The Revolving Chain.

By Sir G. GREENHILL.

(*Read* 13*th November* 1908. *Received* 26*th January* 1909).

The chain is supposed to be rotating bodily about a vertical axis with constant angular velocity, and to have taken up a shape of relative equilibrium ; the links are like separate pendulums jointed together, and the condition is investigated where the chain is composed of one, two, three, or any number of such pendulum links; finally the passage is considered to the ultimate case of a chain of small links, which may be considered as a continuous flexible cord.

Such a chain looked at sideways appears to oscillate through the straight vertical position; and so when the deviation is small, it may be used to illustrate the theory of small oscillation of a chain of links hanging vertically.

I. (*Illustrate by a chain as in Fig. 6*).

The problem of the pendulum oscillation of two particles attached to a thread is considered in Lagrange's *Mécanique Analytique*, I. p. 353; it was resumed by Sir W. Thomson for its practical application to the "Rate of a Clock or Chronometer as influenced by the mode of suspension," republished in his *Popular Lectures and Addresses*, vol. II.

The treatment can be simplified by considering the analogous case of the two particles rotating round the vertical, as a double conical pendulum.

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1. Begin with the case of a single plummet P of weight W, swinging round in a horizontal circle of radius $NP = y$ at the end of a thread CP of length l, attached at C to the vertical axis of rotation CN (Fig. 1), and denote NC, the height of the conical pendulum by λ, and the constant angular velocity by n.

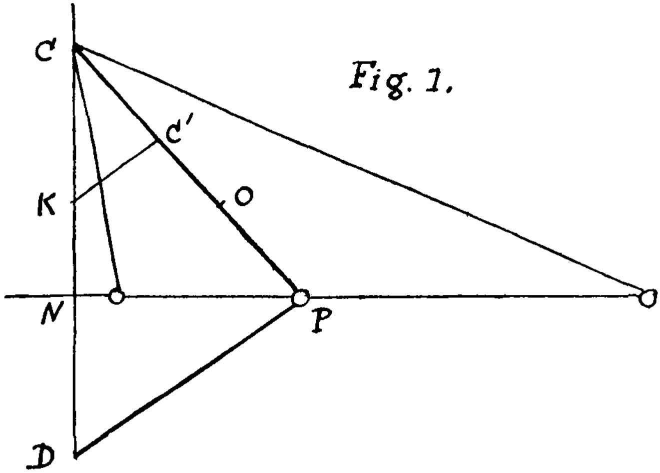


Fig. 1.

II. (*Whirl the pendulum fast, so as to fly out*).

Treating the state of steady motion statically, and employing the statical gravitation unit of force in a field of gravity of strength g, the triangle of force CNP shows that the tension of the thread is Wl/λ, and the centrifugal force (C.F.) or kinetic reaction along NP is Wy/λ; so that the central acceleration of P is gy/λ, and equating this to yn^2,

(1) $$\lambda = \frac{g}{n^2} = g\frac{T^2}{4\pi^2}$$

if T is the period of a revolution in seconds; or

(2) $$T = 2\pi\sqrt{\frac{\lambda}{g}},$$

and when the deviation is small and λ may be replaced by l,

(3) $$T = 2\pi\sqrt{\frac{l}{g}},$$

the ordinary formula for the period or double beat of a pendulum of length l.

III. (Show the pendulum almost vertical).

But g has to be determined experimentally, and this is done by an observation of l and T; and then

$$(4) \qquad g = \frac{4\pi^2}{T^2} l.$$

According to experiment incorporated in the Act of Parliament, with the idea of being utilised in case the standard of length was destroyed, the length of the pendulum which beats the second in a vacuum at sea-level in London is

$$(5) \qquad L = 39{\cdot}1396 \text{ inches},$$

and putting $T = 2$, this makes

$$(6) \qquad g = \pi^2 L = 386{\cdot}28 \text{ inches/sec}^2 = 32{\cdot}19\text{ft./sec.}$$

The formula (1) may now be written, more simply as not involving π,

$$(7) \qquad \lambda = \tfrac{1}{4} L T^2, \quad T = 2\sqrt{\frac{\lambda}{L}},$$

and this is logically the proper form, as it is L which is determined experimentally and not g in accurate measurement, and g is derived from l or L by the formula (4) and (6).

The thread CP may be supposed held at any point O, instead of C on the axis, without altering the conditions; but λ the height of the equivalent conical pendulum is still NC.

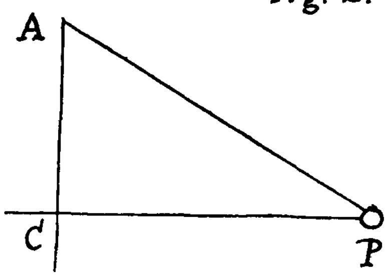
It is supposed that there is no air resistance, so that the thread OP lies in the vertical plane through the axis.

But if the thread is attached to an arm CA, shown in plan in Fig. 2, the conical pendulum may be used as a Whirling Machine, for measuring the resistance R of the air; and

$$(8) \qquad \frac{R}{W} = \frac{CA}{CN},$$

when the thread takes up the direction AP, and the length is adjusted so that CA is at right angles to the vertical plane CNP.

Fig. 2.



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Taking the deflection from the vertical as small, the circular motion of O in a small circle of radius a will cause P at the end of a thread OP of length l to describe a circle of radius $NP = y$, where

$$\frac{y}{a} = \frac{CP}{CO} = \frac{\lambda}{\lambda - l}; \tag{9}$$

but $\lambda - l$ must not be small, or the approximation breaks down, in consequence of *resonance.*

When $\lambda - l$ is negative in consequence of an increase in n, the plummet P will cross to the other side of the vertical axis.

IV. (Illustrate experimentally with rotating watch chain, twirled by a string).

This curious reversal of centrifugal force with increase of angular velocity is utilised in the balance of the Laval steam turbine, where the shaft carrying a wheel slightly out of balance recovers its stability at full speed.

In the equivalent plane oscillation of small extent, the forcing rectilinear vibration, $a\sin nt$ of O, will give a forced vibration

$$\frac{\lambda}{\lambda - l} a\sin nt$$

to P provided $\lambda - l$ is not small as in resonance, say of troops marching on a suspension bridge; and P will oscillate with O or against it, according as λ is greater or less than l.

