107.45 A surprising property of a tennis-like game

Introduction

In essence, a tennis game is won by the first side to win four points with a margin of at least two points. If the game becomes tied at 3-3 (deuce), play will continue until one side achieves the two point margin. Assuming the sides are equally matched, the serving side generally has the advantage of winning each point because the server alone determines the initial placement, velocity and spin of the ball. The receiver can only react defensively. Since one side serves the entire game, the serving side is favoured to win the game itself. (For professional players the serving side is expected to win roughly 60% to 80% of its games, assuming the sides are equally matched. This advantage is neutralised when playing a full set, where the winner must win at least six games by a margin of at least two games and the serve alternates after each game.)

The intent here is not to offer an improvement of the standard game. Rather, it is to reveal the highly counterintuitive nature of server bias for a contrived mini-game similar to tennis.

The mini-game

Denote the equally matched sides (players, teams) by A and B. Assume play begins with A serving. To decrease the server's advantage, the serve will alternate after each point played. The winner is to be the first of A or B to score two consecutive points. (Note this is not the same as winning by a margin of at least two points.) We assume A is the first to serve and we denote the probability of the server (be it A or B) winning the point by p, $0 < p < 1$. As noted, it is generally true that $p > 0.5$.

A sequence of As and Bs will specify the order by which the sides win the points. For example, the sequence ABAA represents A winning the first point, B winning the second followed by A winning the next two consecutive points, resulting in A winning the mini-game. Let $A(p)$ denote the probability (as a function of p) of A, the first server, winning the mini-game.

Is the mini-game fair?

The serve changing sides after each point gives the sides alternating advantages as play proceeds and potentially neutralises the server advantage characteristic of the standard game. On the other hand, A is the first to serve and is expected to win the first point, thus putting victory within A's reach when the next point is played. Furthermore, A will serve one more than B for mini-games ending after an odd number of total points. For mini-games ending after an even number of total points, A and B serve the same number. Overall, A will serve at least as often as B, possibly one more. This too seems to favour A. We need only calculate $A(p)$ to determine if the mini-game is fair or if it favours A, the first server.

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Mini-games won by A are of two types—those where A wins the first point and those where B wins the first point. Those where A wins the first point are of the form AA or ABAA or ABABAA … (even number of total points played) and occur with probability

$$p(1-p) + p^3(1-p) + p^5(1-p) + \ldots = \sum_{i=1}^{\infty} p^{2i-1}(1-p).$$

Those games where B wins the first point are of the form BAA or BABAA or BABABAA or … (odd number of total points played) and occur with probability

$$(1-p)^2 p + (1-p)^4 p + (1-p)^6 p + \ldots = \sum_{i=1}^{\infty} p(1-p)^{2i}.$$

Overall,

$$A(p) = \sum_{i=1}^{\infty} p^{2i-1}(1-p) + \sum_{i=1}^{\infty} p(1-p)^{2i} = (1-p)\sum_{i=1}^{\infty} p^{2i-1} + p\sum_{i=1}^{\infty}(1-p)^{2i}.$$

Both sums converge geometrically, and therefore

$$A(p) = \frac{p(1-p)}{1-p^2} + \frac{p(1-p)^2}{2p-p^2} = \frac{p}{1+p} + \frac{(1-p)^2}{2-p}, \quad p \neq 0,1. \quad (1)$$

The graph of $A(p)$ given in Figure 1 appears symmetric about $\left(\frac{1}{2}, \frac{1}{2}\right)$ as one might expect. This is confirmed by verifying

$$A\left(\frac{1}{2} + c\right) + A\left(\frac{1}{2} - c\right) = 1.$$

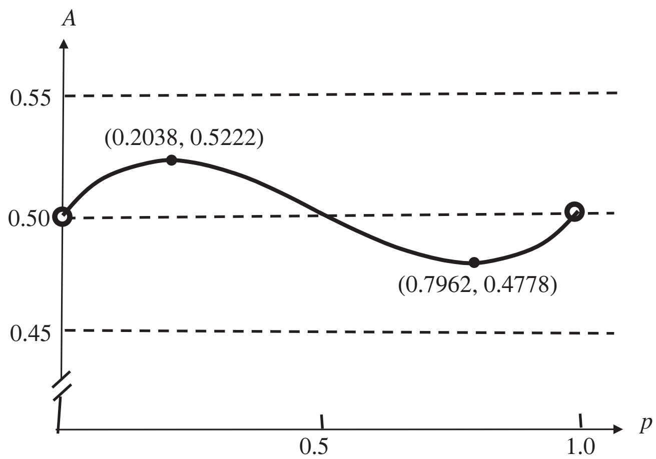

FIGURE 1: A vs p

Critical points are found by differentiating $A(p)$:

$$\frac{dA}{dp} = 0 \Rightarrow p^4 - 2p^3 - 5p^2 + 6p - 1 = 0.$$

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Letting $q = p - \frac{1}{2}$ yields a 'quadratic in form' quartic

$$16q^4 - 104q^2 + 9 = 0$$

which, by letting $u = q^2$, becomes

$$16u^2 - 104u + 9 = 0.$$

Solving for u, back substituting to get p and ignoring extraneous solutions yields

$$p = \frac{1}{2} \pm \sqrt{\frac{13}{4} - \sqrt{10}}.$$

The coordinates of both critical points, rounded off to four decimal places, are given in Figure 1.

Surprisingly, when the server's advantage is greater than 0.5, as is generally the case, it is *B*, the first to receive, and not A, the first to serve, that is favoured to win. This is highly counterintuitive. A deeper dive yields more surprises.

