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BANACH SPACES IN WHICH LARGE SUBSETS OF SPHERES CONCENTRATE

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Abstract We construct a nonseparable Banach space $\mathcal{X}$ (actually, of density continuum) such that any uncountable subset $\mathcal{Y}$ of the unit sphere of $\mathcal{X}$ contains uncountably many points distant by less than 1 (in fact, by less then $1-\varepsilon$ for some $\varepsilon>0$). This solves in the negative the central problem of the search for a nonseparable version of Kottman's theorem which so far has produced many deep positive results for special classes of Banach spaces and has related the global properties of the spaces to the distances between points of uncountable subsets of the unit sphere. The property of our space is strong enough to imply that it contains neither an uncountable Auerbach system nor an uncountable equilateral set. The space is a strictly convex renorming of the Johnson–Lindenstrauss space induced by an $\mathbb{R}$-embeddable almost disjoint family of subsets of $\mathbb{N}$. We also show that this special feature of the almost disjoint family is essential to obtain the above properties.

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1. Introduction

All Banach spaces considered in this paper are infinite dimensional and over the reals. For unexplained terminology, see Section 2. If $\mathcal{X}$ is a Banach space, $\mathcal{Y}\subseteq\mathcal{X}$ and $r>0$, we say that $\mathcal{Y}$ is

- $(r+)$-separated if $\|y-y'\|>r$ for any two distinct $y,y'\in\mathcal{Y}$,
- r-separated if $\|y-y'\|\geq r$ for any two distinct $y,y'\in\mathcal{Y}$,
- r-concentrated if $\|y-y'\|\leq r$ for any two distinct $y,y'\in\mathcal{Y}$,
- r-equilateral if $\|y-y'\|=r$ for any two distinct $y,y'\in\mathcal{Y}$,
- equilateral if it is r-equilateral for some $r>0$.

The classical Riesz lemma of 1916 (If $\mathcal{Y}$ is a closed proper subspace of $\mathcal{X}$ and $\varepsilon>0$, then there is x in the unit sphere of $\mathcal{X}$ such that the distance of x from $\mathcal{Y}$ does exceed $1-\varepsilon$, [44]) allows one to construct $(1-\varepsilon)$-separated sets of the cardinality equal to the density of a Banach space and in its unit sphere. By the compactness of the balls in finite-dimensional spaces it also yields infinite 1-separated sets in any infinite

dimensional Banach space. Kottman proved in 1975 [30] that the unit sphere of every infinite-dimensional Banach space admits an infinite (1+)-separated subset, which was improved in 1981 by Elton and Odell to $(1+\varepsilon)$-separated for some $\varepsilon > 0$ [9][1] who also noted that $c_0(\omega_1)$ does not admit an uncountable $(1+\varepsilon)$-separated set. The Kottman constant of a Banach space (the supremum over $\delta > 0$ such that there is an infinite δ-separated subset of the unit sphere) turned out to be an important tool used for investigating the geometry of the space (e.g., [3, 32, 40, 34, 1, 41]). In fact, it is related to many aspects of Banach spaces such as, for example, packing balls, measures of noncompactness, fixed points, average distances and infinite dimensional convexity (see, e.g., the papers citing [31] or [9]).

It has been clear, at least since the paper [38] of Mercourakis and Vassiliadis, that the nature of separation in uncountable subsets of the unit sphere of a nonseparable Banach space could be equally indicative of the global and diverse properties of the space as in the separable case. The question of whether the unit sphere of a nonseparable Banach space must contain an uncountable (1+)-separated set and if so, of what cardinality compared to the density of the space, has been studied for various classes of Banach spaces, for exampl, in [5, 25, 37, 38] and recently culminated in the paper [19], where it is highlighted as a central question. Notably, the existence of $(1+\varepsilon)$-separated sets of the size equal to the density of the space was proved for superreflexive Banach spaces by Kania and Kochanek in [25] and for $C(K)$ spaces, where K is compact Hausdorff and totally disconnected by Mercourakis and Vassiliadis. Moreover Hájek, Kania and Russo proved in [19] that uncountable (1+)-separated sets exist in any Banach space of sufficiently large density. However, we provide quite a strong negative answer to the general question:

Theorem 1. *There is a (strictly convex) Banach space of density continuum where every uncountable subset of the unit sphere of regular cardinality κ includes a subset of the same cardinality which is $(1-\varepsilon)$-concentrated for some $\varepsilon > 0$.*

To prove it, apply Propositions 15 and 10 obtaining the validity of their hypotheses by Lemmas 12 and 7, respectively. The strength of the above property[2] may be appreciated by seeing how easily it yields the next two theorems which provide other natural properties of Banach spaces unknown to occur until now without making some additional consistent but unprovable set-theoretic assumptions.

Recall an observation of Terenzi from [48] that, if $\mathcal{Y}$ is an equilateral set in a Banach space $\mathcal{X}$, by scaling we may assume that it is a 1-equilateral set, and then by considering $\{y_0 - y : y \in \mathcal{Y} \setminus \{y_0\}\}$ for any $y_0 \in \mathcal{Y}$ we obtain a 1-equilateral set consisting of elements of the unit sphere of $\mathcal{X}$. So Theorem 1 yields:

[1] In fact there is a recent stronger result of T. Russo ([45]) which say that the sphere of any infinite-dimensional Banach space contains infinite $\mathcal{Y}$ such that $\{\pm y : y \in \mathcal{Y}\}$ is $(1+\varepsilon)$-separated for some $\varepsilon > 0$.

[2] Another striking property of such spaces is that one cannot pack uncountably many pairwise disjoint open balls of radius $\frac{1}{3}$ into the unit ball, as such a packing would yield an uncountable 1-separated subset of the unit sphere (cf. [31]). Packing of uncountably many balls of radius $\frac{1}{3} - \varepsilon$ into the unit ball is possible in any inseparable Banach space for any $\varepsilon > 0$ by the Riesz lemma.

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Theorem 2. *There is a (strictly convex) Banach space of density continuum which does not admit an uncountable equilateral set.*

Banach spaces with this property were previously consistently constructed by the author in [27]. These spaces were of the form $C(K)$ for K compact Hausdorff, while it was also proved in [27] that consistently every Banach space of the form $C(K)$ admits an uncountable equilateral set.

Different examples of Banach spaces satisfying Theorem 2 which also do not require any additional set-theoretic assumptions are being presented at the same time in a joint paper of the author with H.M. Wark [29]. They are renormings of $\ell_1([0,1])$, and moreover they do not admit even any infinite equilateral set (However, all equivalent renormings of $\ell_1(\Gamma)$ for Γ uncountable admit $(1+\varepsilon)$-separated set of size Γ by Remark 3.16 of [19]). The above theorem solves Problem 293 of [16].