If the plummet P in Fig. 1 is held by another thread PD so that it cannot fly out to its full distance when free under the influence of the rotation n, NK must be measured upward of length $\lambda = g/n^2$, now less than NC; KC′ is drawn parallel to DP, and PC′KN is taken as a polygon of force in which the sides KN, NP, PC′, C′K represent the weight W, the C.F. Wy/λ, and the tension in PC and PD.

2. Two or more plummets may be fastened on the same thread, and whirled round in a vertical plane with constant angular velocity n.

Starting with the lowest plummet at P_1 in Fig. 3, describing a horizontal circle of radius $N_1P_1 = y_1$, round the centre N_1 on the vertical axis of rotation, measure N_1C_1 vertically upward of length $\lambda = g/n^2$, the height of the equivalent conical pendulum; then P_1C_1 is the direction of the supporting thread as there is nothing below P_1 to control it.

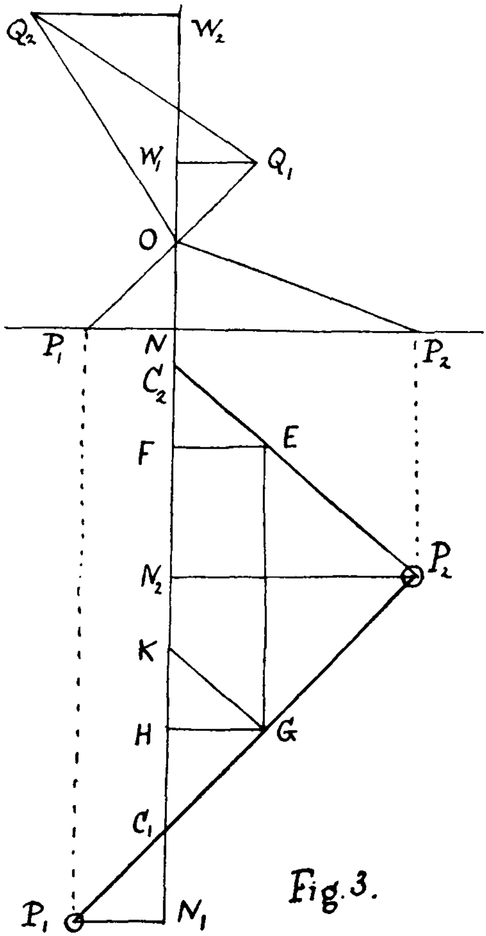

Fig. 3.

If P_2 is the plummet at the upper end of the first thread, and G the C.G. of P_1 and P_2, of weight W_1 and W_2, describing a horizontal circle of radius HG round the centre H, measure $HK = \lambda$ vertically upward, then the second thread above P_2 must take a direction P_2P_3 parallel to GK, meeting the vertical axis in C_2 suppose.

Thus for these two particles, suspended at C_2, and whirling round the vertical (as in Fig. 3).

(1) $$N_1C_1 = HK = FC_2 = \lambda,$$

and denoting the length of thread P_1P_2 and P_2C_2 by l_1 and l_2, and by δ_1, δ_2 the inclination of the thread to the vertical

$$(2) \qquad C_1P_2 = \lambda\sec\delta_1 - l_1,\ GP_2 = \frac{W_1}{W_1+W_2}l_1,\ P_2E = \lambda\sec\delta_2 - l_2\,;$$

and by similar triangles

$$(3) \qquad \frac{P_2E}{GP_2} = \frac{P_2C_2}{C_1P_2}, \quad \frac{\lambda\sec\delta_2 - l_2}{\dfrac{W_1}{W_1+W_2}l_1} = \frac{l_2}{\lambda\sec\delta_1 - l_1}$$

leading to

$$(4) \qquad (\lambda\sec\delta_1 - l_1)(\lambda\sec\delta_2 - l_2) = \frac{W_1}{W_1+W_2}l_1l_2,$$

a quadratic equation for λ.

When each thread is nearly vertical so that $\sec\delta_1$ and $\sec\delta_2$ may be replaced by unity, the same equation will serve for small plane oscillation to give λ the length of the equivalent simple pendulum, and so we arrive at the fundamental equation of Lagrange

$$(5) \qquad (\lambda - l_1)(\lambda - l_2) = \frac{W_1}{W_1+W_2}l_1l_2\,;$$

and the discussion of this equation is the object of Sir W. Thomson's paper, also of a communication by Professor Stokes to vol. I., p, 1, of the *Messenger of Mathematics*, 1871, with the title "Explanation of a Dynamical Paradox," being the solution of a question proposed in the Smith's Prize Examination, 1871.

Put $l_1 + l_2 = 2a$, the total length of the thread, and $l_1 - l_2 = 2x$, so that x is the distance of P_2 from the mid point, the quadratic (5) becomes, on putting $W_1/W_2 = \mu$,

$$(6) \qquad \lambda^2 - 2a\lambda + \frac{a^2 - x^2}{1+\mu} = 0,\ (\lambda - a)^2 = \frac{\mu a^2 + x^2}{1+\mu},$$

so that it is not possible for the roots of the quadratic to be equal unless μ is negative.

Consequently the arguments of Professor Stokes's paper apply only to the case where the lower plummet P_1 is a float immersed in water; and the limits of x are such as to lie outside $a\sqrt{(-\mu)}$ for a vibration to exist, in which the thread P_2P_1 transmits a thrust which urges P_1 back to the vertical against its buoyancy.