Consider those mini-games ending after an even number of total points. For example, a four-point mini-game is of the form ABAA or BABB, the first being won by A and the second by B. The probability of ABAA is $p^3(1 - p)$ and that of BABB is $(1 - p)^3 p$. Clearly A is more likely than B to win a four-point mini-game, assuming $p > 0.5$. This is easily generalised to show A is more likely than B to win any mini-game with an even number (greater than 2) of total points.

So from where does B derive its overall advantage of winning the mini-game? It must come from mini-games involving an odd number of total points. To investigate, consider three-point mini-games, necessarily of the form ABB (B wins) or BAA (A wins). The probability of the first is $p^2(1 - p)$ and that of the second is $(1 - p)^2 p$ showing B is more likely than A to win such a mini-game, assuming $p > 0.5$. This too is easily generalised to show B is more likely than A to win any mini-game with an odd number of total points and it is from these mini-games that B derives its overall advantage. And the advantage is significant. The unconditional probability of B, the first receiver, winning a mini-game of n (odd) total points is $p^{n-1}(1 - p)$ whereas the unconditional probability of A winning such a mini-game is $p(1 - p)^{n-1}$. Thus B is more likely than A to win such a mini-game by a factor of

$$\frac{p^{n-1}(1 - p)}{p(1 - p)^{n-1}} = \left(\frac{p}{1 - p}\right)^{n-2},$$

which can be quite large for p near 1 or large n. As an example, for mini-games of $n = 7$ total points where the server wins the point with probability $p = 0.8$, B, the first receiver, is over 500 times more likely to win than A, the first server. And for all such mini-games, A will be serving more than B! This is startling. Table 1 gives additional winning probabilities for even and odd total point mini-games corresponding to $p = 0.8$.

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Even number of points				Odd number of points			
A wins		B wins		A wins		B wins	
AA	0.16	BB	0.16	BAA	0.032	ABB	0.128
ABAA	0.1024	BABB	0.0064	$(BA)^2A$	0.0013	$(AB)^2B$	0.0819
$(AB)^2AA$	0.0665	$(BA)^2BB$	0.0003	$(BA)^2A$	0.0001	$(AB)^2B$	0.0524
…	…	…	…	…	…	…	…
Total	0.4444	Total	0.1667	Total	0.0333	Total	0.3556

TABLE 1: Winning probabilities for $p = 0.8$

Table 1 ($p = 0.8$) reveals yet another surprise. Since B derives its advantage from those mini-games ending in an odd number of total points, one might presume odd point mini-games (won by either player) are more prevalent than even point mini-games which favour A. But from Table 1 this is not the case, at least for $p = 0.8$. Using the approximate probabilities from the table,

$$P(\text{a mini-game ends in an odd number of points}) \approx 0.0333 + 0.3556 = 0.3899.$$

That is, less than 40% of mini-games where $p = 0.8$ consist of an odd number of points. In fact, for any $p > 0.5$ even point mini-games (favouring A) are expected to outnumber odd point mini-games (favouring B). To see why this is true in general, note that from (1), the probability of a mini-game being won by A and ending in an odd number of points is

$$\frac{(1 - p)^2}{2 - p}. \tag{2}$$

The probability of a mini-game being won by B and ending in an odd number of points is simply the expression in (2) with p replaced with $1 - p$ giving

$$\frac{[1 - (1 - p)]^2}{2 - (1 - p)} = \frac{p^2}{1 + p}. \tag{3}$$

The overall prevalence of mini-games ending in an odd number of points is obtained by summing the probabilities of (2) and (3):

$$\frac{(1 - p)^2}{2 - p} + \frac{p^2}{1 + p} = \frac{p^2 - p + 1}{-p^2 + p + 2}$$

which is clearly less than 50% since $p^2 - p < 0$ and $-p^2 + p > 0$.

Closing remarks

Counterintuitive properties of the mini-game –

- Player B is favoured despite player A being the first to serve.
- Player B wins the vast majority of mini-games with an odd number of total points, despite player A serving one more time than B for all such mini-games.

- A mini-game consisting of an odd number of total points, where B is favoured, is less likely to occur than one consisting of an even number of total points where A is favoured. Yet, B is favoured overall.

To put it simply, but perhaps not convincingly, B is favoured to win the mini-game because the proportion of mini-games B wins of those consisting of an odd number of total points far exceeds the proportion of mini-games A wins of those consisting of an even number of total points. This is quite apparent from the probabilities given in Table 1.

From Figure 1, $A(p)$ is minimised near $p = 0.80$. There $A(p) \approx 0.48$, giving B only a slight advantage (0.52 probability) to win the mini-game. If the mini-game were played in reality, the first receiver's advantage would surely go unnoticed and might even be denied, if brought to the players' attention. For this admittedly contrived scenario, a mathematical analysis is required to reveal that which would otherwise be masked by misleading intuition.

Each day we make hundreds of choices based on the situation before us. Some are unimportant. Coffee or tea? Turn left here? Others have significant consequences. Most decisions are made instinctively for convenience, with no in-depth analysis.

Listen to our intuition? Of course!

Trust our intuition? Not always a good idea!

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107.46 Illustrating complex mappings with Excel

Introduction

Just recently I became aware that Excel has most of the elementary functions of complex analysis included in its library of functions. I guess there are more people out there than me who are not familiar with this feature of Excel, so I would like to give a couple of applications. Since the graphical capabilities of Excel are great, these complex functions could be a good starting point for illustrating complex mappings. The fact that use of Excel is so widespread makes simple complex calculations easily accessible for most students.

The complex functions in Excel can by no means make up for a