As in the case of (1+)-separated or $(1+\varepsilon)$-separated sets, the existence of infinite or uncountable equilateral sets (by the above argument of Terenzi one may assume that such sets are subsets of the unit sphere) was proved by various authors for many particular classes of Banach spaces. For example, infinite equilateral sets exist in any Banach space which contains an isomorphic copy of c_0 [37] or any uniformly smooth space [10]. However, in contrast to the case of (1+)-separated sets, there are infinite-dimensional Banach spaces with no infinite equilateral subsets [48, 47, 11]. Uncountable equilateral sets in nonseparable Banach spaces have been investigated as well, for example, in [5, 27, 37, 38]. Also the strict convexity of the norm of our example should be compared with the results saying that the unit sphere of an infinite-dimensional uniformly convex space admits a $(1+\varepsilon)$-separated set of the cardinality equal to the density of the space [40], Proposition 4.16 of [19].

As stressed in [19], (1+)-separated subsets of the sphere are related to Auerbach bases. Recall that for a Banach space $\mathcal{X}$ the system $(x_i,x_i^*)_{i\in\mathcal{I}} \subseteq \mathcal{X}\times\mathcal{X}^*$ is called a biorthogonal system when $x_j^*(x_i)=0$ if $i\neq j$ and $x_i^*(x_i)=1$ for each $i,j\in I$. It is called an Auerbach system if it is biorthogonal and $\|x_i\|=1=\|x_i^*\|$ for every $i\in I$. It is clear that the elements of $\mathcal{X}$ in an Auerbach system $(x_i,x_i^*)_{i\in\mathcal{I}}$ form a 1-separated subset of the unit sphere as $\|x_i-x_j\|\geq |x_i(x_i-x_j)|=|x_i^*(x_i)|=1$ for any distinct $i,j\in I$. Thus, Theorem 1 yields the following:

Theorem 3. *There is a (strictly convex) Banach space of density continuum with no uncountable Auerbach system.*

A Banach space with this property was previously only constructed under the assumption of the continuum hypothesis CH in [19] (in fact, this is a renorming of $c_0(\omega_1)$ so WLD, the property not shared by our space). By a result of Day, every separable Banach space admits an infinite Auerbach system [6]. Constructions of Banach spaces of density continuum with no fundamental Auerbach systems were presented in [13, 14, 43, 15].

Our Banach space is an equivalent renorming $(\mathcal{X}_{\mathcal{A}},\|\ \|_T)$ of the subspace $(\mathcal{X}_{\mathcal{A}},\|\ \|_\infty)$ of ℓ_∞ which is spanned by c_0 and characteristic functions of elements of an uncountable almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$. Recall that $\mathcal{A}\subseteq\wp(\mathbb{N})$ is called almost

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disjoint if $A \cap A'$ is finite for any distinct $A, A' \in \mathcal{A}$. Such spaces $\mathcal{X}_\mathcal{A}$ were first considered by Johnson and Lindenstrauss in [23, 24]. The renorming $\| \ \|_T$ is obtained in a standard way using a bounded injective operator $T : \ell_\infty \to \ell_2$ (see Section 2.3). As is well known, such a space $(\mathcal{X}_\mathcal{A}, \| \ \|_\infty)$ is isometric to the space of the form $C_0(K_\mathcal{A})$, where $K_\mathcal{A}$ is locally compact, totally disconnected, scattered, separable Hausdorff space known as Ψ-space or Mrówka-Isbell space or Alexandroff–Urysohn space (see, e.g., [21, 28]). In particular, it is c_0-saturated (by [42]) so admits infinite equilateral sets by [37]. The main point, which actually allows one to conclude directly the previous theorems, is to obtain the behaviour of separated sets of the unit sphere as in $c_0(\Gamma)$ and at the same time to have the underlying locally compact space separable (which allows the construction an injective operator from $C(K_\mathcal{A})$ into the separable ℓ_2). This is summarized in the following theorem which is proved by applying Lemma 12 and Proposition 15:

Theorem 4. *There is a separable locally compact Hausdorff space K of weight continuum such that for every $\varepsilon > 0$ and every subset $\mathcal{Y}$ of the unit sphere of $C_0(K)$ of a regular uncountable cardinality there is a subset $\mathcal{Z} \subseteq \mathcal{Y}$ of the same cardinality which is $(1+\varepsilon)$-concentrated.*

To obtain the above property of the space $(\mathcal{X}_\mathcal{A}, \| \ \|_\infty)$ and consequently the properties of its renorming $(\mathcal{X}_\mathcal{A}, \| \ \|_T)$ we need some special property of the almost disjoint family $\mathcal{A}$. A known property that is sufficient for us is the $\mathbb{R}$-embeddability of $\mathcal{A}$ (Definition 11). On the other hand, we show that certain almost disjoint families which are not $\mathbb{R}$-embeddable, known as Luzin families (Definition 18), induce the Banach space $\mathcal{X}_\mathcal{A}$ such that the sphere of $(\mathcal{X}_\mathcal{A}, \| \ \|_\infty)$ admits an uncountable 2-equilateral subset (Proposition 20) and the sphere of $(\mathcal{X}_\mathcal{A}, \| \ \|_T)$ admits an uncountable $(2-\varepsilon)$-separated subset for any $\varepsilon > 0$ (Proposition 21). For more on $\mathbb{R}$-embeddability, see [17]. Note that A. Dow showed in [8] that assuming PFA (the proper forcing axiom) every maximal almost disjoint family contains a Luzin subfamily. However, $\mathbb{R}$-embeddable families of cardinality continuum are abundant and elementary to construct with no additional set-theoretic assumptions (like Luzin families of cardinality ω_1).

Finally, let us comment on the isomorphic theory structure of our spaces (For the Kottman constant in the isomorphic context, see [4]). Using the arguments of Section 5 of [2] (cf. [35]) one can see that there are 2^{2^ω} pairwise nonisomorphic spaces satisfying Theorem 1. It should be clear that the almost disjoint family of branches of the Cantor tree is $\mathbb{R}$-embeddable and such families are Borel in the product topology on $2^\mathbb{N}$ (see, e.g., Lemma 30 of [28]), so it follows from the results of [36] that the spaces $\mathcal{X}_\mathcal{A}$ satisfying Theorem 1 can be isomorphically of the form $C(K_\mathcal{A})$, where $K_\mathcal{A}$ is a Rosenthal compact space. The results of [12] imply that such spaces are representable in the sense of that paper, so they should be considered relatively constructive and nonpathological. $\mathcal{X}_\mathcal{A}$ in the norm $\| \ \|_\infty$ admits a 1-equilateral set of cardinality continuum which is an Auerbach system (just characteristic functions of the elements of the almost disjoint family $\mathcal{A}$). Although, as mentioned above, there are many nonisomorphic Banach spaces of the form $\mathcal{X}_\mathcal{A}$ for an almost disjoint families $\mathcal{A}$, under Martin's axiom MA and the negation of CH all spaces $\mathcal{X}_\mathcal{A}$ are pairwise isomorphic for $\mathcal{A}$ of cardinality ω_1, in particular $\mathcal{X}_\mathcal{A}$ can be isomorphic to $\mathcal{X}_{\mathcal{A}'}$ with $\mathcal{A}$ being $\mathbb{R}$-embeddable and $\mathcal{A}'$ being Luzin ([2]). This, for

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example, provides equivalent renormings of $(\mathcal{X}_A, \| \ \|_\infty)$ for any Luzin family $\mathcal{A}$ which satisfy Theorem 1. In fact, at least consistently, every nonseparable Banach space can be renormed so that the new unit sphere admits uncountable 2-equilateral sets (Theorem 3 [37]). It is unknown at the present moment if this can be proved in ZFC alone.