The form of the quadratic (6) shows that the four cases discussed by Sir W. Thomson are shown better with the figure altered, as in Fig. 4, drawn for $W_2 = 2W_1$, $\mu = \frac{1}{2}, \frac{x}{a} = \pm\frac{1}{2}$,

$$\frac{\lambda}{a} = 1 \pm \tfrac{1}{2}\sqrt{2}, \quad \lambda = C_1P_1.$$

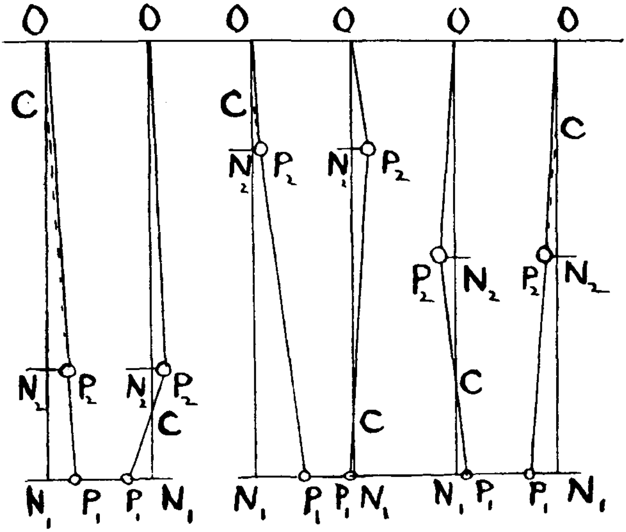

Fig. 4.

Also for $x = 0$, $\frac{\lambda}{a} = 1 \pm \frac{1}{3}\sqrt{3}$, or $1 \pm \sqrt{\frac{\mu}{1+\mu}}$, in general, when the ratio of the periods, $\sqrt{(1+\mu)} + \sqrt{\mu}$, is least, and the two values of λ are as near together as possible, but they do not cross, as asserted by Stokes. So also with $W_2 = 3W_1$ and $x = 0$, $\lambda = \frac{3}{2}a$, $\frac{1}{2}a$.

V. (Experiment with a thread and two weights).

An appropriate suspension of a body to swing as a pendulum is the simplest way to realise experimentally the vibration of the body as controlled by a spring; if the resistance of the spring is Ex for a displacement x, E being called the strength or stiffness of the spring, a weight W will vibrate like a pendulum of length

$$(7) \qquad l = \frac{W}{E}, \quad \text{or} \quad Ex = W\frac{x}{l}.$$

Suppose, for instance, the point of support of the pendulum is controlled by a light spring which yields horizontally; the equivalent pendulum length l is increased by c, the deflection of the spring vertical produced by the weight of the pendulum.

Any elastic yielding of the support of a pendulum requires careful investigation in accurate experiments; Kater's experiments are not considered now to possess the accuracy they claim, from imperfect knowledge of the influence of the support; and it is the effect of the support on the rate of a clock or chronometer which is the object of Sir W. Thomson's investigations, cited above.

The double pendulum of the two plummets P_1 and P_2 will represent the vibration of the weights W_1 and W_2, when the spring connecting W_1 and W_2 is of strength W_1/l_1, and the spring which connects W_2 with the support has a strength W_2/l_2; we see this realised when two railway carriages with spring coupling impinge on an elastic buffer stop; the system is seen to deal a double blow.

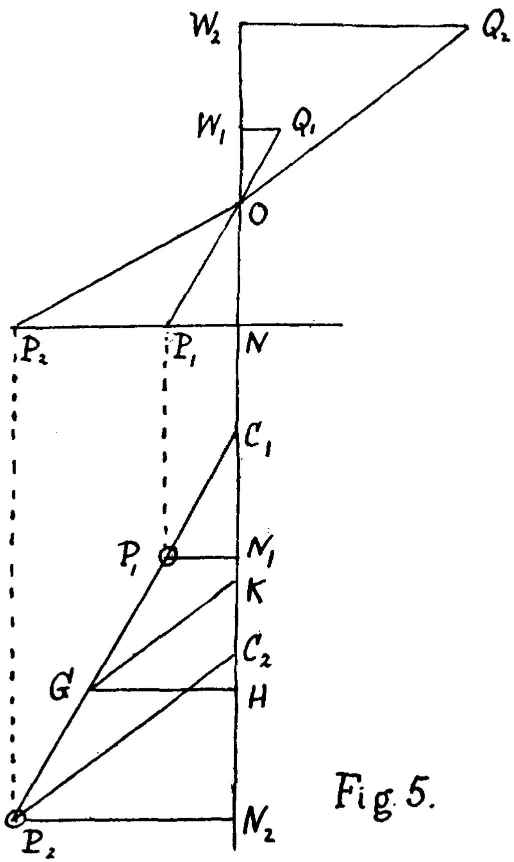

Fig. 5.

The thread P_1P_2 may be replaced by a light stiff wire capable of transmitting a thrust, as in the modification of Stokes's pendulum, and now it is possible for P_1 to be balanced with P_2P_1 pointing upward, as in fig 5, and the condition changes to

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(8) $$(l_1 + \lambda\sec\delta_1)(l_2 - \lambda\sec\delta_2) = \frac{W_1}{W_1 + W_2} l_1 l_2,$$

by a change of sign of l_1 in (4).

It is possible then to balance a rod AB at a constant inclination to the vertical by giving the lower end B a circular motion, and when the rod is nearly vertical, the motion of B may be changed into a vibration to and fro; the rod being replaced in the calculation by two particles kinetically equivalent, one at B for P_2, and the other at P_1, the centre of oscillation of the rod with respect to B.

VI. (Experiment with a billiard cue or broomstick, the easier with the longest).

It should be possible in this way to make a small pendulum P_2P_1 balance almost upright, while P_2 is carried to and fro by a large pendulum; like a man standing up freely on his feet in a swing, or an acrobat on a wire; also to balance a jar of water on the head.

3. The same method will apply to any number of plummets revolving steadily in one vertical plane with uniform angular velocity n; the direction of the thread above any plummet P_m is obtained by determining G the C.G. of P_m and all the plummets below, and drawing GH horizontal to meet the axis in H, and measuring HK vertically upward of length $\lambda = g/n^2$, the height of the equivalent conical pendulum, and drawing GK: the thread above P_m must be in a direction parallel to GK; in this way, starting from the lowest weight, the figure can be constructed.

A diagram of force is obtained in Fig. 3 by taking an origin O, and drawing $ON = \lambda$ vertically downward, and supposing each weight P lifted vertically on to the horizontal line through N; then OP represents the resultant of gravity and C.F. when ON represents the weight W.

