 -topology as $\tau \rightarrow 0$. Moreover, one has*

$$\widetilde{d}(v, \widetilde{X}_\tau(v)) \leq L \cdot |\tau|, \quad \text{for all } v \in \theta^*(F^s \oplus F^u)(\delta_0/C_0).$$

(2) *There exists an \widetilde{f}_τ -invariant C^1 -smooth section $\sigma_\tau: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow \theta^*(F^s \oplus F^u)(\delta_0)$ which is defined as*

$$\sigma_\tau \left(\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \right) = \bigcap_{n \in \mathbb{Z}} \widetilde{f}_\tau^n(\theta^*(F^s \oplus F^u)(\delta_0/C_0)),$$

and which varies continuously with respect to τ . The two C^1 -manifolds,

$$W_\tau^{cs} = \bigcap_{n=0}^{+\infty} \tilde{f}_\tau^{-n}(\theta^*(F^s \oplus F^u)(\delta_0/C_0)) \quad \text{and} \quad W_\tau^{cu} = \bigcap_{n=0}^{+\infty} \tilde{f}_\tau^n(\theta^*(F^s \oplus F^u)(\delta_0/C_0)),$$

intersect transversely at $\sigma_\tau(\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i) = W_\tau^{cs} \cap W_\tau^{cu}$. Moreover, the section σ_τ converges to σ_0 in the C^1 -topology as $\tau \rightarrow 0$, where $\sigma_0(\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i) = \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ is the 0-section.

- (3) For any $t \in \mathbb{R}_i, i \in \mathbb{Z}$ and any C^1 map $\varphi: E_{\theta_i(t)}^s(\delta_0) \oplus E_{\theta_i(t)}^u(\delta_0) \rightarrow E_{\theta_i(t)}^c(\delta_0)$ satisfying $\varphi(0) = 0, \|\partial\varphi/\partial\alpha\| \leq \eta_0$ for $\alpha = s, u$, the C^1 -submanifold

$$\begin{aligned} D_{\theta_i(t)}^{su} &= \exp_{\theta_i(t)}(\text{Graph}(\varphi)) \\ &= \exp_{\theta_i(t)}(\{v^{su} + \varphi(v^{su}): v^{su} \in E_{\theta_i(t)}^s(\delta_0) \oplus E_{\theta_i(t)}^u(\delta_0)\}) \end{aligned}$$

intersects $\Phi(\sigma_\tau((t - \delta_0, t + \delta_0)))$ at a unique point $q = q(t, \tau, \varphi)$. Moreover, let $v_q^{su} \in E_{\theta_i(t)}^s(\delta_0) \oplus E_{\theta_i(t)}^u(\delta_0)$ be such that $q = \exp_{\theta_i(t)}(v_q^{su} + \varphi(v_q^{su}))$, then

$$\|v_q^{su}\| + \|\varphi(v_q^{su})\| \leq L \cdot |\tau|,$$

which in particular implies that $\sigma_\tau(\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i) \subset \theta^*(F^s \oplus F^u)(L \cdot |\tau|)$.

Remark 2.11. Note that the map $\tilde{f}_\tau = \tilde{X}_\tau \circ \theta^* f$ in Theorem 2.10 is the lift of $f_\tau = X_\tau \circ f$. In particular, when $\tau = 0$, one has $\tilde{f}_0 = \theta^* f$ and as a consequence (recall Remark 2.8), when restricted to the 0-section $\theta^*(F^s \oplus F^u)(0) = \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$, it satisfies:

$$\tilde{f}_0|_{\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i} = \theta^* f|_{\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i} = \hat{f}: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i.$$

2.4.1. *Leaf conjugacy.* Given a constant $\eta \in (0, \eta_0]$ and an $(s + u)$ -dimensional C^∞ -sub-bundle F of TM that is η - C^0 -close to $E^s \oplus E^u$. For a point $z \in M$ and a constant $0 < \delta \leq \delta_0$, let

$$D_z^F(\delta) = \exp_z(F_z(\delta)).$$

Then following the arguments in [12, §4.2] (different to our setting here, as [12, §4.2] deals with a bundle $E^s \oplus F$ where E^s is the contracting sub-bundle in the partially hyperbolic splitting and F is an u -dimensional C^∞ -sub-bundle that is C^0 -close to the expanding sub-bundle E^u), there exists a constant $\delta_1 = \delta_1(F, \eta) \in (0, \delta_0]$ such that the following properties are satisfied.

- (1) For any point $z \in M$, the set $D_z^F(\delta_1)$ is a C^r -smooth local manifold whose tangent bundle is 2η -close to $E^s \oplus E^u$. To be precise,

$$\angle(T_w D_z^F(\delta_1), (E^s \oplus E^u)_w) < 2\eta, \quad \text{for all } w \in D_z^F(\delta_1).$$

As a consequence, there exists a C^r -map $\varphi_z^{su}: E^s(\delta_1/2) \oplus E^u(\delta_1/2) \rightarrow E^c(\delta_1/2)$ satisfying that:

- $\text{Graph}(\varphi_z^{su}) = \exp^{-1}(D_z^F(\delta_1)) \cap T_z M(\delta_1/2)$;
- $\varphi_z^{su}(0_z^{su}) = 0_z^c$ and $\|\partial\varphi_z^{su}/\partial\alpha\| < 2\eta$ for $\alpha = s, u$.