2. Preliminaries and terminology

2.1. Notation

The notation attempts to be standard, for unexplained terminology see [18]. In particular $C(K)$ denotes the set of all continuous functions on a compact Hausdorff space K and $C_0(K)$ denotes all continuous functions f on a locally compact Hausdorff K such that for each $\varepsilon > 0$ there is a compact $L \subset K$ such that $|f(x)| < \varepsilon$ for all $x \in K \setminus L$. Both of these types of linear spaces are considered as Banach spaces with the supremum norm which is denoted by $\| \ \|_\infty$. The notation 1_A denotes the characteristic function of a set A. When $\mathcal{Y}$ is a subset of a Banach space, $\overline{span}(\mathcal{Y})$ denotes the norm closure of the linear span of $\mathcal{Y}$. The density of an infinite-dimensional Banach space is the minimal cardinality of a norm dense subset of the space. By ω_1, we mean the first uncountable cardinal. All almost disjoint families of subsets of $\mathbb{N}$ considered in this paper are uncountable and consist of infinite sets. A cardinal κ is said to be of uncountable cofinality if the union of countably many sets of cardinalities smaller than κ has cardinality smaller than κ. If A is a set, then $[A]^2$ denotes the family of all two-element subsets of A. The cardinals like ω_1 or the continuum are of uncountable cofinality ([22]). If $(x_n)_{n\in\mathbb{N}}$ is a sequence of reals and $A \subseteq \mathbb{N}$ is infinite by $\lim_{n\in A} x_n$ we mean $\lim_{k\to\infty} x_{n_k}$, where $(n_k)_{k\in\mathbb{N}}$ is any (bijective) renumeration of A.

2.2. Banach spaces $\mathcal{X}_\mathcal{A}$

Given an infinite almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$, we consider the subspace of $(\ell_\infty, \| \ \|_\infty)$ defined by

$$\mathcal{X}_\mathcal{A} = \overline{span}(\{1_A : A \in \mathcal{A}\} \cup c_0).$$

Note that for a fixed $A_0 \in \mathcal{A}$ the set

$$\ell_\infty^{A_0} = \{f \in \ell_\infty : \lim_{n\in A_0} f(n) \text{ exists}\}$$

is a closed (in the $\| \ \|_\infty$-norm) linear subspace of ℓ_∞. As for every $A_0 \in \mathcal{A}$ all the generators $\{1_A : A \in \mathcal{A}\} \cup c_0$ of $\mathcal{X}_\mathcal{A}$ are in this space, it follows that the entire $\mathcal{X}_\mathcal{A}$ is contained in every ℓ_∞^A for $A \in \mathcal{A}$.

Lemma 5. *Suppose that $\mathcal{A}$ is an almost disjoint family of infinite subsets of $\mathbb{N}$. Let $k \in \mathbb{N}$, $g \in c_0$, $A_1, \dots, A_k \in \mathcal{A}$ be distinct, $q_1, \dots q_k \in \mathbb{R}$ and*

$$f = g + \sum_{1\le i\le k} q_i 1_{A_i}.$$

Then for every $1 \le j \le k$, we have $\lim_{n\in A_j} f(n) = q_j$.

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Proof. We have $\lim_{n\in A_j} g(n)=0$ as $g\in c_0$ and $\lim_{n\in A_j} 1_{A_i}(n)=0$ for $i\neq j$ since $A_i\cap A_j$ is finite. □

2.3. Equivalent renormings induced by bounded operators

Recall that two norms $\|\ \|$ and $\|\ \|'$ on a Banach space $\mathcal{X}$ are equivalent if the identity operator between $(\mathcal{X},\|\ \|)$ and $(\mathcal{X},\|\ \|')$ is an isomorphism, that is, when there are positive constants c,C such that $c\|x\|\leq\|x\|'\leq C\|x\|$ for every $x\in\mathcal{X}$. For more on renormings, see [7]. It is clear that if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces with norms $\|\ \|_{\mathcal{X}}$ and $\|\ \|_{\mathcal{Y}}$, respectively, and $T:\mathcal{X}\to\mathcal{Y}$ is a bounded linear operator. Then

$$\|x\|_T=\|x\|_{\mathcal{X}}+\|T(x)\|_{\mathcal{Y}}$$

is a norm on $\mathcal{X}$ which is equivalent to the norm $\|\ \|_{\mathcal{X}}$. In this paper, besides the supremum norm $\|\ \|_\infty$, we will consider norms of the form $\|\ \|_T$. Recall that a norm $\|\ \|$ on a Banach space $\mathcal{X}$ is called strictly convex if $\|x+y\|=\|x\|+\|y\|$ for $x,y\in\mathcal{X}\setminus\{0\}$ implies that $x=\lambda y$ for some $\lambda>0$. An example of a strictly convex norm is the standard $\|\ \|_2$ norm on ℓ_2.

Lemma 6. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. If $T:\mathcal{X}\to\mathcal{Y}$ is an injective bounded linear operator, $(\mathcal{Y},\|\ \|_{\mathcal{Y}})$ is strictly convex, then $(\mathcal{X},\|\ \|_T)$ is strictly convex.*

Proof. For nonzero $x,y\in\mathcal{X}$, by the triangle inequality, the condition

$$\|x+y\|_{\mathcal{X}}+\|T(x)+T(y)\|_{\mathcal{Y}}=\|x\|_{\mathcal{X}}+\|T(x)\|_{\mathcal{Y}}+\|y\|_{\mathcal{X}}+\|T(y)\|_{\mathcal{Y}}$$

implies that $\|x+y\|_{\mathcal{X}}=\|x\|_{\mathcal{X}}+\|y\|_{\mathcal{X}}$ and $\|T(x)+T(y)\|_{\mathcal{Y}}=\|T(x)\|_{\mathcal{Y}}+\|T(y)\|_{\mathcal{Y}}$. The latter implies $T(x)=\lambda T(y)$ for some $\lambda>0$, which gives $x=\lambda y$ since T is injective. □