Next prolong NO vertically upward to represent the line of load, the segments representing successively the weight of the plummets; and drawing the horizontal lines such as WQ, draw OQ_1, $\dot{Q}_1Q_{21}Q_2Q_{31}\ldots$ parallel to OP_1, OP_2, $OP_3,\ldots$; then OQ_1, OQ_2, $OQ_3\ldots$ will represent the tension of each thread in succession, in direction and magnitude, as shown in Figs. 3 and 5.

Suppose $P_1P_2P_3\ldots$ is a funicular polygon, the transference of the weights to the horizontal line through N will give a string of weights, held on a thread of uniform tension equal to the horizontal component in the funicular polygon, which will imitate the oscillation of the weights in the funicular polygon perpendicular to its plane.

4. When the string of weights is replaced by a continuous chain, the same principle will apply; the tangent PT at any point P is parallel to GK, where G is the C.G. of the chain PA from P down to the free end A, and AC being the tangent at A (Fig. 6),

$$HK = OC = \lambda = g/n^2.$$

VII. (Experiment with a revolving chain).

The curve thus satisfies the relation

$$(1) \qquad \frac{dy}{dx} + \frac{\int y\,ds}{\lambda s} = 0.$$

$$(2) \qquad \lambda s \tan\psi + \int\int \sin\psi\, ds^2 = 0$$

$$(3) \qquad \lambda \frac{d^2}{ds^2}(s\tan\psi) + \sin\psi = 0\,;$$

but this is intractable analytically when the deviation from the vertical is considerable (E. B. Wilson, *Bulletin of the American Mathematical Society*, 1908).

But when the chain is coincident with the vertical and straight so nearly, that s may be replaced by x,

$$(4) \qquad \lambda x\frac{dy}{dx} + \int y\,dx = 0,$$

or, putting $x = \frac{1}{4}\lambda z^2$,

$$(5) \qquad z\frac{dy}{dz} + \int 2yaz = 0$$

$$(6) \qquad \frac{d}{dz}\left(z\frac{dy}{dz}\right) + zy = 0$$

$$(7) \qquad \frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + y = 0,$$

which is identified with Bessel's differential equation

$$(8) \qquad \frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{n^2}{z^2}\right)y = 0,$$

with $n=0$; so that the solution, subject to the condition that y is finite when x and z is zero, is

(9) $$y = bJ_0(z) = bJ_0\left(2\sqrt{\frac{x}{\lambda}}\right).$$

According to the graph of $J_0(z)$ at the end of *Bessel Functions* by Gray and Mathews, the first roots of $J_0(z)=0$ are, by eye,

(10) $$z = 2.4,\ 5.5,\ 8.6,\ 11.8,\ 14.9,\ 18.0,\ \ldots$$

the increment difference approximating to π, so that if N is the number of revolutions per second of a conical pendulum, composed of a plummet at the end of a thread of length l when the deviation from the vertical is small, then for a chain of length l the revolutions R per second for a steady motion at a small inclination to the vertical will be

(11) $$R = \tfrac{1}{2}Nz = \tfrac{1}{2}N(2.4,\ 5.5,\ 8.6,\ \ldots)$$

and the nodes are given by

(12) $$\frac{n}{\lambda} = \tfrac{1}{4}z^2 = 1.44,\ 7.56,\ 18.49,\ 34.81,\ 55.5,\ 81,\ \ldots.$$

VIII. (Experiment with the chain)

The chain is vertical where

(13) $$J_1(z)=0,\ z = 3.8,\ 7.0,\ 10.2,\ 13.3,\ \ldots$$

(14) $$\frac{x}{\lambda} = 3.61,\ 12.25,\ 26.01,\ 44.22,\ \ldots$$

and then the chain below has its C.G. on the axis.

Stretch the scale of z in the Gray-Mathews figure for $J_0(z)$ so that the extension at any point is proportional to z, and we have the curve given for the profile of the rotating chain (Fig. 6) by

$$J_0\left(2\sqrt{\frac{x}{\lambda}}\right).$$

5. When gravity is neglected, as we may suppose if the rotation is rapid, the form of the chain, a skipping rope for instance, is given in the general case by an elliptic function,

(1) $$\frac{y}{b} = \operatorname{sn}K\frac{x}{a},$$

(2) $$\kappa = \frac{b}{\sqrt{(4\lambda h + b^2)}},\quad K = \frac{a\sqrt{(4\lambda h + b^2)}}{2\lambda h},\quad \kappa K = \frac{ab}{2\lambda h},$$

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(*Applications of the Elliptic Function*, p. 68) where $2a$ is the distance between successive nodes on the axis, and h is the *tension-length* of the constant axial component of the tension.

When the chain is nearly straight, so that b and κ are small, and K may be replaced by $\frac{1}{2}\pi$,

$$(3)\qquad \lambda h = \frac{a^2}{(\frac{1}{2}\pi)^2},\ \mathrm{R} = \tfrac{1}{4}\sqrt{\frac{gh}{a^2}},$$

where R denotes the number of revolutions or double vibrations per second of a chain of length $2a$ stretched to a tension length h; this is equivalent to saying that the velocity of propagation along the chain of a transversal vibration is $\sqrt{(gh)}$, a velocity due to a head of half the tension length.

The proof may be given in an elementary manner by saying that the transverse component of the tension at the nodes must balance the C.F. of the chain between, of length $2a$ and weight $2wa$ with line density w; and the chain may be concentrated into a particle at its C.G., at a distance $b/\frac{1}{2}\pi$ from the axis, equal to the mean-average y in the curve

