ON PSEUDO-ANOSOV MAPS WHICH EXTEND OVER TWO HANDLEBODIES

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We show that there is a pair of handlebodies H_1 and H_2 with common boundary F with the properties:
(a) There is no essential simple closed curve in F bounding a disc in both H_1 and H_2.
(b) Given any positive number, there are essential simple curves C_i $i = 1, 2$ on F, bounding discs in H_i whose distance apart in the Hausdorff topology on F is less than this positive number.
Such an example has consequences for Heegaard splittings and recognising the 3-sphere.

0. Introduction

Let $\theta: F \rightarrow F$ be an orientation preserving homeomorphism of a closed orientable surface of genus at least two. We say that the map θ *extends* over the 3-manifold M if there is an identification $\phi: F \rightarrow \partial M$, and a homeomorphism $\Theta: M \rightarrow M$ so that $\theta = \phi^{-1} \Theta | \partial M \phi$. A case of particular interest is when M is a cube-with-handles or "handlebody". A question, attributed to Andrew Casson, which has been around for some time is the following:

Suppose that θ is a pseudo-Anosov map which extends over handlebodies M_1 and M_2, that is to say, there are handlebodies M_1 and M_2, identifications $\phi_i: F \rightarrow \partial M_i$ and homeomorphisms

$$\Theta_i: M_i \rightarrow M_i \text{ so that } \theta = \phi_i^{-1} \Theta_i | \partial M_i \phi_i \qquad i = 1, 2.$$

Must there be an essential simple closed curve C on F which bounds a disc in both M_1 and M_2?

We shall indicate some reasons why this is of interest below. The purpose of this note is to show that the answer to this question is negative. It is of course necessary to restrict the curve C to be simple; since nontrivial normal subgroups in a surface group always intersect, so that there is always a (possibly non-simple) loop which is null homotopic in both handlebodies.

One reason for searching for such an example was that it obstructed a possible approach to deciding the question of when a Heegaard splitting is reducible; this was

the hope that if for every ε, there were embedded discs D_1 in M_1 and D_2 in M_2 whose boundaries were ε close, then there would be a simple curve C on F bounding a disc in both M_1 and M_2. More precisely, we have the following. Fix a hyperbolic metric on F, and use this to define the Hausdorff metric, $\chi(.,.)$, on the closed subsets of F. (See [2].) Combining some knowledge of the action of pseudo-Anosov maps with a result of [9], we deduce:

Corollary 1.4. *It is possible to find a pair of handlebodies M_1 and M_2 with $\partial M_1 = \partial M_2$ which have the following properties.*

(a) *There is no simple closed curve bounding a disc in M_1 and M_2.*

(b) *Given any $\varepsilon > 0$, there is a pair of essential simple closed curves C_1 and C_2 in F, with C_i bounding a disc in M_i and $\chi(C_1, C_2) < \varepsilon$.*

One of the reasons that such an example is hard to find is that it is by no means easy to give examples of pseudo-Anosov maps which extend a handlebody in essentially different ways. There are two trivial constructions: one involving periodic maps which commute with the pseudo-Anosov, and the other by using pairs of curves which fill the surface and bound discs in more than one handlebody. The second of these is doomed to failure, and the first always seems to fail. Moreover, in general the checking of any potential counterexample is impeded by the fact that there appears to be no algorithm known to determine if a pair of handlebodies share a simple closed curve bounding a disc in each.
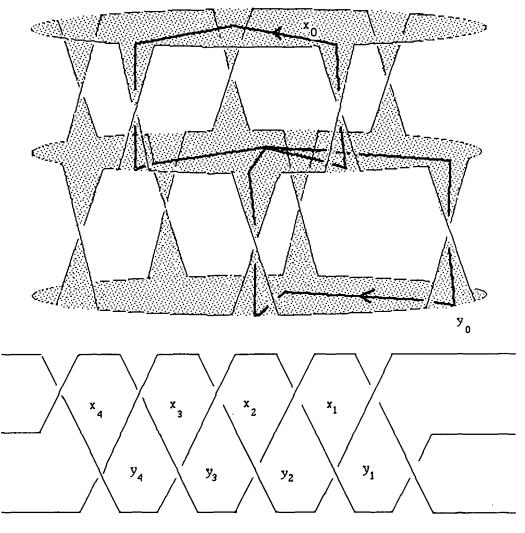
We say that ξ *exchanges two handlebodies*, if there are handlebodies M_1 and M_2 so that ξ does not extend over either handlebody, but ξ^2 extends over both. Then we can also answer the following questions, raised in [10]. Suppose that ξ is a pseudo-Anosov which exchanges a pair of handlebodies. Is there a periodic map commuting with ξ which exchanges them? Part of our construction produces a pseudo-Anosov ξ which exchanges M_1 and M_2. However, a calculation shows:

