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Cohomology of Complex Projective Stiefel Manifolds

L. Astey, S. Gitler, E. Micha and G. Pastor

Abstract. The cohomology algebra mod p of the complex projective Stiefel manifolds is determined for all primes p. When $p = 2$ we also determine the action of the Steenrod algebra and apply this to the problem of existence of trivial subbundles of multiples of the canonical line bundle over a lens space with 2-torsion, obtaining optimal results in many cases.

1 Introduction

Let $W_{n,k}$ denote the complex Stiefel manifold of orthonormal complex k-frames in complex n-space. The complex projective Stiefel manifold $\mathrm{PW}_{n,k}$ is the quotient space of the free circle action on $W_{n,k}$ given by $z(v_1, \dots, v_k) = (zv_1, \dots, zv_k)$. For example, $\mathrm{PW}_{n,1}$ is $(n-1)$-dimensional complex projective space and $\mathrm{PW}_{n,n}$ is the projective unitary group $\mathrm{PU}(n)$. In this paper we determine the cohomology algebra mod p of $\mathrm{PW}_{n,k}$ for all primes p. It turns out that the case of the prime 2 is the most interesting and difficult to understand. In this case we also determine the action of the Steenrod algebra and apply this to the problem of existence of trivial subbundles of multiples of the canonical line bundle over a lens space (see (1.3) below).

In [5] C. Ruiz studied the integral cohomology ring of the complex projective Stiefel manifolds, but unfortunately the theorem which states its structure is incorrect; according to Ruiz's description the top cohomology class is a torsion class, but $\mathrm{PW}_{n,k}$ is a simply connected closed manifold. Although in his paper Ruiz also provides a description of the mod p cohomology which is correct in most cases, we feel that the result requires a correct account.

Let $x \in H^2(\mathrm{PW}_{n,k}; \mathbb{Z}/p)$ be the mod p Euler class of the complex line bundle associated with the principal bundle $\pi\colon W_{n,k} \longrightarrow \mathrm{PW}_{n,k}$ and let N be the smallest integer such that $N > n - k$ and $\binom{n}{N} \not\equiv 0 \bmod p$. We shall prove the following theorems.

Theorem 1.1 *Let $p > 2$ be prime. There exist classes $y_j \in H^{2j-1}(\mathrm{PW}_{n,k}; \mathbb{Z}/p)$ for $n - k < j \leq n$ such that*

$$H^*(\mathrm{PW}_{n,k}; \mathbb{Z}/p) = \mathbb{Z}/p[x]/(x^N) \otimes \Lambda(y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n).$$

Theorem 1.2 *Let $p = 2$. There exist classes $y_j \in H^{2j-1}(\mathrm{PW}_{n,k}; \mathbb{Z}/2)$ for $n - k < j \leq n$ that satisfy the following conditions*

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(i) *If* $n \not\equiv 2 \bmod 4$ *or* $k < n$ *then*

$$H^*(\mathrm{PW}_{n,k}; \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^N) \otimes \Lambda(y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n).$$

(ii) *If* $n \equiv 2 \bmod 4$ *and* $k = n$ *then*

$$H^*(\mathrm{PW}_{n,k}; \mathbb{Z}/2) = \mathbb{Z}/2[y_1]/(y_1^4) \otimes \Lambda(y_3, \dots, y_n)$$

and $x = y_1^2$.

(iii) *We have*

$$\mathrm{Sq}^1 y_j = \tfrac{1}{2}\tbinom{n}{j} x^j,$$

where the right hand side makes sense because $x^j = 0$ *if* $\binom{n}{j}$ *is odd.*

(iv) *If* $i < j$ *then*

$$\mathrm{Sq}^{2i} y_j = \sum_{r=0}^{i} \tbinom{j-1-r}{i-r}\tbinom{n}{r} x^r y_{j+i-r} + \tbinom{n}{j} \sum_{s=0}^{N-1} \tbinom{j-N}{i-s} \sum_{t=0}^{s} \tbinom{N-1-t}{s-t}\tbinom{n}{t} x^{j+i-N-s+t} y_{N+s-t}$$

where $y_N = 0$.