$$(4)\qquad \frac{y}{b} = \sin\tfrac{1}{2}\pi\frac{x}{a},$$

so that the condition is

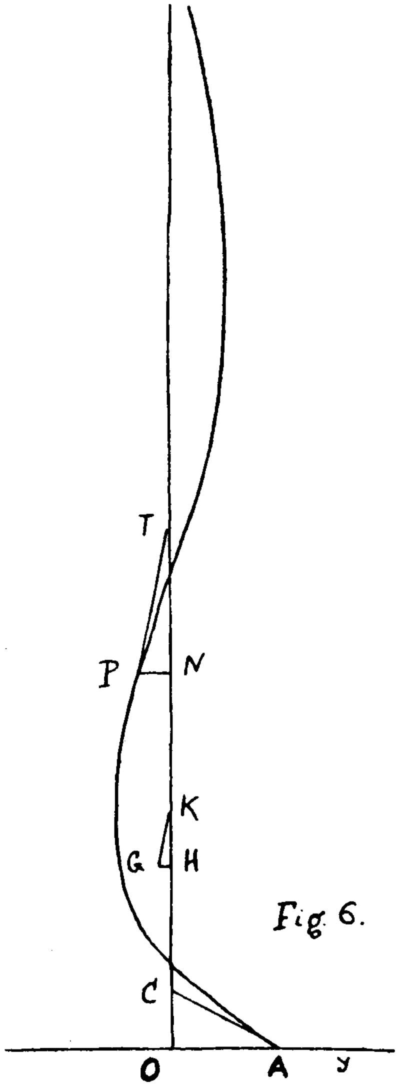

Fig 6.

(5) $$2wh\tfrac{1}{2}\pi\frac{b}{a} = 2wa\frac{b}{\tfrac{1}{2}\pi\lambda}, \text{ or } \lambda h = \frac{4a^2}{\pi^2},$$

as before in (3).

So also in an endless chain in the form of a circle, moving with circumferential velocity v like the rim of a wheel, the tension length h is given by

(6) $$v^2 = gh,\ h = \frac{v^2}{g},$$

so that the bursting velocity, as of a tire, is due to a head of half the breaking length.

The relative velocity of transversal waves in this chain will be $\pm v$; combined with the velocity of the chain this gives two wave sets, one propagated with double velocity, which pass too swift to be noticed, and the other stationary and visible, as in the Aitken-Thomson experiment described in Perry's *Spinning Tops*.

Two or three lengths of bicycle chain with the ends joined would be suitable for this experiment, suspended over a bicycle wheel.

The form of the revolving chain as given in (1) may be deduced by the Calculus of Variations from the condition that the chain assumes the shape of maximum moment of inertia about the axis; in the more general case where the chain is a tortuous curve, not lying in one plane through the axis, the Third Elliptic Integral is required.

Spin the chain, and then draw the ends apart gradually; the angular velocity varies inversely as the moment of inertia and so increases, a simple experimental illustration of the Conservation of Angular Momentum.

IX. (Experiment with a whirling chain, gradually drawn out straight).

A piece of chain should be employed in these experiments on the catenary curve, and not the string of the text books; string makes a very bad funicular catenary, and a much better realisation is seen in the telephone or telegraph wire.

6. Maxwell's principle employed above in §2, by which a body in uniplanar motion is treated as the pair of particles kinetically equivalent, is useful when the thread and plummet is replaced by a link, and these links build up a chain.

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Begin in Fig. 7 with a single pendulum link AB, movable about a horizontal axle through C, and swung round a vertical axis Cx with constant angular velocity n; it can assume a position inclined to the vertical at an angle δ, where gravity and C.F. have equal moment about C.

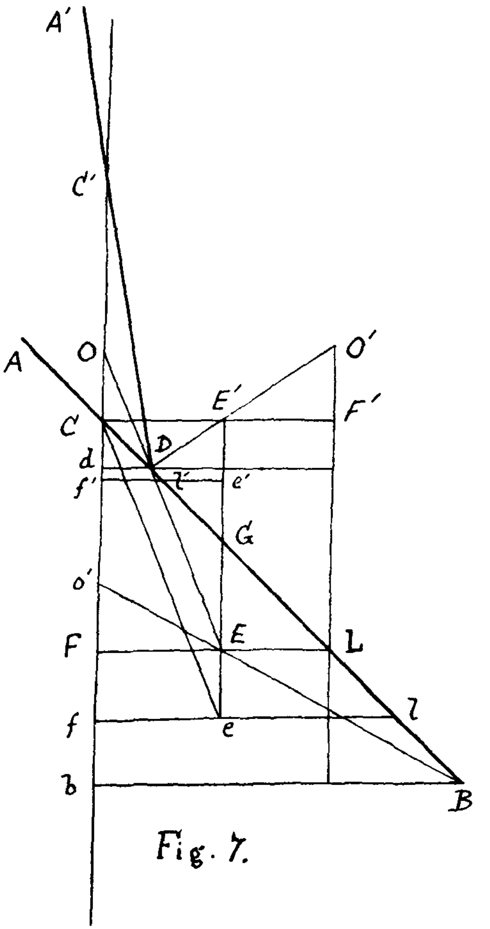

Fig. 7.

Denoting CG by h, and CL the equivalent pendulum length l, the two particles kinetically equivalent may be placed one at L, of weight $W\frac{h}{l}$, and the other at C, of weight $W\left(1-\frac{h}{l}\right)$, W denoting the weight of the pendulum.

The pendulum then swings round the vertical at the same angle as a particle plummet at L, where (Fig. 7)

(1) $$CF=\lambda=\frac{g}{n^2},\ \cos\delta=\frac{CF}{CL}=\frac{\lambda}{l}=\frac{g}{ln^2},$$

and so it cannot leave the vertical till $\lambda < l$.

The reaction at C is the resultant of the C.F.

(2) $$W\frac{h}{l}\cdot\frac{FL}{\lambda}=W\frac{h}{\lambda}\sin\delta,$$

acting horizontally in FEL, and of $W\frac{h}{l}$ vertical through L, and $W\left(1-\frac{h}{l}\right)$ vertical through C, equivalent to W vertical through G, so that the resultant pull on the axle at C is along CE, where E is the intersection of the vertical through G and the horizontal EL.

Any other line EDO through E may be taken as a line of constraint, say of a thread fastened to the pendulum at D and to the vertical axis at O; but now the angular velocity n for this configuration must be changed so that

$$\text{(3)} \qquad \mathrm{OF} = \lambda = g/n^2,$$

and the tension of the thread is $W\sec\phi$, where ϕ denotes the angle EOF.

Thus if AB is to be moved round the vertical by a support at B, the force must be applied along BE, and the angular velocity n must be such that

$$\text{(4)} \qquad bo' = g/n^2.$$

The convertibility of C and L shows that the body can revolve at the same inclination about the vertical axis through L, and then the supporting thread at D must take the direction DE′O′, and LO′ must be the height of the equivalent conical pendulum.