Lemma 7. *There is an injective bounded linear operator $T:\ell_\infty\to\ell_2$. In particular, the equivalent renorming $(\ell_\infty,\|\ \|_T)$ is strictly convex.*

Proof. Consider $T:\ell_\infty\to\ell_2$ given by

$$T(f)=\Big(\frac{f(n)}{2^{n/2}}\Big)_{n\in\mathbb{N}}.$$

$\|f\|_\infty\leq\|f\|_\infty+\|T(f)\|_2=\|f\|_\infty+\sqrt{\sum_{n\in\mathbb{N}}\frac{f(n)^2}{2^n}}\leq(1+\sqrt{2})\|f\|_\infty$. □

Lemma 8. *Suppose that κ is a cardinal of uncountable cofinality and $\mathcal{Y}$ is a subset of cardinality κ of a separable Banach space $\mathcal{X}$. Then for every $\varepsilon>0$ there is $\mathcal{Y}'\subseteq\mathcal{Y}$ of cardinality κ such that $\|y-y'\|\leq\varepsilon$ for every $y,y'\in\mathcal{Y}'$.*

Proof. Let $\{x_n:n\in\mathbb{N}\}$ be norm dense countable subset of $\mathcal{X}$. Balls of diameter ε with the centers in the points x_n cover $\mathcal{X}$ and so cover $\mathcal{Y}$. By the uncountable cofinality of κ, one of the balls must contain κ elements of $\mathcal{Y}$. □

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Lemma 9. *Suppose that* $0 < a \le \|x\| \le \|x'\| \le b < 1$ *for some* $a,b \in \mathbb{R}$ *and* x,x' *in a Banach space* $\mathcal{X}$*. Then*

$$\|x - x'\| \le b\left\| \frac{x}{\|x\|} - \frac{x'}{\|x'\|} \right\| + (b-a)$$

Proof.

$$\|x - x'\| \le \|x - (\|x\|/\|x'\|)x'\| + \|(\|x\|/\|x'\|)x' - x'\| \le$$

$$\le \|x\|\left\| \frac{x}{\|x\|} - \frac{x'}{\|x'\|} \right\| + \left| \frac{\|x\|}{\|x'\|} - 1 \right| \|x'\| \le$$

$$\le b\left\| \frac{x}{\|x\|} - \frac{x'}{\|x'\|} \right\| + (1 - a/b)b.$$

□

The following reduction result allows us to infer the main properties of the final space $(\mathcal{X}_\mathcal{A}, \| \ \|_T)$ as stated in Theorem 1 from the properties of the space $(\mathcal{X}_\mathcal{A}, \| \ \|_\infty)$ which are proved in Proposition 15.

Proposition 10. *Suppose that* $\mathcal{X}$ *is a nonseparable Banach space and* κ *is a cardinal of uncountable cofinality such that for every subset* $\mathcal{Y}$ *of the unit sphere of* $(\mathcal{X}, \| \ \|_\mathcal{X})$ *of cardinality* κ *and every* $\varepsilon > 0$ *there is* $\mathcal{Y}' \subseteq \mathcal{Y}$ *of cardinality* κ *which is* $(1+\varepsilon)$*-concentrated. If* $T : \mathcal{X} \to \mathcal{Z}$ *is a bounded linear injective operator into a Banach space* $\mathcal{Z}$ *with separable range, then* $(\mathcal{X}, \| \ \|_T)$ *has the following property: For every subset* $\mathcal{Y}$ *of the unit sphere of* $(\mathcal{X}, \| \ \|_T)$ *of cardinality* κ*, there is* $\delta > 0$ *and* $\mathcal{Y}' \subseteq \mathcal{Y}$ *of cardinality* κ *which is* $(1-\delta)$*-concentrated.*

Proof. Let $\mathcal{Y}$ be a subset of the unit sphere of $(\mathcal{X}, \| \ \|_T)$ of cardinality κ. As T is injective, we have $0 < \|y\|_\mathcal{X} < \|y\|_T = 1$ for every $y \in \mathcal{Y}$. The interval $(0,1)$ can be covered by intervals of the form $(q-(1-q)/4, q)$ for rationals q satisfying $0 < q < 1$. As κ has uncountable cofinality by passing to a subset of cardinality κ, we may assume that there is a rational $0 < q < 1$ such that

$$q - (1-q)/4 < \|y\|_\mathcal{X} < q$$

for every $y \in \mathcal{Y}$. Now, apply the property of $\mathcal{X}$ with the norm $\| \ \|_\mathcal{X}$ to $\{y/\|y\|_\mathcal{X} : y \in \mathcal{Y}\}$ and $\varepsilon = (1-q)/4q$ obtaining $\mathcal{X}' \subseteq \{y/\|y\|_\mathcal{X} : y \in \mathcal{Y}\}$ of cardinality κ such that $\mathcal{X}'$ is $(1+\varepsilon)$-concentrated in the $\| \ \|_\mathcal{X}$ norm. By Lemma 9 for $a = q-(1-q)/4 \le \|y\|_\mathcal{X} \le \|y'\|_\mathcal{X} \le q = b$, we obtain that

$$\|y' - y''\|_\mathcal{X} \le q(1 + (1-q)/4q) + (1-q)/4 =$$
$$= (q + (1-q)/4) + (1-q)/4 = q + (1-q)/2$$

for every $y', y'' \in \mathcal{Y}' = \{y \in \mathcal{Y} : y/\|y\|_\mathcal{X} \in \mathcal{X}'\}$. Again using the countable cofinality of κ and Lemma 8, we find $\mathcal{Y}'' \subseteq \mathcal{Y}'$ of cardinality κ such that $\|T(y) - T(y')\|_\mathcal{Z} \le (1-q)/4$ for all $y, y' \in \mathcal{Y}''$. So

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$$\|y-y'\|_T = \|y-y'\|_{\mathcal{X}} + \|T(y)-T(y')\|_{\mathcal{Z}} \leq q + 3(1-q)/4 = 1-(1-q)/4$$

obtaining that $\mathcal{Y}''$ is $(1-\delta)$-concentrated for $\delta = (1-q)/4$ as required. □

3. Concentration in spheres of Banach spaces $(\mathcal{X}_{\mathcal{A}}, \| \ \|_\infty)$ induced by $\mathbb{R}$-embeddable almost disjoint families

Definition 11. An almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$ is called $\mathbb{R}$-embeddable if there is a function $\phi : \mathbb{N} \to \mathbb{R}$ such that the sets $\phi[A]$ for $A \in \mathcal{A}$ are ranges of sequences converging to distinct reals.

So $\mathbb{R}$-embeddable families of cardinality continuum are examples of perhaps most standard almost disjoint families.

