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FACTORIZATION OF INVERTIBLE MATRICES OVER RINGS OF STABLE RANK ONE

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Abstract

Every invertible n-by-n matrix over a ring R satisfying the first Bass stable range condition is the product of n simple automorphisms, and there are invertible matrices which cannot be written as the products of a smaller number of simple automorphisms. This generalizes results of Ellers on division rings and local rings.

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1. Introduction

In various situations it is instructive to represent a matrix as a product of matrices of a special nature. For example, every orthogonal n-by-n matrix is the product of at most n reflections [1], [2, Proposition 5, Chapter IX, §6, section 4] (see [4], for further work on reflections). In linear algebra, one writes an invertible matrix as a product of elementary matrices. One can ask how many elementary matrices (or commutators) are needed to represent any product of elementary matrices (respectively, commutators); see [3]. In multiplicative simplex methods, one writes an invertible matrix over a field as the product of matrices each of which differs from the identity matrix by one

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column. These matrices are simple in the sense of the following definition of Ellers [5].

An invertible matrix β over a (possibly non-commutative) field K is *simple*, if $\mathrm{rank}(\beta - 1_V) = 1$, that is, β fixes every vector of some hyperplane in V. Examples of simple matrices include reflections, involutions, transvections, axial affinities and hyperreflections.

Motivated partly by geometric applications, Ellers showed that if β is an element of $\mathrm{Aut}(V)$ and $\mathrm{rank}(\beta - 1_V) = t$, there are simple mappings β_i in $\mathrm{Aut}(V)$ such that $\beta = \beta_1\beta_2 \cdots \beta_t$, and t is the smallest number for which such a factorization of β exists.

Later Ellers generalized these results to commutative local rings R [6] and then to non-commutative local rings R [7].

In this paper, we extend these results to any ring R satisfying the first Bass stable range condition. Along with local rings R, this includes all semilocal rings R, all Artinian rings R, all 0-dimensional commutative rings R (that is, every prime ideal of R is maximal), and many other rings [8], [9], [12].

2. Statement of results

First, we introduce some definitions and notations.

Let R be an associative ring with 1, V a right R-module,

$$V^* = \mathrm{Hom}_R(V, R)$$

the dual module, $\mathrm{End}(V) = \mathrm{Hom}_R(V, V)$ the ring of all R-linear endomorphisms of V, and $\mathrm{Aut}(V)$ the group of all automorphisms of V ($\mathrm{Aut}(V) \subset \mathrm{End}(V)$). A vector $\nu \in V$ is called *unimodular* if $f\nu = 1$ for some $f \in V^*$.

When R is a division ring, the rank of $\alpha \in \mathrm{End}(V)$ is defined as the dimension of αV. In general, there are different ways to extend the notion of rank. In this paper we use two different definitions of rank.

DEFINITION 1. The rank, $\mathrm{rank}(\alpha)$, is the least integer $s \geq 0$ such that $\alpha = \nu_1 f_1 + \cdots + \nu_s f_s$ with $\nu_i \in V$ and $f_i \in V^*$.

In other words, $\alpha: V \to V$ can be decomposed as $V \to R^S \to V$, where R^S is the R-module of s-columns over R.

DEFINITION 2. The unimodular rank, u-$\mathrm{rank}(\alpha)$ is the least integer $s \geq 0$ such that $\alpha = \nu_1 f_1 + \cdots + \nu_s f_s$ with unimodular $\nu_i \in V$ and $f_i \in V^*$.

Both ranks could be infinite (when no such s exists). Clearly, $\mathrm{rank}(\alpha) \leq u$-$\mathrm{rank}(\alpha)$ always. When R is a division ring, both definitions coincide with the usual definition of the rank as the dimension of αV.

An automorphism β in $\mathrm{Aut}(V)$ is called *simple* (respectively, u-simple), if $\mathrm{rank}(\beta - 1_V) = 1$ (respectively, u-$\mathrm{rank}(\beta - 1_V) = 1$). That is, $\beta = 1_V + \nu f$

with $\nu \in V$ (ν is unimodular in the case of u-simple β) and $f \in V^*$. Invertibility of such β is equivalent [10, Section 2] to $1 + f\nu \in \mathrm{GL}_1 R$.

Examples of simple automorphisms include transvections (when $f\nu = 0$) and reflections (or involutions, when $f\nu = -2$). More generally, a hyper-reflection can be defined [5] as a simple $\beta = 1_V + \nu f$ with $f\nu$ having a finite order modulo the commutator subgroup $[\mathrm{GL}_1 R, \mathrm{GL}_1 R]$.

Recall that the first Bass stable range condition on R is:

If $a, b \in R$ and $Ra + Rb = R$ then there is $c \in R$ such that $R(a+cb) = R$.

We write $\mathrm{sr}(R) = 1$ if R satisfies this condition and $R \neq 0$. See [8], [9], [12] for various examples of such rings.