Corollary 1.5. *There is no periodic map commuting with ξ which exchanges M_1 and M_2.*

Thus the answer to this question is also negative. Our example has genus$(F) = 4$, and arises from a somewhat complicated extension which arises in nature as a result of a certain knot being ribbon. The proof that our example works relies heavily on two pieces of luck.

1. The example

In this section we describe our example; some of the details involved in the computation are deferred until Section 2. The construction involves the monodromy of the (3.5) Turk's head knot, the knot 10* of Conway's tables; displayed as the boundary of its Seifert surface in Fig. 1. This knot is hyperbolic, so that the monodromy $\theta' : F' \to F'$

FIGURE 1

of the Seifert surface is pseudo-Anosov. Since we are interested in a map of a closed surface to itself, we cap off F' and extend θ' across this disc, to obtain an automorphism $\theta: F \to F$. We shall prove in Section 2 that this extended map continues to be pseudo-Anosov. Observe that the genus of F is four.

We need to describe our generators for the fundamental group of $\pi_1(F')$; this is also done in Fig. 1. The basepoint is considered to be on the central disc and the loops are all of the type shown, with the convention shown in the bottom figure for their numbering.

In the notation of Fig. 1, it is convenient to write:

$$\pi_1(F) = \, <x_0, x_1, \ldots, x_4, y_0, y_1, \ldots, y_4 | x_0 \cdot x_1, \ldots \cdot x_4 = y_0 \cdot y_1, \ldots \cdot y_4 = I, R>.$$

Here R is the relation which comes from capping off the knot; one verifies from Fig. 1 that this is the relation:

$$1 = y_0 x_1 y_3 x_4 y_1 x_2 y_4 x_0 y_2 x_3.$$

Lemma 1.1. *Use the symbol* $\langle \cdot \rangle_n$ *to denote normal closure in* $\pi_1(F)$.
Set

$$N_1 = \langle y_2 x_2 x_0, \ y_3 x_4 x_2^{-1}, \ y_1 x_0^{-1} x_4^{-1} x_2^{-1} x_1 x_2, \ y_4 x_2^{-2} x_1^{-1} x_2 \rangle_n$$

$$N_2 = \langle y_2 x_2 x_3 x_0^{-1} x_3^{-1}, \ y_3 x_3 x_0 x_1, \ y_1 x_2 x_3, \ y_4 x_0 x_4 x_3^{-1} \rangle_n$$

Then there are handlebodies M_1 *and* M_2, *whose boundary is* F *and for which*

$$N_i = \ker\{i_* : \pi_1(F) \to \pi_1(M_i)\} \ i = 1, 2.$$

Proof. One checks that in both cases the four words given describe disjoint simple closed curves on the surface F. Then the handlebodies are formed by by sewing in 2-handles along thin neighbourhoods of these curves and capping off the resulting 2-sphere boundary component with a 3-handle.   □

Lemma 1.2. *The map* θ *extends over both* M_1 *and* M_2.

Proof. It is shown in [3], that this is equivalent to the map $\theta *$ preserving the subgroups N_1 and N_2. (The subgroups are normal, so that the choice of basepoint is irrelevant here.) Using the calculations of Section 2, this can be verified.

Remark. We shall indicate where this pair of handlebodies came from and why they satisfy 1.1 and 1.2 in Section 2.
Our example is completed by the following:

Theorem 1.3. *There is no simple closed curve representing an element of the subgroup* $N_1 \cap N_2$.