Lemma 12 (Folklore). *There exist almost disjoint families of infinite subsets of* $\mathbb{N}$ *of cardinality continuum which are* $\mathbb{R}$*-embeddable.*

Remark 13. In fact, an almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is $\mathbb{R}$-embeddable if and only if there is an injective $\phi : \mathbb{N} \to \mathbb{Q}$ such that the sets $\phi[A]$ for $A \in A$ are ranges of sequences converging to distinct irrational reals. This follows from Lemma 1 and Lemma 2 of [17].

Lemma 14. *Suppose that* κ *is a cardinal of uncountable cofinality not bigger than continuum,* $X \subseteq [0,1]$ *is uncountable and that* $\mathcal{A} = \{A_x : x \in X\}$ *is an* $\mathbb{R}$*-embeddable almost disjoint family of infinite subsets of* $\mathbb{N}$*. Then given*

(1) $k \in \mathbb{N}$,

(2) *a finite* $F \subseteq \mathbb{N}$,

(3) *a collection* $\{a_\xi : \xi < \kappa\}$ *of pairwise disjoint finite subsets of* X *with* $a_\xi = \{x_1^\xi, \ldots, x_k^\xi\}$ $(x_i^\xi \neq x_j^\xi$ *for* $1 \leq i < j \leq k)$ *for any* $\xi < \kappa$ *such that*

$$A_{x_i^\xi} \cap A_{x_j^\xi} \subseteq F$$

for any $\xi < \kappa$ *and any* $1 \leq i < j \leq k$,

there is a subset $\Gamma \subseteq \kappa$ *of cardinality* κ *such that for every* $\xi, \eta \in \Gamma$ *we have*

$$A_{x_i^\xi} \cap A_{x_j^\eta} \subseteq F$$

for every $1 \leq i < j \leq k$.

Proof. Let $\phi : \mathbb{N} \to \mathbb{R}$ be as in the definition of $\mathbb{R}$-embeddability. By composing it with a homeomorphism from $\mathbb{R}$ onto $(0,1)$, we may assume that $Y = \phi[\mathbb{N}] \subseteq [0,1]$. As the properties stated in the lemma do not change if we relabel the elements of $\mathcal{A}$, we may assume that $\phi[A_x] = \{q_n^x : n \in \mathbb{N}\} \subseteq [0,1] \cap Y$ is such that $(q_n^x)_{n\in\mathbb{N}}$ converges to x.

Fix $k \in \mathbb{N}$ and a finite $F \subseteq \mathbb{N}$. Let $\{a_\xi : \xi < \omega_1\}$ be a collection of pairwise disjoint finite subsets of X with $a_\xi = \{x_1^\xi, \ldots, x_k^\xi\}$ as in the lemma. Using the uncountable cofinality of κ, by passing to subset of cardinality κ, we may assume that for all $\xi < \kappa$ we have $\delta_\xi > \delta$

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for some $\delta > 0$, where

$$\delta_\xi = \min(\{|x_i^\xi - x_j^\xi| : 1 \le i < j \le k\}.$$

Now, note that there is $(x_1,\dots,x_k) \in [0,1]^k$ such that every Euclidean neighbourhood of $(x_1,\dots,x_k)$ contains κ-many points $(x_1^\xi,\dots,x_k^\xi)$ for $\xi < \kappa$. This is because otherwise we can cover $[0,1]^k$ by open sets each containing less than κ-many of the points $(x_1^\xi,\dots,x_k^\xi)$, and the existence of a finite subcover would contradict the fact that all k-tuples $(x_1^\xi,\dots,x_k^\xi)$ are distinct as the sets $\{x_1^\xi,\dots,x_k^\xi\}$ are pairwise disjoint.

Since $\delta_\xi > \delta$ for every $\xi < \kappa$, we have that $\min(\{|x_i - x_j| : 1 \le i < j \le\} \ge \delta$, and in particular all elements x_i for $1 \le i \le k$ are distinct. Let $I_1,\dots I_k$ be open intervals such that $x_i \in I_i$ and $I_i \cap I_j = \emptyset$ for all $1 \le i < j \le k$. By the choice of $(x_1,\dots,x_k)$, there is $\Gamma' \subseteq \kappa$ of cardinality κ such that

$$x_i^\xi \in I_i$$

for all $\xi \in \Gamma'$ and all $1 \le i \le k$.

Since the sequence $(q_n^{x_i^\xi})_{n\in\mathbb{N}}$ converges to x_i^ξ, for every $\xi \in \Gamma'$, there is a sequence $(F_1^\xi,\dots F_k^\xi)$ of finite subsets of $\mathbb{N}$ such that for each $1 \le i \le k$ we have

(1) $F_i^\xi \subseteq A_{x_\xi^i}$

(2) $\phi[A_{x_\xi^i} \setminus F_i^\xi] \subseteq I_i$.

As $I_i \cap I_j = \emptyset$ for distinct $i,j \le k$ condition (2) implies that

(3) $(A_{x_\xi^i} \setminus F_i^\xi) \cap (A_{x_\eta^j} \setminus F_j^\eta) = \emptyset$ for any $\xi,\eta < \kappa$ and $1 \le i < j \le k$.

As the cofinality of κ is uncountable and there are countably many such k-tuples of sets $(F_1^\xi,\dots F_k^\xi)$ (as they are all subsets of $\mathbb{N}$), we may choose $\Gamma \subseteq \Gamma'$ of cardinality κ such that $(F_1^\xi,\dots F_k^\xi)$ is equal to some fixed $(F_1,\dots F_k)$.

It remains to prove the property of Γ stated in the lemma. Fix distinct $\xi,\eta \in \Gamma$ and $1 \le i < j \le k$. Then by (1) and (3) for all $1 \le i < j \le k$, we have

$$A_{x_i^\xi} \cap A_{x_j^\eta} = (F_i \cap F_j) \cup (A_{x_i^\xi} \cap F_j) \cup (A_{x_j^\eta} \cap F_i).$$

By (1), this set is included in $(A_{x_i^\xi} \cap A_{x_j^\xi}) \cup (A_{x_j^\eta} \cap A_{x_i^\eta})$ which is included in F by the hypothesis of the lemma. So we obtain $A_{x_i^\xi} \cap A_{x_j^\eta} \subseteq F$ as required. □

Proposition 15. *Suppose that $\mathcal{A}$ is an $\mathbb{R}$-embeddable almost disjoint family of subsets of $\mathbb{N}$. Whenever $\varepsilon > 0$ and $\mathcal{Y}$ is a subset of of the unit sphere of $(\mathcal{X}_\mathcal{A}, \|\ \|_\infty)$ of regular uncountable cardinality κ, there is $\mathcal{Y}' \subseteq \mathcal{Y}$ of cardinality κ which is $(1+\varepsilon)$-concentrated.*

Proof. We may label elements of $\mathcal{A}$ as A_x for $x \in X$ for some $X \subseteq [0,1]$.