X. (Experiment with a stick, whirled round at the end of a string).

Draw Ce parallel to OE to meet the vertical through G in e, and draw fel horizontal; then

$$\text{(5)} \qquad \mathrm{CO} = \mathrm{E}e = \mathrm{F}f, \ \mathrm{C}f = \mathrm{OF} = \lambda;$$

$$\text{(6)} \qquad \frac{\mathrm{CG}}{\mathrm{DG}} = \frac{\mathrm{G}e}{\mathrm{GE}} = \frac{\mathrm{G}l}{\mathrm{GL}}, \ \mathrm{CG}\,.\,\mathrm{GL} = \mathrm{DG}\,.\,\mathrm{G}l,$$

so that Dl is the pendulum length when D is the point of suspension, and then Cf is the height of the equivalent conical pendulum. Also

$$\text{(7)} \qquad \frac{\mathrm{OD}}{\mathrm{CD}} = \frac{\mathrm{DE}}{\mathrm{DG}}, \ \mathrm{CD}\,.\,\mathrm{DE} = \mathrm{OD}\,.\,\mathrm{DG};$$

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and now putting $OD = a$, $DG = h$, $DL = l$,

$$(8) \quad CD = Cl - Dl = \lambda\sec\delta - l$$

$$(9) \quad DE = OE - OD = \lambda\sec\phi - a,$$

so that

$$(10) \quad (\lambda\sec\delta - l)(\lambda\sec\phi - a) = ah,$$

the condition for steady motion, reducing for small value of δ and ϕ to

$$(11) \quad (\lambda - l)(\lambda - a) = ah,$$

the quadratic for λ requisite in small conical motion, or is the plane oscillation seen sideway, due either to a suspension by the thread DC, as in a boat hung from the davits, or to a forced vibration given by D, as a hand carrying a bag.

If B had been the support, instead of D and on the opposite side of G, the sign of l and h must be changed, so that

$$(12) \quad (l + \lambda)(a - \lambda) = ah, \text{ or } \lambda^2 + (l - a)\lambda - a(l - h) = 0, \text{ with } l - h = \frac{k^2}{h},$$

so that the quadratic for λ has real roots; and this gives the condition for the balance of the rod AB on its lowest point B, by a thread of length $Bo' = a$.

A small circular or to and fro movement of a hand supporting the rod at B, of given frequency, must be adjusted so that $Cl' = \lambda\sec\delta$, where l' is the centre of oscillation of the rod with respect to B, and the thrust of the hand must then be directed to o', where $Bo' = a$.

XI. (Experiment with a uniform rod like a broomstick.)

We may take the rod of length $2h$, and then

$$(13) \qquad l = \frac{4}{3}h, \quad a = \frac{l + \lambda}{l + \lambda - h}\lambda = \frac{\frac{4}{3}h + \lambda}{\frac{1}{3}h + \lambda}\lambda.$$

The experiment may also be made in a swing or on a wire, though this is dangerous for an amateur; it will be noticed that the acrobat arranges the experiment so that his head is nearly stationery during the oscillation, as at C; and he can alter his l by a movement of the arms.

7. When this pendulum AB is displaced from its position of relative equilibrium where $\delta = \cos^{-1}\frac{\lambda}{l}$ so that its inclination varies between α and β, as in Watt's governor, the motion is found to be given by the equation

$$\text{(1)} \qquad \frac{d^2\theta}{dt} - \frac{g}{\lambda}\sin\theta\cos\theta = -\frac{g}{l}\sin\theta,$$

$$\text{(2)} \qquad \left(\frac{d\theta}{dt}\right)^2 = \frac{g}{\lambda}(\cos\theta - \cos\alpha)(\cos\beta - \cos\theta),$$

$$\text{(3)} \qquad \frac{\lambda}{l} = \cos\delta = \tfrac{1}{2}(\cos\alpha + \cos\beta),$$

so that L oscillates to an equal extent vertically above and below its mean position; and the solution of (2) is given by the elliptic function in the form

$$\text{(4)} \qquad \tan\tfrac{1}{2}\theta = \tan\tfrac{1}{2}\alpha\,\mathrm{dn}\,mt,$$

$$\text{(5)} \qquad \kappa' = \frac{\tan\frac{1}{2}\beta}{\tan\frac{1}{2}\alpha}, \quad m = n\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta.$$

In small oscillation, where α and β close in upon δ,

$$\text{(6)} \qquad m = \tfrac{1}{2}n\sin\delta = \tfrac{1}{2}\sqrt{\frac{g}{\lambda}}\sqrt{\left(1 - \frac{\lambda^2}{l^2}\right)},$$

so that the period of oscillation of the pendulum becomes

$$\text{(7)} \qquad 4\pi\sqrt{\frac{\lambda}{g}} \cdot \frac{1}{\sqrt{\left(1 - \frac{\lambda^2}{l^2}\right)}} = 2\pi\sqrt{\frac{l}{g}} \cdot \frac{2\sqrt{(\lambda l)}}{\sqrt{(l^2 - \lambda^2)}};$$

we may suppose this realised in a clock which is spun about a vertical axis, or swung round as a conical pendulum.

When the pendulum just reaches the lowest position, $\beta = 0$, $\kappa' = 0$, and the motion is given by

$$\text{(8)} \qquad \tan\tfrac{1}{2}\theta = \tan\tfrac{1}{2}\alpha\,\mathrm{sech}\,mt,$$

and this changes to

$$\text{(9)} \qquad \tan\tfrac{1}{2}\theta = \tan\tfrac{1}{2}\alpha\,\mathrm{cn}\,mt,$$

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when the pendulum passes through the downward vertical, and swings to an equal distance on each side, as in ordinary pendulum motion, except that with a rotation about a vertical axis the modular angle is no longer $\frac{1}{2}\alpha$, but we find

$$(10) \qquad \kappa^2 = \sin^2\tfrac{1}{2}\alpha \frac{2\lambda + l(1-\cos\alpha)}{2(\lambda - l\cos\alpha)},$$

in which $\lambda = \infty$ and $\kappa = \sin\frac{1}{2}\alpha$, when there is no rotation.