Proof. Form a 3-manifold X by glueing together M_1 and M_2 along F. By the Seifert–Van Kampen theorem, we obtain a presentation of $\pi_1(X) = G$, which after eliminating the y_i's has the shape:

Generators. $x_0, x_1, \ldots, x_4$

Relations.

$$x_0 \cdot x_1 \ldots \cdot x_4 = 1$$

$$R$$

$$x_2 x_0 = x_2 x_3 x_0^{-1} x_3^{-1} \tag{1}$$

$$x_4 x_2^{-1} = x_3 x_0 x_1 \tag{2}$$

$$x_0^{-1} x_4^{-1} x_2^{-1} x_1 x_2 = x_2 x_3 \tag{3}$$

$$x_2^{-2} x_1^{-1} x_2 = x_0 x_4 x_3^{-1} \tag{4}$$

It follows directly from (1) that we have an equivalent relation:

$$x_0 = x_3 x_0^{-1} x_3^{-1} \tag{1$'$}$$

From (2), (1) we deduce:

$$x_4 x_0 = x_3 x_0 x_1 x_2 x_3 x_0^{-1} x_3^{-1} = x_3 (x_0 x_1 x_2 x_3) x_0^{-1} x_3^{-1} = x_3 x_4^{-1} x_0^{-1} x_3^{-1}.$$

Now using (1)$'$, we deduce that (2) is equivalent to:

$$x_4 = x_3 x_4^{-1} x_3^{-1}. \tag{2$'$}$$

Now from (4), we may write $x_1 = x_2 x_3 x_4^{-1} x_0^{-1} x_2^{-2}$ and use this to substitute into (3), yielding:

$$x_0^{-1} x_4^{-1} x_2^{-1} x_2 x_3 x_4^{-1} x_0^{-1} x_2^{-2} x_2 = x_2 x_3.$$

Using (1)$'$ and (2)$'$ and (3)$'$ we deduce from this relation, the new relation:

$$x_2 = x_3 x_2^{-1} x_3^{-1}. \tag{3$'$}$$

Finally, we remove x_1 from the relations (1),...,(4) altogether by using $x_0 \cdot x_1 \ldots x_4 = 1$ in (4), which becomes:

$$x_3^2 x_2 x_4 x_0 x_2 x_4 x_0 = 1. \tag{4$'$}$$

This gives an equivalent presentation for G:

Generators. x_0, x_2, x_4, x_3

Relations.

$$R$$

$$x_0 = x_3 x_0^{-1} x_3^{-1}$$

$$x_2 = x_3 x_2^{-1} x_3^{-1}$$

$$x_4 = x_3 x_4^{-1} x_3^{-1}$$

$$x_3^2 x_2 x_4 x_0 x_2 x_4 x_0 = 1.$$

We now may rewrite the knot relation, by substituting for the y_i's and x_1. A routine calculation reveals that R is equivalent to:

$$x_4 x_0 x_2 x_4 x_0 x_2 x_3^2 = 1.$$

Observe that x_3^2 is central in G; so that the knot relation is equivalent to the last relation in the above list. This was bound to happen; the fundamental group of X does not notice in this construction whether we use F or F' since they differ only in 3-handles.

We may now prove Theorem 1.3. First observe that $H_1(X) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$. It follows that the minimal number of generators of G is four. Further, any embedded S^2 in X is separating, since $H_1(X)$ is finite. Recalling that the genus of F is four, we see that if there were a simple closed curve bounding a disc in M_1 and M_2 the induced reduction of $X = X_1 \# X_2$ could not have either piece simply connected, else the induced Heegaard splitting on the non simply-connected piece would exhibit G as $a \leqq 3$ generator group.

It follows that if such a curve existed, we could write G as a nontrivial free product $G_1 * G_2$. However, as we have already observed, the element x_3^2 is central in G; this cannot happen in a nontrivial free product unless $x_3^2 = 1$.

To rule out this possibility, observe that the group G surjects onto the quaternion group Q_8 which has presentation:

$$\langle -1, i, j, k \mid i^2 = j^2 = k^2 = -1, ij = -ji, ik = -ki, jk = -kj \rangle$$

where we map

$$x_3 \rightarrow i \quad x_0 \rightarrow 1 \quad x_2 \rightarrow k \quad x_4 \rightarrow j.$$

In this group x_3 has order 4, completing the proof.   □

We can identify the manifold X with little difficulty. The presence of centre shows that it is likely to be a Seifert fibred space. We can exhibit the fundamental group in the form given in Hempel's book [7] as follows:

Set $x_2 x_3 = c_2, x_4 x_3 = c_4, x_0^{-1} x_3 = c_0, x_3 = c_3, c = c_4 c_0 c_2 c_3$ and $t = x_3^2$.

Then G has presentation with generators c_0, c_2, c_4, c_3, c, t with relations:

(a) t is central.

(b) $c_i^2 = t$ and $c^2 = t^{-1}$

(c) $c_4 c_0 c_2 c_3 = c$.