Remark If $k = n$ then $\mathrm{PW}_{n,k}$ is the projective unitary group $\mathrm{PU}(n)$. This particular case of Theorem 1.2 is due to Baum and Browder [1] and Borel [2]. For $p > 2$ the action of the Steenrod algebra on Chern classes, knowledge of which is needed to compute its action on the cohomology of $\mathrm{PW}_{n,k}$, is given by very complicated formulae [6]. We shall not enter into this computation in this paper. We also remark that the *integral* cohomology of $\mathrm{PW}_{n,k}$ appears to be very difficult to calculate. For example, it is not too hard to see that

$$H^*(\mathrm{PW}_{n,2}; \mathbb{Z}) = \mathbb{Z}[u, y]/(nu^{n-1}, u^n, u^{n-1}y, y^2)$$

with u and y in degrees 2 and $2n-1$, so that $H^{2n-2}(\mathrm{PW}_{n,2}; \mathbb{Z})$ is cyclic of order n, and this is the only torsion here. However, for $k > 2$ the cohomology of $\mathrm{PW}_{n,k}$ seems to contain increasingly complicated torsion, and to have a ring structure quite complicated to describe, not at all like the simple structure claimed by Ruiz in [5]. We remark finally that other questions about the manifolds $\mathrm{PW}_{n,k}$ arise naturally, for example the question of their parallelizability. We settle this issue in a forthcoming paper to appear in the Proceedings of the American Mathematical Society.

The space $\mathrm{PW}_{n,k}$ is universal for n-fold Whitney multiples of complex line bundles that admit trivial complex k-dimensional subbundles, in a sense that will be made precise at the beginning of Section 5. As an application of this and of Theorem 1.2 we study the existence of trivial complex subbundles of Whitney multiples of the canonical complex line bundle over lens spaces with 2-torsion.

Let $L^d(m)$ denote the quotient of the $(2d+1)$-sphere by the standard action of the m-th roots of unity. Let λ be the complex line bundle associated with the principal bundle $S^{2d+1} \to L^d(m)$, and let ν be the numerical function defined by $t = 2^{\nu(t)} \cdot$ (odd integer).

Theorem 1.3 *Assume there exists j such that $\nu\binom{n}{j} = \nu(m) > 0$ and such that j is even if n is even. Then the bundle $n\lambda$ over $L^{j+1}(m)$ does not admit a trivial complex subbundle of dimension $n - j + 1$.*

An interesting special case is the following, in which the result of 1.3 is optimal.

Corollary 1.4 *Let $\nu(n) > s > 0$ and $j = 2^{\nu(n)-s}$. Then the bundle $n\lambda$ over $L^{j+1}(2^s)$ admits a trivial complex subbundle of dimension $n - j$ but not of a larger dimension.*

To prove the existence of the subbundle mentioned in (1.4) we identify the relevant obstruction with the value of a secondary cohomology operation in the manner of [7].

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2 The mod p Cohomology of $\mathrm{PW}_{n,k}$

In what follows cohomology will be understood to have coefficients in $\mathbb{Z}/p$ with p prime.

Consider the fibration

(2.1)
$$W_{n,k} \xrightarrow{\pi} \mathrm{PW}_{n,k} \xrightarrow{g} \mathrm{CP}^\infty$$

where CP^∞ is infinite dimensional complex projective space and g classifies the principal S^1-bundle π. In analogy with (1.3) of [3] there is, up to homotopy, a pullback diagram

(2.2)
$$\begin{array}{ccc} \mathrm{PW}_{n,k} & \xrightarrow{f} & \mathrm{BU}(n-k) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ \mathrm{CP}^\infty & \xrightarrow{f_0} & \mathrm{BU}(n) \end{array}$$

where i is the canonical map, f_0 classifies the Whitney sum of n copies of the Hopf bundle over CP^∞, and f classifies the bundle whose fibre over a point u in $\mathrm{PW}_{n,k}$ is the orthogonal complement in $\mathbb{C}^n$ of the subspace generated by a k-frame representing u.

It is well known [2] that

$$H^*(W_{n,k}) = \Lambda(z_{n-k+1}, \ldots, z_n)$$

where z_j has degree $2j-1$ and is characterized in the Serre spectral sequence of the universal fibration

$$W_{n,k} \longrightarrow \mathrm{BU}(n-k) \xrightarrow{i} \mathrm{BU}(n)$$

as the element that transgresses to (the mod p reduction of) the universal Chern class c_j. It follows from this, applying naturality of transgression to (2.2), that

$$\tau z_j = \binom{n}{j} w^j$$

where τ denotes transgression in the fibration (2.1) and w is the generator in $H^2(\mathrm{CP}^\infty)$. It is now clear that the limit term E_∞ of the Serre spectral sequence for (2.1) is

$$E_\infty = \mathbb{Z}/p[w]/(w^N) \otimes \Lambda(z_{n-k+1}, \dots, z_{N-1}, z_{N+1}, \dots, z_n) \tag{2.3}$$

where

$$N = \min\{ j \mid n-k < j \leq n \text{ and } \binom{n}{j} \not\equiv 0 \mod p \}. \tag{2.4}$$

Lemma 2.5 *Let $x \in H^2(\mathrm{PW}_{n,k})$ be the Euler class of the complex line bundle associated with the principal bundle $\pi \colon W_{n,k} \longrightarrow \mathrm{PW}_{n,k}$ and let N be as in (2.4), then*