- (2) Given a C^1 -center curve γ^c with radius δ_1 and centered at a point $z \in M$, the set

$$B_z^F(\gamma^c, \delta_1) = \bigcup_{w \in \gamma^c} D_w^F(\delta_1)$$

is a neighborhood of z in M and the family

$$\mathcal{D}^F(\gamma^c) = \{D_w^F(\delta_1) : w \in \gamma^c\}$$

forms a C^0 -foliation of $B_z^F(\delta_1)$. Moreover, the foliation $\mathcal{D}^F(\gamma^c)$ is transverse to any C^1 -curve $\gamma \in B_z^F(\delta_1)$ that is tangent to the η -cone field of E^c .

- (3) By the uniform transversality and compactness, for any constant $\delta \in [0, \delta_1]$, there exists $\delta' \in [0, \delta_1]$ such that given a C^1 -center curve γ^c with radius δ_1 and centered at a point $z \in M$, if γ_1 and γ_2 are two C^1 -curves tangent to the η_0 -cone field of E^c in $B_z^F(\gamma^c, \delta_1)$ with endpoints $w_i, w'_i, i = 1, 2$ that satisfy

$$w_1, w_2 \in D_{w_0}^F(\delta_1) \text{ and } w'_1, w'_2 \in D_{w'_0}^F(\delta_1) \text{ where } w_0, w'_0 \in \gamma^c,$$

then $\text{length}(\gamma_1) \geq \delta$ implies $\text{length}(\gamma_2) \geq \delta'$.

Denote $\tilde{D}_t^F(\delta) = \Phi^{-1}(D_{\theta(t)}^F(\delta))$ for $0 < \delta < \delta_1$. Then

$$\mathcal{D}^F(\delta_1/10) = \left\{ \tilde{D}_t^F(\delta) \cap \theta^*(F^s \oplus F^u)(\delta_1/10) : t \in \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \right\}$$

forms a C^0 -foliation of $\theta^*(F^s \oplus F^u)(\delta_1/10)$ that is transverse to $(\Phi^{-1})^*(E^c)$ everywhere. Following the idea of [19, §7], one can define a leaf conjugacy as follows. There exists a constant $\tau_1 = \tau_1(\eta, F, \delta_1) \in (0, \tau_0]$ where τ_0 is from Theorem 2.10, such that for every $\tau \in [-\tau_1, \tau_1]$, the leaf conjugacy

$$h_\tau : \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow \bigsqcup_{i \in \mathbb{Z}} \sigma_\tau(\mathbb{R}_i),$$

where

$$h_\tau(t) = \sigma_\tau(\mathbb{R}_i) \cap \tilde{D}_t^F(\delta_1), \quad \text{for all } t \in \mathbb{R}_i,$$

is well defined. In particular, when $\tau = 0$, the map $h_0|_{\mathbb{R}_i}$ is the identity map $\text{Id}|_{\mathbb{R}_i}$.

The constant η and the bundle F (thus $\delta_1 = \delta_1(\eta, F)$ and $\tau_1 = \tau_1(\eta, F, \delta_1)$) will be determined in Proposition 3.1 in the next section.

3. Proofs of main theorems

In this section, we prove Theorem A and Corollary B. Let $f \in \text{DPH}'_1(M)$, $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ and $X \in \mathcal{X}^r(M)$ be a vector field that is positively transverse to $E^s \oplus E^u$. For the proof of Theorem A, all the settings and assumptions are taken from §2.

3.1. *A global perturbation along the center direction.* One has the following proposition that realizes perturbations along center directions. The idea essentially follows from arguments in [12, Proposition 4.17] with the difference that [12, Proposition 4.17] deals with periodic leaves while here we consider general cases. We will include the proof for completeness. Recall in Remarks 2.8 and 2.11, the 0-section $\theta^*(F^s \oplus F^u)(0)$ is identified

to be $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$ and thus is associated with an order from $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i$. Also restricted on $\theta^*(F^s \oplus F^u)(0)$, one has $\tilde{f}_0|_{\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i} = \theta^* f|_{\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i} = \hat{f}$.