Theorem 3. *If* $\mathrm{sr}(R) = 1$, $\beta \in \mathrm{Aut}(V)$ *and* $\mathrm{rank}(\beta - 1) = s < \infty$, *then* β *is the product of* s *simple automorphisms, and it cannot be factored into any product of a smaller number of simple automorphisms.*

Theorem 4. *If* $\mathrm{sr}(R) = 1$, $\beta \in \mathrm{Aut}(V)$ *and* u-$\mathrm{rank}(\beta - 1) = s < \infty$, *then* β *is the product of* s *u-simple automorphisms, and it cannot be factored into any product of a smaller number of u-simple automorphisms.*

Theorem 3 will be proved in the next section. The proof of Theorem 4 is so similar that we leave it to the reader.

3. Proof of Theorem 3

Let $\mathrm{GL}_n R$ denote the group of all n-by-n invertible matrices over R. It can be identified with $\mathrm{Aut}(R^n)$, where R^n is the R-module of n-columns over R.

Lemma 5. *Assume that* $\mathrm{sr}(R) = 1$. *Let* $n \geq 1$ *be an integer, and* $\beta = (b_{i,j}) \in \mathrm{GL}_n R$. *Then there is a simple matrix* $\gamma \in \mathrm{GL}_n R$ *such that* $(\gamma\beta\gamma^{-1})_{n,n} \in \mathrm{GL}_1 R$.

Proof. Consider the last row $(b_{n,1}, \dots, b_{n,n})$ of the matrix $\beta = (b_{i,j}) \in \mathrm{GL}_n R$. Since β is invertible $\sum b_{n,i} R = R$. The first Bass stable range condition implies all higher Bass conditions for R as well as for the opposite ring [11]. So there are $c_i \in R$ such that

$$(b_{n,n} + b_{n,1}c_1 + \cdots + b_{n,n-1}c_{n-1})R = R.$$

Since $\mathrm{sr}(R) = 1$, every one-sided unit in R is a unit (a result of Kaplansky, see [12]). So $b_{n,n} + b_{n,1}c_1 + \cdots + b_{n,n-1}c_{n-1} \in \mathrm{GL}_1 R$. Let γ be the simple

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matrix which differs from the identity matrix 1_n only in the last column, the entries of the last column of γ being $-c_1, -c_2, \dots, -c_{n-1}, 1$. Then $(\gamma\beta\gamma^{-1})_{n,n} = b_{n,n} + b_{n,1}c_1 + \dots + b_{n,n-1}c_{n-1} \in \mathrm{GL}_1 R$.

Let us prove now the first conclusion of Theorem 3. So let $\beta = 1_V + \nu_1 f_1 + \dots + \nu_s f_s \in \mathrm{Aut}(V)$ with $\nu_i \in V$ and $f_i \in V^*$. We want to prove that β is a product of s simple matrices. We proceed by induction on s. Set $b_{i,j} = f_i\nu_j \in R$ and consider the matrix $\beta' = 1_s + (b_{i,j})$. By [10, Section 2], $\beta' \in \mathrm{GL}_s R$. By Lemma 5 above, there is $\gamma \in \mathrm{GL}_s R$ such that $(\gamma\beta'\gamma^{-1})_{s,s} \in \mathrm{GL}_1 R$. Replacing $(\nu_1, \dots, \nu_s)$ by $(\nu_1, \dots, \nu_s)\gamma^{-1}$ and $(f_1, \dots, f_s)^{\mathrm{T}}$ by $\gamma(f_1, \dots, f_s)^{\mathrm{T}}$, we do not change β, but replace $\beta' = 1+s+(f_1, \dots, f_s)^{\mathrm{T}}(\nu_1, \dots, \nu_s)$ by $\gamma\beta'\gamma^{-1}$. So we can assume that $1+f_s\nu_s = (\beta')_{s,s} \in \mathrm{GL}_1 R$. By [10, Section 2], $\delta = 1_V + \nu_s f_s \in \mathrm{Aut}(V)$. So δ is a simple matrix. We have $\beta = \delta(1_V + (\delta^{-1}\nu_1)f_1 + \dots + (\delta^{-1}\nu_{s-1})f_{s-1})$. By the induction hypothesis, the second factor, $(1_V + (\delta^{-1}\nu_1)f_1 + \dots + (\delta^{-1}\nu_{s-1})f_{s-1})$ is the product of $s-1$ simple automorphisms. So β is the product of s simple automorphisms.

Let us prove now the second conclusion of Theorem 3. That is, we want to prove that if $\beta = \delta_1 \cdots \delta_t$ is the product of t simple automorphisms δ_i, then $\mathrm{rank}(\beta - 1_V) \le t$. We write $\delta_i = 1_V + \nu_i f_i$ with $\nu_i \in V$ and $f_i \in V^*$. By induction on m, we see easily that $\delta_1 \cdots \delta_m = 1_V + \nu_1 g_1 + \dots + \nu_m g_m$, where $g_i \in V^*$ depend on m. So $\mathrm{rank}(\beta - 1_V) \le t$.