Thus X is Seifert fibred over a two sphere with 5 exceptional fibres—as an orbifold these are cone points with angle π, so that the orbifold Euler characteristic is $-1/2$.

Proof of 1.4. The result of [4] implies that there is an essential curve bounding a disc in M_1 which is the union of an arc $\alpha_+ \cup \alpha_-$ where $\alpha_\pm$ are arcs in the invariant laminations for θ. Similarly, such a curve exists for M_2. Moreover, one can check that these arcs can be chosen so that they do not cut across the principal regions for the laminations, that is, they are not subarcs of isolated leaves. (The existence of such curves can also be deduced from [9].) Denoting these two curves by A_1 and A_2 we now have from [2], Theorem 4.1, that $\theta^k A_i \to L^+$ for $i = 1, 2$, where L^+ is the unique perfect stable lamination left invariant by θ, and the convergence is in the Hausdorff topology. The corollary now follows. □

Proof of 1.5. Suppose that such a map existed; call it ω say. Since ω commutes with θ, it must fix the invariant foliations of θ; moreover, it must also preserve the measure on these foliations. The latter is because if μ is the invariant measure on the stable foliation say, we can form an ω-invariant measure by $M = \mu + \omega^*\mu + \cdots + (\omega^*)^{n-1}\mu$. Since this is an invariant measure on the stable foliation, unique ergodicity gives $M = k\mu$. Thus ω is an isometry of the affine structure defined by the union of the stable and unstable measure foliations. In particular, the curve A_1 constructed above must be carried to a curve which bounds a disc in M_2 and is the union of two straight lines of the same length. One can check explicitly that no such curve exists. □

2. Some computations

In this section we show how one can perform the computations alluded to in the previous sections.

The knot K which we consider is the (3.5) Turk's Head knot of Fig. 1. It is alternatively described as the braid closure of $(\sigma_1 \sigma_2^{-1})^5$. This is a homogeneous braid in the sense of [11], so that K is fibred, with fibre surface F', visible in Fig. 1. The knot of Fig. 1 clearly has a 5-fold symmetry coming from a rotation about a vertical axis; this symmetry induces a map $\omega: F' \to F'$ which has three fixed points and carries the boundary circle to itself, so that ω extends as a map of the closed surface F to itself. Clearly ω and the monodromy of K commute.

We now prove that the monodromy is pseudo-Anosov in the following way. There is a branched covering $p: F \to F/\omega$; where there are four branch points, each of order 5. The surface F/ω is easily seen to be topologically a 2-sphere and can be considered metrically as an orbifold with 4 cone points, each of angle π. This orbifold is flat; in fact denoting the flat torus by T, this has a central involution τ, and we have a metric covering $T \to T/\tau = F/\omega$. This gives a correspondence between maps of the torus to itself and orbifold maps of F/ω.

The methods of Goldsmith [6] apply to this situation (indeed, she works out this example, though not in quite this notation) and we deduce that the monodromy $\theta: F \to F$ comes from lifting the linear map:

$$\theta = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : F/\omega \to F/\omega.$$

Since this map is a hyperbolic map of the torus, the lifted map θ is pseudo-Anosov; it has the same local properties. (See [5].) This description enables one to visualise the monodromy on F in a very concrete way, but we shall not need to do this, we refer to [8].

More importantly, the knot K is ribbon, so that by a theorem of Casson–Gordon [3], the map θ extends over a handlebody; and by examining the ribbon move, one sees that the element $x_2 y_0$ bounds a disc in this handlebody. Below we shall give a description of the action of θ_* on $\pi_1(F)$; knowledge of one nonseparating curve defines the whole handlebody, since the characteristic polynomial of $\theta_*: H_1(F) \to H_1(F)$ factorises as $f(t)f(t^{-1})$, with $f(t)$ irreducible over $\mathbf{Z}$. This gives handlebody M_1 of Lemma 1.1.

One of the surprising features given the highly symmetric nature of K is that the handlebody M_1 is so asymmetric. We digress to observe that this is actually forced to be the case, but since we make no use of this fact, we only sketch the proof:

Proposition 2.1. *There is no compression body over which both θ and ω extend.*

Proof. One computes easily that ω_* has characteristic polynomial $g(t)^2$, where $g(t) = t^4 + r^3 + \cdots + 1$; which is $\mathbf{Z}$ irreducible. If follows that if ω compresses at all, it must either be a handlebody or involve just one 2-handle. The latter is impossible if the pseudo-Anosov is also to extend over the compression, since the core of the 2-handle would have to be preserved up to free homotopy. But now an examination of the fixed point data [1] shows that ω cannot extend over any 3-manifold.   □

Notice that this already gives five handlebodies over which θ extends, since $\omega^r N_1$ must be different from N_1 for $0 < r < 5$. However the reader may check that every such pair gives a reducible Heegaard splitting.