PROPOSITION 3.1. *There exist a constant $\eta = \eta(f, X) \in (0, \eta_0]$, an $(s + u)$ -dimensional C^∞ sub-bundle $F \subset TM$ that is η -close to $E^s \oplus E^u$ and $\tau_2 = \tau_2(f, X, \eta, F) \in (0, \tau_1]$ such that the following properties are satisfied. For $\tau \in [-\tau_2, \tau_2]$, let $h_\tau: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow \bigsqcup_{i \in \mathbb{Z}} \sigma_\tau(\mathbb{R}_i)$ be the leaf conjugacy*

$$h_\tau(t) = \sigma_\tau(\mathbb{R}_i) \cap \tilde{D}_t^F(\delta_1) \quad \text{for all } t \in \mathbb{R}_i,$$

then for any $\tau \in [-\tau_2, \tau_2]$, there exists $\Delta_\tau > 0$ such that for any $t \in \mathbb{R}_i$, one has:

(1) if $0 < \tau < \tau_2$, then

$$\tilde{f}_\tau \circ h_\tau(t) > h_\tau(\theta^* f(t) + \Delta_\tau);$$

(2) if $-\tau_2 < \tau < 0$, then

$$\tilde{f}_\tau \circ h_\tau(t) < h_\tau(\theta^* f(t) - \Delta_\tau).$$

Here the order on each $\sigma_\tau(\mathbb{R}_i)$ is inherited from that on \mathbb{R}_i . The constants $\delta_1 = \delta_1(\eta, F)$ and $\tau_1 = \tau_1(\eta, F, \delta_1)$ are from §2.4.1.

Proof. We only prove the case when $\tau > 0$ since the other case when $\tau < 0$ is symmetric. □

3.1.1. *The lifting vector field \widehat{X} on TM .* One first lifts X to TM in the following way: given a point $z \in M$, let $\widehat{X}_z = D \exp_z^{-1}(X)$. Then \widehat{X}_z is a vector field which is well defined in a neighborhood of the zero vector $0_z \in T_z M$. Note that for $v \in T_z M$, its tangent space $T_v(T_z M) \cong T_z M$, thus it admits a direct sum $T_v(T_z M) = E_z^s(v) \oplus E_z^c(v) \oplus E_z^u(v)$ induced by $T_z M = E_z^s \oplus E_z^c \oplus E_z^u$. For $v \in T_z M$ close to 0_z , denote

$$\widehat{X}_z(v) = \widehat{X}_z^s(v) \oplus \widehat{X}_z^c(v) \oplus \widehat{X}_z^u(v),$$

where $\widehat{X}_z^\alpha(v) \in E_z^\alpha(v)$, $\alpha = s, c, u$. Recall that X is positively transverse to $E^s \oplus E^u$, thus there exist constants $\delta_2 > 0$, $a_0 > 0$, and $b_0 > 0$ such that for any $z \in M$:

- for any $v \in T_z M(\delta_2) = E_z^s(\delta_2) \oplus E_z^c(\delta_2) \oplus E_z^u(\delta_2)$, it satisfies

$$\|\widehat{X}_z^c(v)\| \geq b_0, \quad \text{and} \quad \|\widehat{X}_z^s(v) + \widehat{X}_z^u(v)\| < a_0 \|\widehat{X}_z^c(v)\| \leq 2a_0 \|\widehat{X}\|;$$

- for any $v \in E_z^s(\delta_2) \oplus E_z^u(\delta_2)$, the curve $\exp_z(v \times E_z^c(\delta_2))$ is tangent to the η_0 -cone field of E^c .

Recall that one has defined an order on E_z^c that is induced from the orientation of E^c in §2.1. One thus can associate the order of E_z^c to $E_z^c(v)$ for each $v \in T_z M$. Then we have $\widehat{X}_z^c(v) \geq b_0 > 0$ for every vector $v \in T_z M(\delta_2)$.

3.1.2. *The choices of constants η , τ' and the C^∞ bundle F .* Let $\tau_0 > 0$ and $L > 1$ be the two constants given by Theorem 2.10. Now we take the constant $\eta = \eta(f, X) \in (0, \eta_0]$ and the $(s + u)$ -dimensional C^∞ sub-bundle $F \subset TM$ as follows.

- Take η to satisfy

$$0 < \eta < \min \left\{ \eta_0, \frac{b_0}{8 \cdot [(\|Df\| + 1) \cdot L + 2a_0 \|\widehat{X}\|]} \right\}.$$

- Take an $(s + u)$ -dimensional C^∞ sub-bundle $F \subset TM$ that is η -close to $E^s \oplus E^u$, that is, $\angle(F, E^s \oplus E^u) < \eta$.

Associated to η and F , there exist $\delta_1 = \delta_1(\eta, F) \in (0, \delta_0]$ and $\tau_1 = \tau_1(\eta, F, \delta_1) \in (0, \tau_0]$ that are from §2.4.1. Set $\delta' = 10^{-2} \min\{\delta_1, \delta_2\}$ and take a constant $\tau' = \tau'(f, X, \eta, F) \in (0, \tau_0]$ that satisfies

$$0 < \tau' < \min \left\{ \tau_1, \frac{\delta'}{100L(\|X\| + 1)\|Df\|} \right\},$$

where $\|X\| = \sup_{x \in M} \|X(x)\|$ and $\|Df\| = \sup_{x \in M} \|Df(x)\|$.