When the pendulum just reaches the upward vertical, $\alpha = \pi$, we find

$$(11) \qquad \tan\tfrac{1}{2}\theta = \mathrm{Csh}\,mt,$$

changing to

$$(12) \qquad \tan\tfrac{1}{2}\theta = \mathrm{Ctn}\,mt,$$

for complete revolutions; and there is still the motion given by

$$(13) \qquad \tan\tfrac{1}{2}\theta = \tan\tfrac{1}{2}\beta\,\mathrm{nc}\,mt,$$

in which the pendulum oscillates through the highest position to an angle β on each side of the downward vertical, but this implies a negative g, and so is merely the motion in (9) upside down.

8. When the thread OD in fig. 7 is replaced by another pendulum DA′, the two may be considered the lowest pair of a chain composed of jointed links, each a pendulum.

Reckoning upward from the lowest, distinguish by the suffix m the weight W of the m^{th} link, the length a of the link as the distance $P_{m-1}P_m$ of the pivot pins, the radius of gyration k about the centre of gravity G, the length l of the equivalent pendulum when the link is suspended freely from P_m, and h the distance P_mG_m; then

$$(1) \qquad k^2 = h(l-h) = P_mG_m \,.\, G_mP_{m-1}.$$

Denote the height of the equivalent conical pendulum by λ, when the chain is swinging bodily in a vertical plane with angular velocity n, $\lambda = g/n^2$; and let the suffix m distinguish δ the inclination of the m^{th} link to the vertical, and ϕ of the reaction at the pin P_m, and y the distance of P_m from the vertical axis, and C the point on the axis where the line of a link intersects it.

Beginning with the lowest rod, free its lower end, and replacing it by the equivalent particles at P and L,

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(2) $$W\frac{h}{l}\cdot\frac{FL}{\lambda}l\cos\delta - W\frac{h}{l}l\sin\delta = 0,$$

taking moments round P ; so that

(3) $$FL = \lambda\tan\delta,\ CF = \lambda,\ CL = \lambda\sec\delta,$$

(4) $$CP = \lambda\sec\delta - l,\ y = (\lambda\sec\delta - l)\sin\delta,$$

(5) $$\tan\phi = \frac{HG}{\lambda} = \frac{y + h\sin\delta}{\lambda}\ ;$$

all with the suffix 1 to distinguish the first link from the others.

For any intermediate link, constrained by the links below, FC will not be λ for the actual revolution ; but the moment of its own C.F. and weight about its upper pin P will be

(6) $$W\frac{h}{l}\frac{FL}{\lambda}l\cos\delta - W\frac{h}{l}l\sin\delta = Wh\left(\frac{FL}{\lambda\sec\delta} - \sin\delta\right)$$

$$= Wh\left(\frac{y + l\sin\delta}{\lambda\sec\delta} - \sin\delta\right)$$

$$= W\frac{h}{\lambda\sec\delta}\Big[y - (\lambda\sec\delta - l)\sin\delta\Big]$$

and this for the m^{th} link is written

(7) $$W_m\frac{h_m}{\lambda\sec\delta_m}\left[y_m - \frac{\lambda\sec\delta_m - l_m}{a_m}(y_{m-1} - y_m)\right].$$

To this must be added the moment round P_m of the C.F. and weight of all the links below, acting at P_{m-1} ; and the sum must be equated to zero, and thereby a recurring relation is obtained for the y's ; the moment round P_m of the r^{th} link being

(8) $$W_r\frac{y_r + h_r\sin\delta_r}{\lambda}a_n\cos\delta_n - W_r a_m\sin\delta_m$$

$$= W_r\frac{a_m}{\lambda\sec\delta_m}\left[y_r + \frac{h_r}{a_r}(y_{r-}\ \ - y_r)\right] - W_m(y_{m-1} - y_m),$$

(9) $$W_1\frac{a_m}{\lambda\sec\delta_m}\left(1 + \frac{h_1}{\lambda\sec\delta_1 - l_1}\right)y_1 - W_1(y_{m-1} - y_m),$$

for the lowest rod.

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For the lowest link, with nothing suspended below it

$$(10) \qquad W_1 h_1 \frac{y_1 + l_1 \sin\delta_1}{\lambda} - W_1 h_1 \sin\delta_1 = 0,$$

$$y_1 = (\lambda\sec\delta_1 - l_1)\sin\delta_1, \quad CO_1 = \lambda\sec\delta_1 - l_1,$$

as above in (4).

For the second link

$$(11) \qquad W_2 \frac{h_2}{\lambda\sec\delta_2}\left[y_2 - \frac{\lambda\sec\delta_2 - l_2}{a_2}(y_1 - y_2)\right]$$

$$+ W_1 \frac{a_2}{\lambda\sec\delta_2}\left(1 + \frac{h_1}{\lambda\sec\delta_1 - l_1}\right) y_1 - W_1(y_1 - y_2) = 0\,;$$

and multiplying by $\frac{\lambda}{a_2}\sec\delta_2$, and collecting coefficients of y_1 and y_2,

$$(12) \qquad \left[W_1 \frac{\lambda\sec\delta_2}{a_2} + W_2 \frac{h_2}{a_2}\left(1 + \frac{\lambda\sec\delta_2 - l_2}{a_2}\right)\right] y_2$$

$$+ \left[W_1\left(1 + \frac{h_1}{\lambda\sec\delta_1 - l_1}\right) - W_1 \frac{\lambda\sec\delta_2}{a_2} - W_2 \frac{h_2}{a_2}\,\frac{\lambda\sec\delta_2 - l_2}{a_2}\right] y_1 = 0.$$

Thus $y_2 = 0$, if

$$(13) \qquad 1 + \frac{h_1}{\lambda\sec\delta_1 - l_1} - \frac{\lambda\sec\delta_2}{a_2} - \frac{W_2}{W_1}\,\frac{h_2}{a_2}\,\frac{\lambda\sec\delta_2 - l_2}{a_2} = 0,$$

or putting $\frac{W_2}{W_1}\,\frac{h_2}{a_2} = m$,

$$(14) \qquad \frac{a_2 h_1}{\lambda\sec\delta_1 - l_1} = (m+1)(\lambda\sec\delta_2 - l_2) + l_2 - a_2.$$

When $a_2 = l_2$, the relation (14) reduces to

$$(15) \qquad (\lambda\sec\delta_1 - l_1)(\lambda\sec\delta_2 - l_2) = \frac{l_2 h_1}{m+1},$$

analogous to (4) §2 for two plummets.