(i) The subalgebra of $H^*(\mathrm{PW}_{n,k})$ generated by x is a truncated polynomial algebra $\mathbb{Z}/p[x]/(x^N)$.
(ii) If $y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n$ are elements in $H^*(\mathrm{PW}_{n,k})$ such that $\pi^* y_j = z_j$ then there is an additive isomorphism

$$H^*(\mathrm{PW}_{n,k}) \cong \mathbb{Z}/p[x]/(x^N) \otimes \Lambda(y_{n-k+1}, \dots, y_{N-1}, y_{N+1}, \dots, y_n).$$

Proof From standard properties of the Serre spectral sequence, we see that x projects to w in $E_\infty^{2,0}$ and that the condition $\pi^* y_j = z_j$ implies y_j projects to z_j in $E_\infty^{0,2j-1}$. The lemma now follows from (2.3).

Proof of Theorem 1.1 Since squares of elements of odd degree are zero in mod p cohomology if p is odd, the additive isomorphism of (2.5) is an isomorphism of algebras.

A difficulty in the proof of Theorem 1.2 arises from the fact that the argument used to derive the multiplicative structure in (1.1) is not valid when $p = 2$. To deal with this problem we shall make in Section 3 a careful choice of generators y_j and compute in Section 4 the action of the Steenrod algebra on these y_j.

3 A Choice of Generators for $H^*(\mathrm{PW}_{n,k}; \mathbb{Z}/2)$

Until further notice cohomology should be taken with coefficients in $\mathbb{Z}/2$. We construct generators for $H^*(\mathrm{PW}_{n,k})$ by the method introduced by Massey and Peterson in [4].

Let E be a contractible space with a free right action of $U(n)$. We have a commutative diagram

$$\begin{array}{ccccc} W_{n,k} & \xrightarrow{\pi} & \mathrm{PW}_{n,k} & \xrightarrow{f} & \mathrm{BU}(n-k) \\ \downarrow{\scriptstyle p_k} & & \downarrow & & \downarrow{\scriptstyle i_k} \\ E & \xrightarrow{\pi_0} & \mathrm{CP}^\infty & \xrightarrow{f_0} & \mathrm{BU}(n) \end{array} \tag{3.1}$$

in which both squares are pullback squares (*cf.* (2.2)). To be very specific, embed S^1 in $U(n)$ as its centre, and take $\mathrm{CP}^\infty = E/S^1$ and $\mathrm{BU}(n) = E/U(n)$ and let π_0 and f_0 be the obvious maps. Take $\mathrm{BU}(n-k)$ to be the space $E \times_{U(n)} \big(U(n)/U(n-k)\big)$. Then the spaces denoted here by $W_{n,k}$ and $\mathrm{PW}_{n,k}$ are the pullbacks $E \times \big(U(n)/U(n-k)\big)$ and $E \times_{S^1} \big(U(n)/U(n-k)\big)$ which are canonically homotopy equivalent to the manifolds $W_{n,k}$ and $\mathrm{PW}_{n,k}$.

We have, induced by (3.1), a commutative diagram with exact columns

(3.2)
$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
H^{odd}(\mathrm{PW}_{n,k}) & & 0 \\
\downarrow{\scriptstyle \delta} & & \downarrow \\
H^{ev}(\mathrm{CP}^\infty, \mathrm{PW}_{n,k}) & \xleftarrow{f^*} & H^{ev}\big(\mathrm{BU}(n), \mathrm{BU}(n-k)\big) \\
\downarrow{\scriptstyle i^*} & & \downarrow \\
H^{ev}(\mathrm{CP}^\infty) & \xleftarrow{f_0^*} & H^{ev}\big(\mathrm{BU}(n)\big)
\end{array} \quad .$$

We identify $H^*\big(\mathrm{BU}(n), \mathrm{BU}(n-k)\big)$ with the ideal in $H^*\big(\mathrm{BU}(n)\big)$ generated by c_j for $n-k < j \leq n$ and write $u_j = f^* c_j$. Recall that N denotes the smallest integer such that $N > n-k$ and $\binom{n}{N}$ is odd. Since $f_0^* c_j = \binom{n}{j} w^j$ from (2.2), and noting that the expression $\binom{n}{j} w^{j-N} u_N$ makes good sense because $\binom{n}{j}$ is even when $j < N$, we see that $i^*\big(u_j + \binom{n}{j} w^{j-N} u_N\big) = 0$.

Let y_j be the unique element in $H^{2j-1}(\mathrm{PW}_{n,k})$ such that

$$\delta y_j = u_j + \binom{n}{j} w^{j-N} u_N.$$

Note that $y_N = 0$ and that $\delta y_j = u_j$ if $j < N$.