Theorem 3 is proved. We complement it with the following result.

PROPOSITION 6. *For any associative ring R with $\mathrm{sr}(R) = 1$, any integer $n \ge 2$, and any integer s in the interval $0 \le s \le n$, there is a matrix $\beta \in \mathrm{GL}_n R$ with u-$\mathrm{rank}(\beta - 1_n) = \mathrm{rank}(\beta - 1_n) = s$. So this β is the product of s simple matrices and it is not a product of a smaller number of simple matrices.*

To prove this proposition we will need the following two lemmas.

LEMMA 7. *Let R be an associative ring with $\mathrm{sr}(R) = 1$ and $\alpha \in \mathrm{End}(V)$ be such that the R-module αV is a direct summand of V and has a free basis of cardinality s. Then u-$\mathrm{rank}(\alpha) = \mathrm{rank}(\alpha) = s$.*

PROOF. Let $\{e_j\}$ be a free basis for αV of cardinality s.

We prove first that $\mathrm{rank}(\alpha) \le u$-$\mathrm{rank}(\alpha) \le s$. If $s = \infty$, there is nothing to prove, so let $s < \infty$. For every ν in V, we have $\alpha\nu = \sum e_i f_i(\nu)$ with $f_i(\nu) \in R$. Since $\{e_i\}$ is a basis, $f_i \in V^*$. So $\alpha = \sum e_i f_i$, hence

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$\operatorname{rank}(\alpha) \leq u\text{-}\operatorname{rank}(\alpha) \leq s$. (Note that $\operatorname{rank}(\alpha) \leq s$ holds even without the assumption that αV is a direct summand.)

Let us prove now that $\operatorname{rank}(a) \geq s$. Suppose on the contrary that $t = \operatorname{rank}(\alpha) < s$. That is,

$$\alpha = \nu_1 f_1 + \cdots + \nu_t f_t \in \operatorname{Aut}(V)$$

with $\nu_i \in V$ and $f_i \in V^*$. Pick $\pi \in \operatorname{End}(V)$ such that $\pi^2 = \pi$ and $\pi V = \alpha V$. Set $u_i = \pi \nu_i$. We can write $u_i = \sum_j e_j a_{j,i}$ with $a_{j,i} \in R$. Note that $t < \infty$, so only finitely many e_j are involved in all these linear combinations. Say, $u_i = \sum_j^m e_j a_{j,i}$ for $i = 1, \ldots, t$ with $t < m < \infty$. Now we write $e_j = \sum u_i b_{i,j}$ for $j = 1, \ldots, m$ with $b_{i,j} \in R$. We have $\alpha\beta = 1_m$, where $\alpha = (a_{j,i})$ and $\beta = (b_{i,j})$. Complementing α by zero columns and β by zero rows, we obtain two matrices α', β' in the ring $M_m R$ of square matrices over R such that $\alpha'\beta' = \alpha\beta = 1_m$. Since $\operatorname{sr}(R) = 1$, we have $\operatorname{sr}(M_n R) = 1$ by [11]. So, by Kaplansky's result [12], $\beta \in \operatorname{GL}_m R$. But since β has a zero row, this is impossible.

Remark. Lemma 7 holds if the condition $\operatorname{sr}(R) = 1$ is replaced by the condition $R \neq 0$ together with the condition $\operatorname{sr}(R) < \infty$ or the condition that R is commutative.

Lemma 8. *For any $n \geq 2$ there exists an invertible matrix β_n in $\operatorname{GL}_n R$ such that the matrix $\beta_n - 1$ is also invertible.*

Proof. When $n = 2$, we can take

$$\beta_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

When $n = 3$, we can take

$$\beta_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $n \geq 4$, we can write β_n as the direct sum of the above matrices β_2 and β_3. For example, $\beta_4 = \beta_2 \oplus \beta_2$ is the required matrix in $\operatorname{GL}_4 R$, $\beta_5 = \beta_3 \oplus \beta_2$ is the required matrix in $\operatorname{GL}_5 R$, and so on.

Proof of Proposition 6. When $s = 0$, we take $\beta = 1_n$. When $1 \leq s \leq n-1$, we can take $\beta = \gamma + 1_{n-s-1}$, where $\gamma \in \operatorname{GL}_{s+1} R$ is the Jordan matrix with ones along the diagonal. Then $(\beta - 1_n)R^n$ is a direct summand of R^n with s free generators, so $u\text{-}\operatorname{rank}(\beta - 1_n) = \operatorname{rank}(\beta - 1_n) = s$ by Lemma 7.

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Finally, when $s = n$, we find β as in Lemma 8, so $(\beta - 1_n)R^n = R^n$, hence $\text{rank}(\beta - 1_n) = n$ by Lemma 7.

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