Fix a subset $\mathcal{Y} = \{f_\xi : \xi < \kappa\}$ of the unit sphere of $(\mathcal{X}_\mathcal{A}, \|\ \|_\infty)$ of uncountable regular cardinality κ. By the definition of $\mathcal{X}_\mathcal{A}$ for each $\xi < \kappa$, there are finite sets $\{x_1^\xi,\dots x_{m_\xi}^\xi\} \subseteq X$ ($x_i^\xi \ne x_j^\xi$ for $1 \le i < j \le m_\xi$ and $\xi < \kappa$), distinct rationals $q_1^\xi,\dots q_{m_\xi}^\xi$ for all $\xi < \kappa$ and rational

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valued finitely supported $g_\xi \in c_0$ such that

$$\Big\| f_\xi - \big(g_\xi + \sum_{1\leq i\leq m_\xi} q_i^\xi 1_{A_{x_i^\xi}}\big)\Big\|_\infty \leq \varepsilon/3.$$

Using the uncountable cofinality of κ by passing to a subset of cardinality κ, we may assume that for all $\xi<\omega_1$ we have $m_\xi = m$ for some $m\in\mathbb{N}$, $g_\xi = g$ for some finitely supported $g\in c_0$, and that $q_i^\xi = q_i$ for some rationals q_i for all $1\leq i\leq m$ and for all $\xi<\kappa$. Since for a regular cardinal κ any family of κ-many finite sets contains a subfamily of cardinality κ such that the intersection of any two distinct members of the subfamily is a fixed finite $\Delta\subseteq\kappa$ (A version of the Δ-system lemma, Example 25.3 of [26]) by passing to a subset of cardinality κ, we may assume that there is such a Δ satisfying

$$\{x_1^\xi,\dots x_m^\xi\}\cap\{x_1^\eta,\dots x_m^\eta\} = \Delta$$

for every $\xi<\eta<\kappa$. By passing to a subset of cardinality κ, we may assume that there is a fixed $G\subseteq\{1,\dots,m\}$ such that $\Delta = \{x_i^\xi : i\in G\}$ for every $\xi<\kappa$. If $G=\{1,\dots,m\}$, we conclude that $\{f_\xi : \xi<\kappa\}$ is even $2\varepsilon/3$-concentrated, so we will assume that G is a proper subset of $\{1,\dots,m\}$. So by reordering $\{1,\dots,m\}$, we may assume that there is $1\leq k\leq m$ such that the sets $\{x_1^\xi,\dots x_k^\xi\}$ are pairwise disjoint for $\xi<\kappa$ and $\{x_{k+1}^\xi,\dots x_m^\xi\} = \{x_{k+1},\dots x_m\} = \Delta$ for some $x_{k+1},\dots x_m\in X$ for each $\xi<\kappa$. So for every $\xi<\kappa$, we have

$$\Big\| f_\xi - \big(g + \sum_{1\leq i\leq m} q_i 1_{A_{x_i^\xi}}\big)\Big\|_\infty \leq \varepsilon/3. \tag{1}$$

And for every $\xi<\eta<\kappa$, we have

$$\|f_\xi - f_\eta\|_\infty \leq \| \sum_{1\leq i\leq k} q_i 1_{A_{x_i^\xi}} - \sum_{1\leq i\leq k} q_i 1_{A_{x_i^\eta}}\|_\infty + 2\varepsilon/3. \tag{2}$$

Also, as for each $1\leq j\leq k$, we have $\lim_{n\in A_{x_j^\xi}} |f_\xi(n)|\leq 1$ since the elements f_ξ are taken from the unit sphere in the $\|\ \|_\infty$-norm, and $\lim_{n\in A_{x_j^\xi}} g(n) = 0$ as $g\in c_0$ and

$$\lim_{n\in A_{x_j^\xi}} \Big(\sum_{1\leq i\leq m} q_i 1_{A_{x_i^\xi}}\Big)(n) = q_j,$$

by Lemma 5, so we may conclude from (1) that for all $1\leq j\leq k$ we have

$$|q_j|\leq 1+\varepsilon/3. \tag{3}$$

Moreover, since there are countably many finite subsets of $\mathbb{N}$ and κ is of uncountable cofinality, we may assume that there is a fixed finite $F\subseteq\mathbb{N}$ such that

$$A_{x_i^\xi}\cap A_{x_j^\xi}\subseteq F \text{ for all } 1\leq i<j\leq m \text{ and every } \xi<\kappa. \tag{4}$$

Using Lemma 14 and the hypothesis that $\mathcal{A}$ is $\mathbb{R}$-embeddable, there is subset $\Gamma\subseteq\kappa$ of cardinality κ such that

$$A_{x_i^\xi}\cap A_{x_j^\eta}\subseteq F \text{ for any } 1\leq i<j\leq k \text{ and any } \xi,\eta\in\Gamma. \tag{5}$$

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Since there are countably many rational valued functions defined on F, by passing to a subset of Γ of cardinality κ we may also assume that for each $\xi,\eta\in\Gamma$ and each $n\in F$ we have

$$\big(\sum_{1\le i\le k} q_i 1_{A_{x_i^\xi}}\big)(n)=\big(\sum_{1\le i\le k} q_i 1_{A_{x_i^\eta}}\big)(n).$$

So by (5), the following are the only possible cases for $n\in\mathbb{N}$:

$$\Big(\sum_{1\le i\le k} q_i 1_{A_{x_i^\xi}}-\sum_{1\le i\le k} q_i 1_{A_{x_i^\eta}}\Big)(n)=\begin{cases}0 & \text{if } n\in F,\\ q_i-q_i=0 & \text{if } n\in A_{x_i^\xi}\cap A_{x_i^\eta}\setminus F \text{ for some } i,\\ q_i & \text{if } n\in A_{x_i^\xi}\setminus(A_{x_i^\eta}\cup F) \text{ for some } i,\\ -q_i & \text{if } \in A_{x_i^\eta}\setminus(A_{x_i^\xi}\cup F) \text{ for some } i,\\ 0 & \text{if } n\notin\bigcup_{1\le i\le k}(A_{x_i^\xi}\cup A_{x_i^\eta}).\end{cases}$$

By (3), it follows that

$$\Big\|\sum_{1\le i\le k} q_i 1_{A_{x_i^\xi}}-\sum_{1\le i\le k} q_i 1_{A_{x_i^\eta}}\Big\|_\infty\le\max\{|q_i|:1\le i\le k\}\le 1+\varepsilon/3$$

which by (2) implies the required $\|f_\xi-f_\eta\|\le 1+\varepsilon$ for any $\xi,\eta\in\Gamma$. □

Remark 16. We note that in the language of the paper [20] of Hrušák and Guzmán an almost disjoint family $\mathcal{A}$ cannot have both the property from Lemma 14 and contain an n-Luzin gap for some $n\in\mathbb{N}$. They showed that under the assumption of MA and the negation of CH every almost disjoint family of cardinality smaller than continuum which contains no n-Luzin gaps for any $n\in\mathbb{N}$ is $\mathbb{R}$-embeddable.