We show that $\pi^* y_j = z_j$ in $H^*(W_{n,k})$, if $j \neq N$, as follows. Consider the diagram

(3.3)(k)
$$\begin{array}{ccc}
H^{*-1}(W_{n,k}) & & 0 \\
\cong \downarrow{\scriptstyle \delta} & & \downarrow \\
H^*(E, W_{n,k}) & \xleftarrow{(f\pi)^*} & H^*\big(\mathrm{BU}(n), \mathrm{BU}(n-k)\big) \\
& & \downarrow \\
& & H^*\big(\mathrm{BU}(n)\big)
\end{array} \quad .$$

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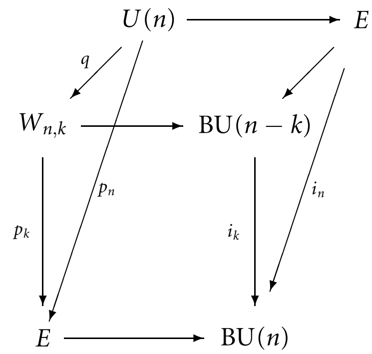
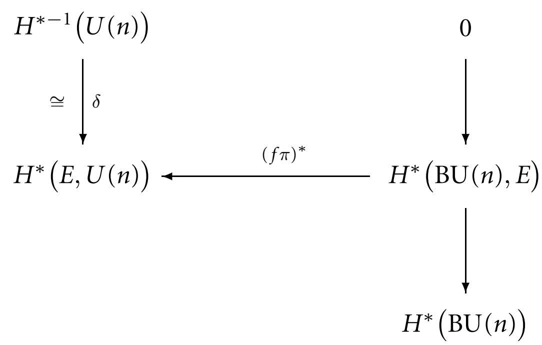
By naturality of connecting homomorphisms and the definition of y_j we have $\delta(\pi^* y_j) = (f\pi)^* c_j$. Consider the commutative diagram

(3.4)



where q is the projection $U(n) \to U(n)/U(n-k)$. It follows from this that diagram (3.3)(k) maps into diagram (3.3)(n). Diagram (3.3)(n) is



and we have $\delta(q^*\pi^* y_j) = (f\pi)^* c_j$ here. Since $f\pi$ is the projection of the universal bundle over $\mathrm{BU}(n)$, this equation shows, by definition of transgression, that $q^*\pi^* y_j$ transgresses to c_j in the universal bundle. Thus, $q^*\pi^* y_j = z_j$ in $H^*\big(U(n)\big)$. Since q^* is a monomorphism and $q^* z_j = z_j$ for $n - k < j \le n$, we conclude that $\pi^* y_j = z_j$ in $H^*(W_{n,k})$.

Proof of (i) and (ii) of Theorem 1.2 By the previous paragraph and Lemma 2.5 we only need to compute y_j^2. Note first that $\delta(y_j^2) = \mathrm{Sq}^{2j-1}\delta y_j = 0$ because δ commutes with Steenrod operations and Sq^{2j-1} acts trivially on $H^*\big(\mathrm{BU}(n), \mathrm{BU}(n-k)\big)$, so y_j^2 must be a multiple of x^{2j-1}, since

$$H^{4j-2}(\mathrm{CP}^\infty) \to H^{4j-2}(\mathrm{PW}_{n,k}) \xrightarrow{\delta} H^{4j-1}(\mathrm{CP}^\infty, \mathrm{PW}_{n,k})$$

is exact. Thus $y_j^2 = 0$ if $2j - 1 \ge N$. If $2j - 1 < N$ we have, assuming parts (iii) and (iv)

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of (1.2), which shall be proved in the next section, that

$$\begin{aligned} y_j^2 &= \mathrm{Sq}^1 \mathrm{Sq}^{2j-2} y_j \\ &= \mathrm{Sq}^1 \sum_{r=0}^{j-1} \binom{n}{r} x^r y_{2j-1-r} \\ &= \tfrac{1}{2} \sum_{r=0}^{j-1} \binom{n}{r} \binom{n}{2j-1-r} x^{2j-1}. \end{aligned}$$

Consider the expansion of $(1+a)^{2n} = (1+a)^n(1+a)^n$. Comparing the coefficients of a^{2j-1} we find

$$\begin{aligned} \binom{2n}{2j-1} &= \sum_{r=0}^{2j-1} \binom{n}{r} \binom{n}{2j-1-r} \\ &= 2 \sum_{r=0}^{j-1} \binom{n}{r} \binom{n}{2j-1-r}, \end{aligned}$$

so that $y_j^2 = \frac{1}{4}\binom{2n}{2j-1} x^{2j-1}$. Recall the numerical functions ν and α defined by $t = 2^{\nu(t)} \cdot$ (odd integer) and $\alpha(t) =$ sum of coefficients in the binary expansion of t, and recall the well known formula

$$\nu\binom{m}{t} = \alpha(t) + \alpha(m-t) - \alpha(m). \tag{3.5}$$

This gives

$$\begin{aligned} \nu\binom{2n}{2j-1} &= \alpha\big(2(j-1)+1\big) + \alpha\big(2(n-j)+1\big) - \alpha(2n) \\ &= \alpha(j-1) + \alpha(n-j) - \alpha(n) + 2 \\ &= \big(\alpha(1) + \alpha(j-1) - \alpha(j)\big) + \big(\alpha(j) + \alpha(n-j) - \alpha(n)\big) + 1 \\ &= \nu\binom{j}{1} + \nu\binom{n}{j} + 1 \\ &= \nu\Big(2j\binom{n}{j}\Big) \end{aligned}$$

and so $y_j^2 = \frac{1}{2} j \binom{n}{j} x^{2j-1}$.