3.1.3. *The estimation of moving along the center direction.* Fix $t \in \mathbb{R}_i$ and denote $z_0 = \theta_i(t)$. For any $\tau \in [-\tau', \tau']$, let $q_\tau = \Phi(h_\tau(t))$. By the choices of constants and Theorem 2.10, one has that $q_\tau \in D_{z_0}^F(\delta')$. Moreover, considering the direct sum $\exp^{-1}(q_\tau) = v_{q_\tau}^{su} + v_{q_\tau}^c \in E_{z_0}^{su} \oplus E_{z_0}^c$, it satisfies that

$$\|v_{q_\tau}^{su}\| \leq L \cdot \tau \leq \frac{\delta'}{100(\|X\| + 1)\|Df\|}.$$

Let $\varphi_{z_0}^{su} : E_{z_0}^s(\delta_1/2) \oplus E_{z_0}^u(\delta_1/2) \rightarrow E_{z_0}^c(\delta_1/2)$ be the C^r -map satisfying that

$$\text{Graph}(\varphi_{z_0}^{su}) = \exp^{-1}(D_{z_0}^F(\delta_1)) \cap T_z M(\delta_1/2),$$

then one has

$$v_{q_\tau}^c = \varphi_{z_0}^{su}(v_{q_\tau}^{su}) < 2\eta \cdot \|v_{q_\tau}^{su}\| \leq 2\eta \cdot L \cdot \tau \leq 2\eta \cdot \frac{\delta'}{100(\|X\| + 1)\|Df\|}.$$

Note that $z_0 = \theta_i(t)$ and $f(z_0) = \theta_{i+1}(\theta^* f(t))$. By the choices of constants above, for any $\tau \in [-\tau', \tau']$, one has

$$f(q_\tau) \in B_{f(z_0)}(\delta'/10) \quad \text{and} \quad f_\tau(q_\tau) = X_\tau \circ f(q_\tau) \in B_{f(z_0)}(\delta').$$

Consider the direct sum

$$\exp_{f(z_0)}^{-1}(f(q_\tau)) = v_{f(q_\tau)}^{su} + v_{f(q_\tau)}^c \in E_{f(z_0)}^{su} \oplus E_{f(z_0)}^c,$$

then one has the following claim.

CLAIM 3.2. *There exist a continuous function $\xi : \mathbb{R} \rightarrow \mathbb{R}^+$ that depends on f and X satisfying that:*

- (1) $\xi(\tau) = o(\tau)$ as $\tau \rightarrow 0$;
- (2) *the following estimates holds:*
 - $\|v_{f(q_\tau)}^{su}\| \leq (\|Df(z_0)\| + \xi(\tau)) \cdot \|v_{q_\tau}^{su}\| \leq (\|Df(z_0)\| + \xi(\tau)) \cdot L\tau$;
 - $\|v_{f(q_\tau)}^c\| \leq (\|Df|_{E_{z_0}^c}\| + \xi(\tau)) \cdot \|v_{q_\tau}^c\| \leq 2\eta \cdot (\|Df|_{E_{z_0}^c}\| + \xi(\tau)) \cdot L\tau$.

The second formula implies that

$$v_{f(q_\tau)}^c \geq -2\eta \cdot (\|Df|_{E_{z_0}^c}\| + \xi(\tau)) \cdot L\tau.$$

Proof of the Claim. The proof is essentially because the map $\exp_{f(z_0)}^{-1} \circ f \circ \exp_{z_0}$ is C^1 -smooth at 0_{z_0} and $D(\exp_{f(z_0)}^{-1} \circ f \circ \exp_{z_0})(0_{z_0}) = Df(z_0)$. See the details in the proof of [12, Claim 4.19]. \square

3.1.4. *The action of \widehat{X}_τ on $\exp_{f(z_0)}^{-1}(f(q_\tau))$.* Note first that

$$\exp_{f(z_0)} \circ \widehat{X}_\tau(\exp_{f(z_0)}^{-1}(f(q_\tau))) = X_\tau \circ f(q_\tau) = f_\tau(q_\tau) = \Phi \circ \widetilde{f}_\tau(h_\tau(t)) =: p_\tau.$$

This gives

$$\widehat{X}_\tau(\exp_{f(z_0)}^{-1}(f(q_\tau))) = \exp_{f(z_0)}^{-1}(\Phi \circ \widetilde{f}_\tau(h_\tau(t))) = \exp_{f(z_0)}^{-1}(p_\tau).$$

We consider the direct sum

$$\widehat{X}_\tau(v_{f(q_\tau)}^{su} + v_{f(q_\tau)}^c) = \widehat{X}_\tau(\exp_{f(z_0)}^{-1}(f(q_\tau))) = v_{p_\tau}^{su} + v_{p_\tau}^c \in E_{f(z_0)}^{su} \oplus E_{f(z_0)}^c.$$

It satisfies that for every $\tau \in [-\tau', \tau']$:

- $\|v_{p_\tau}^{su}\| \leq \|v_{f(q_\tau)}^{su}\| + \sup_{v \in T_{f(z_0)}M(\delta_2)} (\|\widehat{X}_{f(z_0)}^s(v) + \widehat{X}_{f(z_0)}^u(v)\|) \cdot \tau \leq \|v_{f(q_\tau)}^{su}\| + 2a_0\|\widehat{X}\| \cdot \tau$;
- $v_{p_\tau}^c \geq -\|v_{f(q_\tau)}^c\| + \inf_{v \in T_{f(z_0)}M(\delta_2)} \|\widehat{X}_{f(z_0)}^c(v)\| \cdot \tau \geq -\|v_{f(q_\tau)}^c\| + b_0 \cdot \tau$.