Remark 17. By Proposition 15, for every $\varepsilon>0$ the space $(\mathcal{X}_\mathcal{A},\|\ \|_\infty)$ for an $\mathbb{R}$-embeddable almost disjoint family $\mathcal{A}$ admits no uncountable $(1+\varepsilon)$-separated set in its unit sphere. Nevertheless, such spaces always admit (1+)-separated sets of unit vectors of the cardinality equal to the density of $\mathcal{X}_\mathcal{A}$. For $A\in\mathcal{A}$, define

$$f_A=1_A-\sum\{\frac{1}{k+1}1_{\{k\}}:k\in\mathbb{N}\setminus A\}$$

Then given any two distinct $A,A'\in\mathcal{A}$, choose $k\in A\setminus A'$ and observe that we have $\|f_A-f_{A'}\|\ge|f_A-f_{A'}|(k)=1-(-1/(k+1))>1$.

4. Separation in spheres of Banach spaces $(\mathcal{X}_\mathcal{A},\|\ \|_T)$ induced by Luzin almost disjoint families

Definition 18. An almost disjoint family $\{A_\xi:\xi<\omega_1\}$ is called a Luzin family if $f_\alpha:\alpha\to\mathbb{N}$ is finite-to-one for any $\alpha<\omega_1$, where

$$f_\alpha(\beta)=\max(A_\beta\cap A_\alpha)$$

for every $\beta<\alpha<\omega_1$.

Lemma 19 ([33]). *Luzin families exist.*

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Luzin families were first constructed in [33]. See Section 3.1. of [21] for the construction and for more information.

Proposition 20. *Suppose that $\mathcal{L}=\{A_\xi:\xi<\omega_1\}$ is a Luzin family, and ξ_α,η_α satisfy*

$$\xi_\beta<\eta_\beta<\xi_\alpha<\eta_\alpha<\omega_1$$

for all $\beta<\alpha<\omega_1$. Then $(\mathcal{X}_\mathcal{L},\|\ \|_\infty)$ admits an uncountable 2-equilateral set $\{f_\alpha:\alpha\in X\}$ among elements of its unit sphere of the form

$$f_\alpha=1_{A_{\xi_\alpha}}-1_{A_{\eta_\alpha}}$$

for all $\alpha\in\Gamma$ for some uncountable $\Gamma\subseteq\omega_1$.

Proof. It is clear that $\|f_\alpha\|_\infty\leq 1$ for each $\alpha<\omega_1$ as f_α may assume values among $\{-1,0,1\}$. Since $A_{\xi_\alpha}\cap A_{\eta_\alpha}$ is finite and both $A_{\xi_\alpha},A_{\eta_\alpha}$ are infinite, we have that $\|f_\alpha\|=1$ for each $\alpha<\omega_1$. Let $F_\alpha\subseteq\mathbb{N}$ for $\alpha<\omega_1$ be such a finite set that $A_{\xi_\alpha}\cap A_{\eta_\alpha}\subseteq F_\alpha$. By passing to an uncountable subset, we may assume that $F_\alpha=F$ for each $\alpha<\omega_1$ and some finite $F\subseteq\mathbb{N}$. Let $k\in\mathbb{N}$ be such that $F\subseteq\{1,\dots,k\}$. Now, we will use the following version of Erdös-Dushnik-Miller theorem: Whenever κ is regular and uncountable cardinal and $c:[\kappa]^2\to\{0,1\}$, then either there is 1-monochromatic set of cardinality κ or there is a 0-monochromatic set of order type $\omega+1$ (see 24.32 of [26]). Consider a coloring $c:[\omega_1]^2\to\{0,1\}$ defined by $c(\{\beta,\alpha\})=0$ if $f_{\xi_\alpha}(\eta_\beta)\leq k$ for the ordering of the elements $\beta<\alpha$ and $c(\{\beta,\alpha\})=1$ otherwise. Note that there cannot be a 0-monochromatic set $\Delta\subseteq\omega_1$ of order type $\omega+1$ because denoting its biggest element as α we would have that $f_{\xi_\alpha}|\{\eta_\beta:\beta\in\Delta\cap\alpha\}$ is bounded below by k which would contradict the hypothesis that $\mathcal{L}$ is a Luzin family, that is, that the functions f_{ξ_α} are finite-to-one. So it follows that there is an uncountable 1-monochromatic $\Gamma\subseteq\omega_1$ for the coloring c.

For $\alpha,\beta\in\Gamma$ with $\beta<\alpha$, there is $m>k$ such that $m\in A_{\xi_\alpha}\cap A_{\eta_\beta}$. By the choice of F and the fact that $F\subseteq\{1,\dots,k\}$, we have that neither $m\in A_{\eta_\alpha}$ nor $m\in A_{\xi_\beta}$, and so it follows that

$$\|f_\alpha-f_\beta\|_\infty=\|(1_{A_{\xi_\alpha}}-1_{A_{\eta_\alpha}})-(1_{A_{\xi_\beta}}-1_{A_{\eta_\beta}})\|_\infty=2.$$ □

Proposition 21. *Suppose that $\mathcal{L}=\{A_\xi:\xi<\omega_1\}$ is a Luzin family, $\mathcal{A}$ is an almost disjoint family such that $\mathcal{L}\subseteq\mathcal{A}$, $\mathcal{X}$ is a Banach space and $T:\mathcal{X}_\mathcal{A}\to\mathcal{X}$ is a bounded linear operator with separable range. Then for every $\varepsilon\in(0,1)$ the unit sphere of $(\mathcal{X}_\mathcal{A},\|\ \|_T)$ admits an uncountable $(2-\varepsilon)$-separated set.*