Since $\binom{n}{j}$ is even we see that $y_j^2 = 0$ unless j is odd and $\binom{n}{j} \equiv 2 \bmod 4$. Note that the coefficients $\binom{n}{j}, \dots, \binom{n}{2j-1}$ are all even since we assume that $2j-1 < N$. Then parts (i) and (ii) of Theorem 1.2 are a consequence of the following lemma, which will be proved in the appendix.

Lemma 3.6 *Let n and j be positive integers with j odd and $\binom{n}{j} \equiv 2 \bmod 4$. Assume $\binom{n}{j}, \dots, \binom{n}{2j-1}$ are all even. Then $n \equiv 2 \bmod 4$ and $j = 1$.*

4 The Action of the Steenrod Algebra on $H^*(\mathrm{PW}_{n,k};\mathbb{Z}/2)$

In this section we prove parts (iii) and (iv) of Theorem 1.2. We begin by proving a result slightly more general than (iii) which we shall use in the proof of Theorem 1.3.

Lemma 4.1 *Let β be the Bockstein operator associated with the exact sequence*

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{r} \mathbb{Z}/2 \to 0$$

and let $x_0 \in H^2(\mathrm{PW}_{n,k};\mathbb{Z})$ be the Euler class of the complex line bundle associated with $\pi\colon W_{n,k} \to \mathrm{PW}_{n,k}$. Then

$$\beta y_j = \tfrac{1}{2}\tbinom{n}{j} x_0^j$$

provided $j < N$.

Note that for $j < N$ part (iii) of Theorem 1.2 follows from (4.1) because $\mathrm{Sq}^1 = r_*\beta$ and $r_* x_0 = x$. When $j \geq N$ note that $\delta\,\mathrm{Sq}^1 y_j = 0$, because δ commutes with Steenrod operations and $\mathrm{Sq}^1 H^*\big(\mathrm{BU}(n), \mathrm{BU}(n-1)\big) = 0$, so the cohomology exact sequence shows $\mathrm{Sq}^1 y_j$ must be a multiple of x^j. Since x^j is zero, part (iii) of (1.2) is also correct for $j \geq N$.

To prove Lemma 4.1 observe first that with the integral Serre spectral sequence of fibration (2.1), arguing as in Section 2, one can show that for $n-k < i \leq n$ the order of x_0^i is m_i, the greatest common divisor of the binomial coefficients $\binom{n}{s}$ for $n-k < s \leq i$. The spectral sequence also shows that the image of

$$\pi^*\colon H^{2j-1}(\mathrm{PW}_{n,k};\mathbb{Z}) \to H^{2j-1}(W_{n,k};\mathbb{Z})$$

is generated by $(m_{j-1}/m_j)z_j^0$ if $j > n-k+1$ and that $H^{2j-1}(\mathrm{PW}_{n,k};\mathbb{Z}) = 0$ if $j = n-k+1$; here z_j^0 is an exterior algebra generator of $H^*(W_{n,k};\mathbb{Z})$ analogous to z_j and such that $r_* z_j^0 = z_j$; *cf.* Section 2.

By the argument that showed $\delta\,\mathrm{Sq}^1 y_j = 0$ above we now have $\delta\beta y_j = 0$, so βy_j is a multiple of x_0^j, and since $2\beta y_j = 0$ we may write

$$\beta y_j = c \cdot \tfrac{1}{2} m_j x_0^j \tag{4.2}$$

for some integer c, because $\frac{1}{2} m_j x_0^j$ is the element of order 2 in the subgroup of $H^{2j}(\mathrm{PW}_{n,k};\mathbb{Z})$ generated by x_0^j.

If $j = n-k+1$ we have $m_j = \binom{n}{j}$ and $\beta y_j \neq 0$, because $H^{2j-1}(\mathrm{PW}_{n,k};\mathbb{Z}) = 0$ implies y_j is not the reduction of an integral class, so c in (4.2) is odd. This proves (4.1) when $j = n-k+1$.