Combined with Claim 3.2, one has

$$\begin{aligned} \|v_{p_\tau}^{su}\| &\leq [(\|Df(z_0)\| + \xi(\tau)) \cdot L + 2a_0\|\widehat{X}\|] \cdot \tau \quad \text{and} \\ v_{p_\tau}^c &\geq b_0 \cdot \tau - 2\eta \cdot (\|Df|_{E_{z_0}^c}\| + \xi(\tau)) \cdot L \cdot \tau. \end{aligned}$$

3.1.5. *The constant τ_2 .* By item (1) of Claim 3.2, since $\xi(\tau)$ is a positive infinitesimal as $\tau \rightarrow 0$, there exists $\tau_2 \in (0, \tau']$ such that for any $\tau \in [-\tau_2, \tau_2]$, the following holds:

$$0 < \xi(\tau) < 1 \quad \text{and} \quad [(\|Df\| + 1) \cdot L + 2a_0\|\widehat{X}\|] \cdot |\tau| < \delta'.$$

Consider the disk $D_{f(z_0)}^F(\delta')$ and let $\varphi_{f(z_0)}^{su} : E_{f(z_0)}^s(\delta_1/2) \oplus E_{f(z_0)}^u(\delta_1/2) \rightarrow E_{f(z_0)}^c(\delta_1/2)$ be the C^r -map whose graph determines $D_{f(z_0)}^F(\delta')$ through the exponential map $\exp_{f(z_0)}$. When $\tau \in (0, \tau_2]$, by the choice of η , one has

$$\begin{aligned} v_{p_\tau}^c - \varphi_{f(z_0)}^{su}(v_{p_\tau}^{su}) &> b_0 \cdot \tau - 2\eta \cdot (\|Df|_{E_{z_0}^c}\| + 1) \cdot L \cdot \tau - 2\eta\|v_{p_\tau}^{su}\| \\ &> b_0 \cdot \tau - 2\eta \cdot [2(\|Df\| + 1) \cdot L + 2a_0\|\widehat{X}\|] \cdot \tau \\ &> b_0 \cdot \tau - \frac{b_0}{2} \cdot \tau = \frac{b_0}{2} \cdot \tau. \end{aligned}$$

Thus, when $0 < \tau < \tau_2$, the point

$$p_\tau = \exp_{f(z_0)}(v_{p_\tau}^{su} + v_{p_\tau}^c) = f_\tau(q_\tau) = \Phi \circ \widetilde{f}_\tau(h_\tau(t))$$

belongs to $D_{\theta_{i+1}(t)}^F(\delta')$ for some $t' \in \mathbb{R}_{i+1}$ that satisfies

$$t' > \theta^* f(t).$$

Finally, consider the segment

$$\gamma : = \exp_{f(z_0)}(\{v_{p_\tau}^{su} + v^c : \varphi_{f(z_0)}^{su}(v_{p_\tau}^{su}) \leq v^c \leq v_{p_\tau}^c\}).$$

One has that γ is tangent to the η_0 -cone field of E^c with length uniformly bounded from from below (depending on τ). By item (3) of §2.4.1, there exists $\Delta_\tau > 0$ such that $t' - \theta^* f(t) > \Delta_\tau$. As a consequence, one has

$$\tilde{f}_\tau \circ h_\tau(t) > h_\tau(\theta^* f(t) + \Delta_\tau).$$

This completes the proof.

Remark 3.3. The constant Δ_τ depends on f, X, F , and τ but is independent of the choice of center pseudo orbit Γ .

3.2. *Proof of Theorem A.* Based on Proposition 3.1, we are ready to prove Theorem A. As we have mentioned before, we have to deal with general orbits of center leaves rather than periodic ones which are considered in the proof of [12, Theorem 2.3].

Proof of Theorem A. Let $f \in \text{DPH}_1^r(M), r \in \mathbb{N}_{\geq 2}$. Assume $X \in \mathcal{X}^r(M)$ is a vector field that is transverse to $E^s \oplus E^u$ and the constant ε_0 is taken as in §2.1. Let $L > 1$ be from Theorem 2.10. Let the constant $\eta > 0$, the C^∞ -bundle $F \subset TM$, and the constant τ_2 be from Proposition 3.1. For each

$$k > 2 \max \left\{ \frac{1}{\tau_2}, \frac{L+1}{\varepsilon_0} \right\},$$

let $\Delta_{1/k} > 0$ be the constant associated to $\tau = 1/k$ from Proposition 3.1 and take $\varepsilon \in (0, \varepsilon_0)$ satisfying