If $\delta_1 = \delta_2$, the links are in one line as if clamped at rest, as required with a church bell and its clapper. Then C_1 coincides with P_2, and $\lambda\sec\delta_1 - l_1 = a_2$, so that (14) reduces to

$$(16) \qquad h_1 = (m+1)(l_1 + a_2 - l_2) - a_2 + l_2$$
$$= (m+1)l_1 + m(a_2 - l_2).$$

For a uniform rod ACB, suspended at C, and jointed at D, this condition (16) reduces to

$$(17) \qquad CG \,.\, CD = CA^2,$$

where G is the midpoint C.G. of the rod ; because L the centre of oscillation of the rod DB about D must be the centre of oscillation of the whole rod AB about C, to make the bending movement zero at D.

Then C and D must be on opposite sides of G, and C must lie outside the middle third of the rod, and D outside the middle half.

For the third rod

$$(18) \qquad W_3 \frac{h_3}{\lambda\sec\delta_3}\left[y_3 - \frac{\lambda\sec\delta_3 - l_3}{a_3}(y_2 - y_3)\right]$$
$$+ W_2 \frac{a_3}{\lambda\sec\delta_3}\left[y_2 + \frac{h_2}{a_2}(y_1 - y_2)\right] - W_2(y_2 - y_3)$$
$$+ W_1 \frac{a_3}{\lambda\sec\delta_3}\left(1 + \frac{h_1}{\lambda\sec\delta_1 - l_1}\right)y_1 - W_1(y_2 - y_3) = 0$$

Multiply by $\dfrac{\lambda}{a_3}\sec\delta_3$, and collect coefficients of y_1, y_2, y_3,

$$(19) \qquad \left[(W_1 + W_2)\frac{\lambda\sec\delta_3}{a_3} + W_3\frac{h_3}{a_3}\left(1 + \frac{\lambda\sec\delta_3 - l_3}{a_3}\right)\right]y_3$$
$$+\left[W_2\left((1 - \frac{h_2}{a_2}\right) - (W_1 + W_2)\frac{\lambda\sec\delta_3}{a_3} - W_3\frac{h_3}{a_3}\,\frac{\lambda\sec\delta_3 - l_3}{a_3}\right]y_2$$
$$+\left[W_1\left(1 + \frac{h_1}{\lambda\sec\delta_1 - l_1}\right) + W_2\frac{h_2}{a_2}\right]y_1 = 0.$$

Thus if $y_3 = 0$,

$$(20) \qquad \frac{y_1}{y_2} = -\frac{W_2\left(1 - \dfrac{h_2}{a_2}\right) - (W_1 + W_2)\dfrac{\lambda\sec\delta_3}{a_3} - W_3\dfrac{h_3}{a_3}\,\dfrac{\lambda\sec\delta_3 - l_3}{a_3}}{W_1\left(\dfrac{h_1}{1 + \lambda\sec\delta_1 - l_1}\right) + W_2\dfrac{h_2}{a_2}}$$

$$= -\frac{W_1\dfrac{\lambda\sec\delta_2}{a_2} + W_2\dfrac{h_2}{a_2}\left(1 + \dfrac{\lambda\sec\delta_2 - l_2}{a_2}\right)}{W_1\left(1 + \dfrac{h_1}{\lambda\sec\delta_1 - l_1}\right) - W_1\dfrac{\lambda\sec\delta_2}{a_2} - W_2\dfrac{h_2}{a_2}\,\dfrac{\lambda\sec\delta_2 - l_2}{a_2}},$$

a cubic equation for λ.

Proceeding in this way, the relation for the m^{th} link is found to be

$$(21)\quad \left[(W_1+\ldots+W_{m-1})\frac{\lambda\sec\delta_m}{a_m}+W_m\frac{h_m}{a_m}\left(1+\frac{\lambda\sec\delta_m-l_m}{a_m}\right)\right]y_m$$

$$+\left[W_{m-1}\left(1-\frac{h_{m-1}}{a_{m-1}}\right)-(W_1+\ldots+W_{m-1})\frac{\lambda\sec\delta_m}{a_m}-W_m\frac{h_m}{a_m}\frac{\lambda\sec\delta_m-l_m}{a_m}\right]y_{m-1}$$

$$+\left[W_{m-2}\left(1-\frac{h_{m-2}}{a_{m-1}}\right)+W_{m-1}\frac{h_{m-1}}{a_{m-1}}\right]y_{m-2}$$

$$+\quad\ldots\ldots\ldots\ldots$$

$$+\left[W_3\left(1-\frac{h_3}{a_3}\right)+W_4\frac{h_4}{a_4}\right]y_3$$

$$+\left[W_2\left(1-\frac{h_2}{a_2}\right)+W_3\frac{h_3}{a_3}\right]y_2$$

$$+\left[W_1\left(1+\frac{h_1}{\lambda\sec\delta_1-l_1}\right)+W_2\frac{h_2}{a_2}\right]y_1=0,$$

the general recurring relation, in which m is to be replaced by 3, 4, 5, ...

When the L of each link coincides with the lower pin of the link, the revolving chain is kinetically equivalent to a system of plummets concentrated at the pins, the condition for which has been investigated above.

A fly-wheel in rotation may be mounted on each rod as an axis, and the modification investigated of the equations, in the manner given by Sir W. Thomson in the *Proceedings of the London Mathematical Society*, vol. VI., 1875—"Vibration and waves in a stretched uniform chain of symmetrical gyrostats."

For experimental illustration a chain could be constructed of bicycle wheels, hung vertically, and so a mechanical model is realised of the action of a magnetized medium on polarised light, as described in Perry's *Spinning Tops*.