If $j > n-k+1$ we distinguish two cases. If m_{j-1}/m_j is even then $\binom{n}{j}/m_j$ is odd, since m_j is the greatest common divisor of m_{j-1} and $\binom{n}{j}$. Thus

$$\beta y_j = c \cdot \tfrac{1}{2}\tbinom{n}{j} x_0^j$$

by (4.2). To finish the proof of (4.1) in this case we need only show that c is odd. If c is even, so that $\beta y_j = 0$, consider the diagram

$$\begin{array}{ccccc}
H^{2j-1}(\mathrm{PW}_{n,k};\mathbb{Z}) & \xrightarrow{r_*} & H^{2j-1}(\mathrm{PW}_{n,k}) & \xrightarrow{\beta} & H^{2j}(\mathrm{PW}_{n,k};\mathbb{Z}) \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \\
H^{2j-1}(W_{n,k};\mathbb{Z}) & \xrightarrow{r_*} & H^{2j-1}(W_{n,k}) & & .
\end{array}$$

Let u be such that $r_* u = y_j$. Then $\pi^* u$ is a multiple of $(m_{j-1}/m_j) z_j^0$, and hence is even. But then

$$z_j = \pi^* y_j = \pi^* r_* u = r_* \pi^* u = 0;$$

this proves that c must be odd. If m_{j-1}/m_j is odd select integers α_s such that

$$\sum_{s=n-k+1}^{j-1} \alpha_s \binom{n}{s} = m_{j-1}$$

and consider the class

$$A = \sum_{s=n-k+1}^{j-1} \alpha_s w^{j-1-s} u_s$$

in $H^{2j-2}(\mathrm{CP}^\infty, \mathrm{PW}_{n,k};\mathbb{Z})$. We have (*cf.* (3.2) with integral coefficients)

$$i^* A = m_{j-1} w^{j-1}$$

and so

$$i^* \Big((m_{j-1}/m_j) u_j - \Big(\binom{n}{j} / m_j \Big) wA \Big) = 0$$

in $H^{2j}(\mathrm{CP}^\infty;\mathbb{Z})$. This determines an element $a \in H^{2j-1}(\mathrm{PW}_{n,k};\mathbb{Z})$ such that

$$\delta a = (m_{j-1}/m_j) u_j - \Big(\binom{n}{j} / m_j \Big) wA.$$

Then

$$\begin{aligned}
\delta r_* a &= u_j + \Big(\binom{n}{j} / m_j \Big) \sum_{s=n-k+1}^{j-1} \alpha_s w^{j-s} u_s \\
&= \delta \Big(y_j + \Big(\binom{n}{j} / m_j \Big) \sum_{s=n-k+1}^{j-1} \alpha_s x^{j-s} y_s \Big),
\end{aligned}$$

so that

$$r_* a = y_j + \Big(\binom{n}{j} / m_j \Big) \sum_{s=n-k+1}^{j-1} \alpha_s x^{j-s} y_s$$

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in $H^{2j-1}(\mathrm{PW}_{n,k})$. Taking β we have

$$\begin{aligned}\beta y_j &= \Big(\tbinom{n}{j}/m_j\Big) \sum_{s=n-k+1}^{j-1} \alpha_s x_0^{j-s}\Big(\tfrac{1}{2}\tbinom{n}{s}x_0^s\Big)\\ &= \Big(\tbinom{n}{j}/m_j\Big)\big(\tfrac{1}{2}m_{j-1}\big)x_0^j\\ &= \tfrac{1}{2}\tbinom{n}{j}x_0^j\end{aligned}$$

because m_{j-1}/m_j is odd. This ends the proof of Lemma 4.1.

Now the proof of part (iv) of (1.2). From the Wu relations

$$\mathrm{Sq}^{2i}\, c_j = \sum_{r=0}^{i} \tbinom{j-1-r}{i-r} c_r c_{j+i-r}$$

in $H^*\big(\mathrm{BU}(n)\big)$, valid for $i < j$, we obtain (recall (3.2))

$$\mathrm{Sq}^{2i}\, u_j = \sum_{r=0}^{i} \tbinom{j-1-r}{i-r}\tbinom{n}{r} w^r u_{j+i-r} \tag{4.3}$$

in $H^*(\mathrm{CP}^\infty, \mathrm{PW}_{n,k})$.

Let $\lambda_r^{i,j}$ denote $\binom{j-1-r}{i-r}\binom{n}{r}$. We have from the definition of y_j,

$$\delta\, \mathrm{Sq}^{2i}\, y_j = \mathrm{Sq}^{2i}\, u_j + \tbinom{n}{j} \sum_{s=0}^{N-1} \mathrm{Sq}^{2(i-s)}\, w^{j-N}\, \mathrm{Sq}^{2s}\, u_N + \tbinom{n}{j}(\mathrm{Sq}^{2(i-N)}\, w^{j-N}) w^N u_N \tag{4.4}$$