$$0 < \varepsilon < \min\{1/k, \Delta_{1/k}\}.$$

Assume $x \dashv_f y$. By Proposition 2.1, one takes an ε -center pseudo orbit

$$\Gamma = \{x_0, x_1, \dots, x_n\}$$

such that

$$d(x_0, x) < \varepsilon \quad \text{and} \quad d(x_n, y) < \varepsilon.$$

Moreover, one assumes that Γ satisfies Lemma 2.5. Additionally, if $x \in \text{CR}(f)$, that is, $y = x$, one can take Γ to satisfy that $x_0 = x_n$ by Proposition 2.1 and Remark 2.2. Without loss of generality, we assume that $-\varepsilon < t_i \leq 0$ for each $0 \leq i \leq n$ and the other case is symmetric.

Let $\Gamma_\infty = \{x_i\}_{i=-\infty}^{+\infty}$ be the extended bi-infinite center pseudo orbit such that $x_i = f^i(x_0)$ when $i \leq -1$ and $x_i = f^{i-n}(x_n)$ when $i \geq n+1$. Denote by

$$\theta^* f : \theta^*(F^s \oplus F^u)(\delta_0/C_0) \rightarrow \theta^*(F^s \oplus F^u)(\delta_0/2)$$

the bundle dynamics associated to Γ_∞ as in §2.3. Moreover, let

$$\tilde{f}_\tau = \tilde{X}_\tau \circ \theta^* f : \theta^*(F^s \oplus F^u)(\delta_0/C_0) \rightarrow \theta^*(F^s \oplus F^u)(\delta_0)$$

be the action of lifting vector field of X composed with $\theta^* f$ as in §2.4.

From the choice of center pseudo orbit Γ , one knows that

$$\tilde{f}_0^i(0_0) = (\theta^* f)^i(0_0) \leq 0_i \quad \text{for all } 1 \leq i \leq n.$$

In the trivial case when $\tilde{f}_0^n(0_0) = 0_n$, which is equivalent to $f^n(x_0) = x_n$, the collection $\{x_0, x_1, \dots, x_n\}$ is a piece of orbit segment from x_0 to x_n and the conclusion holds automatically.

We now assume that $\tilde{f}_0^n(0_0) < 0_n$. Consider the bundle dynamics $\theta^* f$ on the lifting bundle $\theta^*(F^s \oplus F^u)$ and its perturbation $\tilde{f}_{1/k} = \tilde{X}_{1/k} \circ \theta^* f$ defined on $\theta^*(F^s \oplus F^u)(\delta_0/C_0)$. Let $h_{1/k}: \bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_i \rightarrow \bigsqcup_{i \in \mathbb{Z}} \sigma_{1/k}(\mathbb{R}_i)$ be the leaf conjugacy in §2.4.1. Note that by the choices of constants, one has

$$-\Delta_{1/k} < -\varepsilon < t_i \leq 0 \quad \text{for all } 1 \leq i \leq n.$$

By Proposition 3.1, one has the following estimation for the zero point $0_0 \in \mathbb{R}_0$:

$$\begin{aligned} \tilde{f}_{1/k}^n(h_{1/k}(0_0)) &> \tilde{f}_{1/k}^{n-1}(h_{1/k}(\theta^* f(0_0) + \Delta_{1/k})) = \tilde{f}_{1/k}^{n-1}(h_{1/k}(t_1 + \Delta_{1/k})) \\ &> \tilde{f}_{1/k}^{n-1}(h_{1/k}(0_1)) \\ &> \dots \\ &> \tilde{f}_{1/k}(h_{1/k}(0_{n-1})) \\ &> h_{1/k}(\theta^* f(0_{n-1}) + \Delta_{1/k}) = h_{1/k}(t_n + \Delta_{1/k}) \\ &> h_{1/k}(0_n). \end{aligned}$$

Recall that $\tilde{f}_0^n(0_0) < 0_n$ and σ_τ as well as h_τ varies continuously with respect to τ . Thus, there exists $\tau_k \in (0, 1/k)$ such that

$$\tilde{f}_{\tau_k}^n(h_{\tau_k}(0_0)) = h_{\tau_k}(0_n).$$

Let $p_k = \Phi(h_{\tau_k}(0_0))$ and $q_k = \Phi(h_{\tau_k}(0_n))$, then it satisfies

$$f_{\tau_k}^n(p_k) = (X_{\tau_k} \circ f)^n(p_k) = q_k.$$

Moreover, by Theorem 2.10, one has that

$$\begin{aligned} d(x, p_k) &\leq d(x, x_0) + d(x_0, p_k) < \varepsilon + L \cdot \tau_k < \frac{L + 1}{k}, \\ d(y, q_k) &\leq d(y, x_n) + d(x_n, q_k) < \varepsilon + L \cdot \tau_k < \frac{L + 1}{k}. \end{aligned}$$

In particular, when $x \in \text{CR}(f)$, that is, $y = x$, one has $x_0 = x_n$ as well as $0_0 = 0_n$, which implies that

$$p_k = \Phi(h_{\tau_k}(0_0)) = \Phi(h_{\tau_k}(0_n)) = q_k.$$

As a consequence, one has that $f_{\tau_k}^n(p_k) = (X_{\tau_k} \circ f)^n(p_k) = p_k$.