Proof. As $\mathcal{X}_\mathcal{L}$ is a subspace of $\mathcal{X}_\mathcal{A}$, we can assume that $\mathcal{A}=\mathcal{L}$. By Lemma 8, there is an uncountable subset $\Xi\subseteq\omega_1$ such that

$$\|T(1_{A_\xi}-1_{A_\eta})\|\leq\frac{\varepsilon/2}{1-\varepsilon/2}\qquad(\star)$$

for every $\xi,\eta\in\Xi$. Let $\{\xi_\alpha:\alpha<\omega_1\}$ and $(\eta_\alpha:\alpha<\omega_1)$ be enumerations of two uncountable and disjoint subsets of Ξ satisfying for all $\beta<\alpha<\omega_1$ the following conditions

$$\xi_\beta<\eta_\beta<\xi_\alpha<\eta_\alpha.$$

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By Proposition 20, there is an uncountable $\Gamma \subseteq \omega_1$ such that $\{f_\alpha : \alpha \in \Gamma\}$ is a 2-equilateral set of the unit sphere of $(\mathcal{X}_\mathcal{A}, \| \ \|_\infty)$, where for each $\alpha \in \Gamma$ we have

$$f_\alpha = 1_{A_{\xi_\alpha}} - 1_{A_{\eta_\alpha}}.$$

For $\alpha \in \Gamma$, consider

$$g_\alpha = f_\alpha / \|f_\alpha\|_T.$$

We claim that $\{g_\alpha : \alpha \in \Gamma\}$ is the desired $(2-\varepsilon)$-separated set of the unit sphere of $(\mathcal{X}_\mathcal{A}, \| \ \|_T)$. To prove it take $\beta < \alpha$ with $\alpha, \beta \in \Gamma$. As

$$\|g_\alpha - g_\beta\|_T = \|g_\alpha - g_\beta\|_\infty + \|T(g_\alpha) - T(g_\beta)\|_\mathcal{X},$$

it is enough to prove that $\|g_\alpha - g_\beta\|_\infty \geq 2-\varepsilon$. We have

$$\|g_\alpha - g_\beta\|_\infty = \|\frac{f_\alpha}{\|f_\alpha\|_T} - \frac{f_\alpha}{\|f_\alpha\|_T}\|_\infty \geq \|f_\alpha - f_\beta\|_\infty - \|\frac{f_\alpha}{\|f_\alpha\|_T} - f_\alpha\|_\infty - \|\frac{f_\beta}{\|f_\beta\|_T} - f_\beta\|_\infty.$$

Since $\alpha, \beta \in \Gamma$, we have $\|f_\alpha\|_\infty = \|f_\beta\|_\infty = 1$, $\|f_\alpha - f_\beta\|_\infty = 2$. So now, we estimate

$$\|\frac{f_\alpha}{\|f_\alpha\|_T} - f_\alpha\|_\infty = \Big|\frac{1}{\|f_\alpha\|_\infty + \|T(f_\alpha)\|_\mathcal{X}} - 1\Big| \|f_\alpha\|_\infty = \Big|\frac{1}{1 + \|T(f_\alpha)\|_\mathcal{X}} - 1\Big|.$$

By equation (*), we have $\|T(f_\alpha)\|_\mathcal{X} \leq \frac{\varepsilon/2}{1-\varepsilon/2}$, and so

$$1 - \varepsilon/2 = \frac{1}{1 + \frac{\varepsilon/2}{1-\varepsilon/2}} \leq \frac{1}{1 + \|T(f_\alpha)\|_\mathcal{X}} \leq 1,$$

and so

$$\|\frac{f_\alpha}{\|f_\alpha\|_T} - f_\alpha\|_\infty \leq \varepsilon/2.$$

The same calculation works for $\|\frac{f_\beta}{\|f_\beta\|_T} - f_\beta\|_\infty$, so we conclude that

$$\|g_\alpha - g_\beta\|_\infty \geq 2 - \varepsilon/2 - \varepsilon/2 = 2 - \varepsilon$$

as required. □

5. Final remarks and open problems

Recall that the Ramsey theorem says that given any $k \in \mathbb{N}$ and any coloring $c : [\mathbb{N}]^2 \to \{1, \ldots, k\}$ there is an infinite $A \subseteq \mathbb{N}$ which is i-monochromatic for c for some $1 \leq i \leq k$, that is, $c[[A]^2] = \{i\}$. On the other hand, colorings of all pairs of uncountable cardinals not bigger than continuum may not have uncountable monochromatic sets as already shown by Sierpiński (24.23 of [26]). The phenomena considered in this paper can be seen from a Ramsey theoretic point of view. For example, given a Banach space X one can

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consider a coloring $c:[S_{\mathcal{X}}]^2\to\{-1,0,1\}$ given by

$$c_{\mathcal{X}}(x,y)=\begin{cases}-1 & \text{if } \|x-y\|<1\\ 0 & \text{if } \|x-y\|=1\\ 1 & \text{if } \|x-y\|>1.\end{cases}$$

A(1+)-separated set in $S_{\mathcal{X}}$ is a 1-monochromatic set for $c_{\mathcal{X}}$, and by an observation of Terenzi mentioned in the introduction, the existence of an uncountable equilateral set in $\mathcal{X}$ is equivalent to the existence of an uncountable 0-monochromatic set for $c_{\mathcal{X}}$. On the other hand, our main result says that there is a nonseparable Banach space $\mathcal{X}$ such that every uncountable subset of $S_{\mathcal{X}}$ contains an uncountable (-1)-monochromatic set. It is not accidental that the separable results of Kottman or Elton and Odell employ the Ramsey theorem [30, 9]. However, we are still far from understanding the structure of monochomatic sets for the colorings $c_{\mathcal{X}}$. For example, it is natural to ask if the following dichotomy holds:

Question 22. Is it true that the unit sphere of every nonseparable Banach space $\mathcal{X}$ either contains a (1+)-separated set or every uncountable subset of the sphere $S_{\mathcal{X}}$ contains an uncountable subset which is (1−)-concentrated (or $(1-\varepsilon)$-concentrated for some $\varepsilon>0$)?

Here, by (1−)-concentrated we mean a set $\mathcal{Y}\subseteq\mathcal{X}$ such that $\|x-y\|<1$ for any two distinct $x,y\in\mathcal{Y}$. In fact, if the above dichotomy does not hold, it would be interesting to search for a nonseparable Banach space $\mathcal{X}$ such that $c_{\mathcal{X}}$ has no uncountable 1-monochromatic sets and the family of all uncountable (-1)-monochromatic sets is minimal in some sense. Another aspect of our paper is the following general question:

Question 23. What nonseparable metric spaces can be isometrically embedded in every nonseparable Banach space?

The Riesz lemma implies that for every $\varepsilon>0$ every nonseparable Banach space of density κ contains a metric space (M,d), where $M=\{0\}\cup\{x_\xi:\xi<\kappa\}$, and where $d(0,x_\xi)=1$ for all $\xi<\kappa$ and $d(x_\xi,x_\eta)>1-\varepsilon$ for any $\xi<\eta<\kappa$. In this article, among other results, we have shown in ZFC that uncountable metric spaces, where distances between any two distinct points are the same, do not embed isometrically into all nonseparable Banach spaces. The above question has been considered on the countable level in [39]. On the finite level, for example, Shkarin has proved in [46] that every finite ultrametric space (a metric space where the distance d satisfies $d(x,z)\leq\max(d(x,y),d(y,z))$ for any points x,y,z) isometrically embeds in any infinite-dimensional Banach space.

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