where in the last term we have used the fact (*cf.* (3.2)) that

$$u_N^2 = f_0^* c_N f^* c_N = \tbinom{n}{N} w^N u_N = w^N u_N.$$

Using (4.3) to expand $\mathrm{Sq}^{2i}\, u_j$ and $\mathrm{Sq}^{2s}\, u_N$ and adding

$$2\sum_{r=0}^{i} \lambda_r^{i,j} w^r \tbinom{n}{j+i-r} w^{j+i-r-N} u_N + 2\tbinom{n}{j} \sum_{s=0}^{N-1} \mathrm{Sq}^{2(i-s)}\, w^{j-N} \sum_{t=0}^{s} \lambda_t^{s,N} w^t \tbinom{n}{N+s-t} w^{s-t} u_N,$$

which equals zero, to the right hand side of (4.4), we find, after rearranging terms, that

$$\begin{aligned}\delta\, \mathrm{Sq}^{2i}\, y_j &= \sum_{r=0}^{i} \lambda_r^{i,j} w^r \Big(u_{j+i-r} + \tbinom{n}{j+i-r} w^{j+i-r-N} u_N\Big)\\ &\quad + \tbinom{n}{j} \sum_{s=0}^{N-1} \mathrm{Sq}^{2(i-s)}\, w^{j-N} \sum_{t=0}^{s} \lambda_t^{s,N} w^t \Big(u_{N+s-t} + \tbinom{n}{N+s-t} w^{s-t} u_N\Big)\\ &\quad + \alpha w^{j+i-N} u_N\\ &= \delta\ (\text{right hand side of (iv) of (1.2)}) + \alpha w^{j+i-N} u_N\end{aligned} \tag{4.5}$$

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where we use the well known fact that $\mathrm{Sq}^{2m}\, w^l = \binom{l}{m} w^{l+m}$ and where α is the integer

$$\sum_{r=0}^{i} \lambda_r^{i,j} \binom{n}{j+i-r} + \binom{n}{j} \sum_{s=0}^{N-1} \binom{j-N}{i-s} \sum_{t=0}^{s} \lambda_t^{s,N} \binom{n}{N+s-t} + \binom{n}{j}\binom{j-N}{i-N}.$$

Since $\alpha w^{j+i-N} u_N$ lies in im δ, it goes to zero under i^*; but

$$\begin{aligned} i^*(\alpha w^{j+i-N} u_N) &= \alpha w^{j+i-N} \binom{n}{N} w^N \\ &= \alpha w^{j+i} \end{aligned}$$

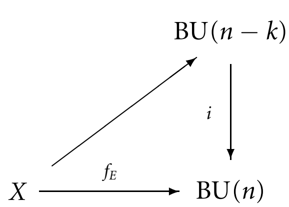
in $H^*(\mathrm{CP}^\infty)$, so we have $\alpha \equiv 0 \bmod 2$. Now part (iv) of (1.2) follows from (4.5) and the fact that δ is a monomorphism.

5 Application to Existence of Trivial Subbundles

The main idea behind the proof of Theorem 1.3 and of its corollary is as follows.

Let E be a complex n-plane bundle over a CW complex X. Then E admits a trivial complex subbundle of dimension k if and only if its associated bundle of k-frames admits a section. This condition is in turn equivalent to the existence of a lift in the diagram

(5.1)



where f_E classifies E, because i is the bundle of k-frames associated with the universal n-plane bundle on $\mathrm{BU}(n)$. If $E = n\lambda$, the Whitney sum of n copies of a complex line bundle λ, then in view of (2.2) this lift must factor through a map f such that the composite

$$X \xrightarrow{f} \mathrm{PW}_{n,k} \xrightarrow{g} \mathrm{CP}^\infty$$

classifies λ. Thus $n\lambda$ admits a trivial complex subbundle of dimension k if and only if there is a map

(5.2) $$f\colon X \to \mathrm{PW}_{n,k}$$

such that $f^* x_0 = c_1(\lambda)$ in $H^2(X;\mathbb{Z})$, where x_0 is the Euler class of the complex line bundle associated with $\pi\colon W_{n,k} \to \mathrm{PW}_{n,k}$.

The following lemma summarizes the facts concerning the cohomology of lens spaces which we shall need.

Lemma 5.3 *Let $L^d(m)$ be a lens space as in (1.3) and λ the associated complex line bundle.*

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(i) *We have*

$$H^{ev}\big(L^d(m);\mathbb{Z}\big) = \mathbb{Z}[c_1(\lambda)]/\big(mc_1(\lambda), c_1^{d+1}(\lambda)\big).$$

(ii) *If m is even then*

$$H^*\big(L^d(m)\big) = \mathbb{Z}/2[u,v]/(u^{d+1}, v^2 - \epsilon u)$$

where u is the reduction $\mod 2$ *of* $c_1(\lambda)$ *and v has degree* 1, *and where* $\epsilon = 1$ *if* $\nu(m) = 1$ *and* $\epsilon = 0$ *if* $\nu(m) > 1$.