By taking $k \rightarrow \infty$, this allows ε to be arbitrarily small for the ε -center pseudo orbit Γ . This finishes the proof of Theorem A. □

3.3. Proof of Corollary B. Based on Theorem A, the proof of item (1) in Corollary B is obtained from the classical Baire arguments, see for instance [4, §5] and [13, §3]. Items (2) and (3) are direct consequences of item (1). We provide a short proof for completeness of the paper. As we have mentioned, the reason that we consider the set $\text{DEPH}_1^r(M)$ in Corollary B is because $\text{DEPH}_1^r(M)$ forms an open set in $\text{PH}^r(M)$.

Proof of Corollary B. As a consequence of [12, Theorem A], there exists a dense G_δ subset \mathcal{R}_0 in $\text{DEPH}'_1(M)$ such that every $f \in \mathcal{R}_0$ satisfies $\Omega(f) = \overline{\text{Per}(f)}$.

Take a countable basis $\{O_n\}_{n \in \mathbb{N}}$ of M and let $\{W_n\}_{n \in \mathbb{N}}$ be the countable collection where every W_n is the union of finitely many elements in $\{O_n\}_{n \in \mathbb{N}}$. For each $n, m \in \mathbb{N}$, one defines the following two open sets $\mathcal{H}_{n,m}$ and $\mathcal{N}_{n,m}$ in $\text{DEPH}'_1(M)$:

- $f \in \mathcal{H}_{n,m}$ if there exists a neighborhood \mathcal{U} of f in $\text{DEPH}'_1(M)$ such that for any $g \in \mathcal{U}$, there exists $k \geq 1$ satisfying $g^k(W_n) \cap W_m \neq \emptyset$;
- $f \in \mathcal{N}_{n,m}$ if there exists a neighborhood \mathcal{U} of f in $\text{DEPH}'_1(M)$ such that for any $g \in \mathcal{U}$ and for any $k \geq 1$, it satisfies $g^k(W_n) \cap W_m = \emptyset$.

It is not difficult to verify that $\mathcal{N}_{n,m} = \text{Int}(\text{DEPH}'_1(M) \setminus \mathcal{H}_{n,m})$, where $\text{Int}(\cdot)$ means taking the interior. Thus, $\mathcal{H}_{n,m} \cup \mathcal{N}_{n,m}$ is open dense in $\text{DEPH}'_1(M)$. Let

$$\mathcal{R} = \mathcal{R}_0 \cap \left(\bigcap_{n,m \in \mathbb{N}} (\mathcal{H}_{n,m} \cup \mathcal{N}_{n,m}) \right).$$

Then \mathcal{R} is a dense G_δ subset in $\text{DEPH}'_1(M)$. We claim that the conclusions of Corollary B hold for $f \in \mathcal{R}$.

Let $f \in \mathcal{R}$. Assume $x \not\prec_f y$. For any neighborhood U of x and any neighborhood V of y , take $n, m \in \mathbb{N}$ such that $W_n \subset U$ and $W_m \subset V$. By Theorem A, for any neighborhood \mathcal{U} of f in $\text{DEPH}'_1(M)$, there exists $g \in \mathcal{U}$ and $k \geq 1$ such that $g^k(W_n) \cap W_m \neq \emptyset$. This implies that $f \in \mathcal{H}_{n,m}$ since $f \in \mathcal{H}_{n,m} \cup \mathcal{N}_{n,m}$. As a consequence, there exists $\ell > 1$ such that $f^\ell(U) \cap V \neq \emptyset$ by the definition of $\mathcal{H}_{n,m}$ together with the fact $W_n \subset U$ and $W_m \subset V$. Since U and V are taken arbitrarily, one has $x \prec_f y$. This proves item (1).

By item (1), for any $x \in \text{CR}(f)$, it satisfies $x \prec_f x$ which is equivalent to say $x \in \Omega(f)$. Thus, one has $\text{CR}(f) = \Omega(f)$. Since $f \in \mathcal{R}$, one has $\Omega(f) = \overline{\text{Per}(f)}$. This verifies item (2).

To prove item (3), assume $f \in \mathcal{R}$ is chain-transitive. For any two non-empty open sets $U, V \subset M$, there exists $x \in U$ and $y \in V$ such that $x \not\prec_f y$. By item (1), one has $x \prec_f y$, thus there exists $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$. This proves that f is transitive. □

Acknowledgments. We are very grateful to the anonymous referees for their valuable suggestions. Y.S. was partially supported by National Key R&D Program of China (2020YFE0204200, 2021YFA1001900), NSFC (12071007, 11831001, 12090015) and the Fundamental Research Funds for the Central Universities. X.W. was partially supported by National Key R&D Program of China (2021YFA1001900), NSFC (12071285) and Innovation Program of Shanghai Municipal Education Commission (No. 2021-01-07-00-02-E00087).