The second handlebody of Lemma 1.1 arises in the following way. Notice that we have an orientation reversing square root in $GL(2, \mathbf{Z})$ for the matrix θ. Writing $\xi^2 = \theta$, it is easy to check that ξ also lifts to F, and that choosing an appropriate lift, also denoted ξ, we may arrange $\xi^2 = \theta$ as maps of F. Thus θ also extends over the handlebody ξN_1. This gives another five handlebodies. In the notation of Lemma 1.1, $N_2 = \omega^2 \xi N_1$. Empirically one finds that this choice is the only pair which yields an irreducible Heegaard splitting.

It remains only to show how to obtain the action of θ_* on $\pi_1(F)$. It turns out that a trick makes this fairly easy. We refer to Fig. 2, and regard the basepoint and surface as living behind the page; so that we are using the Wirtinger under-presentation. Write a^b for $b^{-1}ab$.
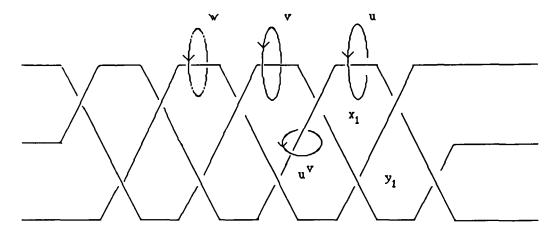
FIGURE 2

Then the action of the monodromy is given by conjugation by a meridian; initially we use v. We see that:

$$x_1 = u^{-1}v \qquad y_2 = (u^{-1})^v \cdot w = v^{-1}u^{-1}vw$$

$$x_2 = v^{-1}w \qquad y_3 = u^v(v^{-1})^w = v^{-1}uvw^{-1}v^{-1}w$$

$$\theta x_2 = vx_2v^{-1} = wv^{-1} = x_2y_3^{-1}x_1^{-1}$$

$$\theta y_2 = vy_2v^{-1} = u^{-1}vwv^{-1} = x_1x_2y_3^{-1}x_1^{-1}.$$

Now we may compute the action of the monodromy on the remainder of the generators by chicanery. Notice that we could have used w to compute the monodromy, and therefore by symmetry we have:

$$\theta_w(x_3) = x_3y_4^{-1}x_2^{-1}.$$

But

$$\theta_w(x_3) = wx_3w^{-1} = (wv^{-1})\,vx_3v^{-1}\,(vw^{-1}) = x_2y_3^{-1}x_1^{-1}\theta x_3(x_2y_3^{-1}x_1^{-1})^{-1}.$$

Hence

$$\theta x_3 = x_1y_3x_2^{-1}x_3y_4^{-1}x_2^{-1} \cdot x_2y_3^{-1}x_1^{-1} = x_1y_3x_2^{-1}x_3y_4^{-1}y_3^{-1}x_1^{-1}.$$

This gives an inductive method of computing the monodromy which actually works for any (3,k) Turk's Head knot. One finds:

$$\theta x_0 = x_1 y_2^{-1} y_1^{-1} x_4^{-1} x_0 y_2 x_1^{-1} \qquad \theta y_0 = x_1 y_2^{-1} y_1^{-1} x_0 y_2 x_1^{-1}$$

$$\theta x_1 = x_1 y_2^{-1} x_0^{-1} \qquad \theta y_1 = x_1 y_2^{-1}$$

$$\theta x_2 = x_2 y_3^{-1} x_1^{-1} \qquad \theta y_2 = x_1 x_2 y_3^{-1} x_1^{-1}$$

$$\theta x_3 = x_1 y_3 x_2^{-1} x_3 y_4^{-1} y_3^{-1} x_1^{-1} \qquad \theta y_3 = x_1 y_3 x_3 y_4^{-1} y_3^{-1} x_1^{-1}$$

$$\theta x_4 = x_1 y_3 y_4 x_3^{-1} x_4 y_1 y_2 x_1^{-1} \qquad \theta y_4 = x_1 y_3 y_4 x_4 y_1 y_2 x_1^{-1}.$$

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