REFERENCES

[1] D. V. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Tr. Mat. Inst. Steklova* **90** (1967), 209pp (in Russian).
 [2] M.-C. Arnaud. Création de connexions en topologie C^1 . *Ergod. Th. & Dynam. Sys.* **21** (2001), 339–381.
 [3] M. Asaoka and K. Irie. A C^∞ closing lemma for Hamiltonian diffeomorphisms of closed surfaces. *Geom. Funct. Anal.* **26** (2016), 1245–1254.

- [4] C. Bonatti and S. Crovisier. Récurrence et genericité. *Invent. Math.* **158** (2004), 33–104.
- [5] C. Bonatti and L. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Ann. of Math. (2)* **143** (1996), 357–396.
- [6] M. Brin. Topological transitivity of a certain class of dynamical systems, and flows of frames on manifolds of negative curvature. *Funktsional. Anal. i Prilozhen.* **9** (1975), 9–19 (in Russian).
- [7] D. Cristofaro-Gardiner, R. Prasad and B. Zhang. Periodic Floer homology and the smooth closing lemma for area-preserving surface diffeomorphisms. *Preprint*, 2022, [arXiv:2110.02925](https://arxiv.org/abs/2110.02925).
- [8] S. Crovisier. Periodic orbits and chain-transitive sets of C^1 -diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **104** (2006), 87–141.
- [9] S. Crovisier. Perturbation de la dynamique de difféomorphismes en petite régularité. *Astérisque* **354** (2013), 1–147.
- [10] O. Edtmair and M. Hutchings. PFH spectral invariants and C^∞ closing lemmas. *Preprint*, 2022, [arXiv:2110.02463](https://arxiv.org/abs/2110.02463).
- [11] S. Gan and Y. Shi. A chain transitive accessible partially hyperbolic diffeomorphism which is non-transitive. *Proc. Amer. Math. Soc.* **146** (2018), 223–232.
- [12] S. Gan and Y. Shi. C^r -closing lemma for partially hyperbolic diffeomorphisms with 1D-center bundle. *Adv. Math.* **407** (2022), Paper no. 108553.
- [13] S. Gan and L. Wen. Heteroclinic cycles and homoclinic closures for generic diffeomorphisms. *J. Dynam. Differential Equations* **15** (2003), 451–471.
- [14] N. Gourmelon. Adapted metrics for dominated splittings. *Ergod. Th. & Dynam. Sys.* **27** (2007), 1839–1849.
- [15] C. Gutierrez. A counter-example to a C^2 -closing lemma. *Ergod. Th. & Dynam. Sys.* **7** (1987), 509–530.
- [16] A. Hammerlindl. Leaf conjugacies on the torus. *Ergod. Th. & Dynam. Sys.* **33** (2013), 896–933.
- [17] A. Hammerlindl and R. Potrie. Pointwise partial hyperbolicity in three-dimensional nilmanifolds. *J. Lond. Math. Soc. (2)* **89** (2014), 853–875.
- [18] S. Hayashi. Connecting invariant manifolds and the solution of the C^1 stability and Ω -stability conjectures for flows. *Ann. of Math. (2)* **145** (1997), 81–137.
- [19] M. Hirsch, C. Pugh and M. Shub. *Invariant Manifolds (Lecture Notes in Mathematics, 583)*. Springer Verlag, Berlin, 1977.
- [20] H. Hu, Y. Zhou and Y. Zhu. Quasi-shadowing for partially hyperbolic diffeomorphisms. *Ergod. Th. & Dynam. Sys.* **35** (2015), 412–430.
- [21] S. Kryzhevich and S. Tikhomirov. Partial hyperbolicity and central shadowing. *Discrete Contin. Dyn. Syst.* **33** (2013), 2901–2909.
- [22] R. Mañé. An ergodic closing lemma. *Ann. of Math. (2)* **116** (1982), 503–540.
- [23] R. Mañé. On the creation of homoclinic points. *Publ. Math. Inst. Hautes Études Sci.* **66** (1988), 139–159.
- [24] R. Mañé. A proof of the C^1 stability conjecture. *Publ. Math. Inst. Hautes Études Sci.* **66** (1988), 161–210.
- [25] S. Martinchich. Global stability of discretized Anosov flows. *Preprint*, 2023, [arXiv:2204.03825](https://arxiv.org/abs/2204.03825).
- [26] R. Potrie. Partial hyperbolicity and foliations in T^3 . *J. Mod. Dyn.* **9** (2015), 81–121.
- [27] C. Pugh. The closing lemma. *Amer. J. Math.* **89** (1967), 956–1009.
- [28] L. Wen and Z. Xia. C^1 connecting lemmas. *Trans. Amer. Math. Soc.* **352** (2000), 5213–5230.
- [29] J.-C. Yoccoz. Recent developments in dynamics. *Proc. Int. Congr. Math.* **1(2)** (1994), 246–265.