(iii) *If m is even then the Bockstein operator*

$$\beta\colon H^1\big(L^d(m)\big) \to H^2\big(L^d(m);\mathbb{Z}\big)$$

is given by $\beta v = 2^{\nu(m)-1} \cdot c_1(\lambda)$.

The proof of parts (i) and (ii) of Lemma 5.3 is a straightforward calculation with the Gysin sequence of the sphere bundle $L^d(m) \to \mathrm{CP}^d$ associated with the m-fold tensor power of the Hopf bundle over CP^d, while part (iii) follows easily from the Bockstein exact sequence. We leave the details to the interested reader.

Suppose that $f\colon L^d(m) \to \mathrm{PW}_{n,k}$ is as in (5.2) with m even and $n-k<d$. Then $f^*x_0^d = c_1^d(\lambda)$ has order m by (5.3), so the order of x_0^d, which is (see the proof of (4.1)) the greatest common divisor of the coefficients $\binom{n}{j}$ for $n-k<j\le d$, must be divisible by m. In particular,

$$\binom{n}{j} \equiv 0 \mod 2^{\nu(m)} \tag{5.4}$$

for $n-k<j\le d$.

Proposition 5.5 *If* $f\colon L^d(m) \to \mathrm{PW}_{n,k}$ *is as in (5.2) with m even and* $n-k<d$ *then*

$$f^*\colon H^*(\mathrm{PW}_{n,k}) \to H^*\big(L^d(m)\big)$$

satisfies $f^*x = u$ *and*

$$f^*y_j = \frac{1}{2^{\nu(m)}}\binom{n}{j}u^{j-1}v \tag{5.6}$$

for $n-k<j\le d$, *where the right hand side makes sense by (5.4).*

To prove (5.5) notice first that $f^*x = u$ follows from $f^*x_0 = c_1(\lambda)$ by reducing mod 2, so we need only prove (5.6). Write $s = \nu(m)$ and note that from (4.1) and (5.2) we have

$$f^*\beta y_j = \tfrac{1}{2}\binom{n}{j}c_1^j(\lambda)$$

in $H^{2j}\big(L^d(m);\mathbb{Z}\big)$. Write $f^*y_j = au^{j-1}v$ for some integer a. Then

$$\beta(au^{j-1}v) = \tfrac{1}{2}\binom{n}{j}c_1^j(\lambda)$$

by naturality of β. Since by (5.3)

$$\begin{aligned}\beta(u^{j-1}v) &= c_1^{j-1}(\lambda)\beta(v)\\ &= 2^{s-1}c_1^j(\lambda),\end{aligned}$$

we must have

$$2^{s-1}a \equiv \tfrac{1}{2}\tbinom{n}{j} \mod m,$$

because $c_1^j(\lambda)$ has order m, again by (5.3). Thus

$$a \equiv \tfrac{1}{2^s}\tbinom{n}{j} \mod \tfrac{m}{2^{s-1}},$$

and, since $m/2^{s-1}$ is even, this implies (5.6).

Proof of Theorem 1.3 Assume that $n\lambda$ does admit a trivial complex subbundle of dimension $n-j+1$, so that we have a map

$$f\colon L^{j+1}(m) \to \mathrm{PW}_{n,n-j+1}$$

as in (5.5). By (5.6) we have, with $s=\nu(m)$,

$$\begin{aligned}\mathrm{Sq}^2 f^* y_j &= \tfrac{1}{2^s}\tbinom{n}{j}\,\mathrm{Sq}^2(u^{j-1}v)\\ &= (j-1)\cdot\tfrac{1}{2^s}\tbinom{n}{j}u^j v\end{aligned}$$

where we use the fact that $\mathrm{Sq}^1 u = 0$ because u is the reduction mod 2 of an integral class. We also have, from (iv) of Theorem 1.2 and from (5.6)

$$\begin{aligned}f^*\,\mathrm{Sq}^2 y_j &= f^*\big((j-1)y_{j+1} + nxy_j\big)\\ &= \Big((j-1)\cdot\tfrac{1}{2^s}\tbinom{n}{j+1} + n\cdot\tfrac{1}{2^s}\tbinom{n}{j}\Big)u^j v.\end{aligned}$$

Since $u^j v$ generates $H^{2j+1}\big(L^{j+1}(m)\big) \cong \mathbb{Z}/2$ we conclude that

$$(n+j-1)\cdot\tfrac{1}{2^s}\tbinom{n}{j} + (j-1)\cdot\tfrac{1}{2^s}\tbinom{n}{j+1} \equiv 0 \mod 2. \tag{5.7}$$

If n is even then by hypothesis j is also even, so (5.7) becomes

$$\tfrac{1}{2^s}\tbinom{n}{j} + \tfrac{1}{2^s}\tbinom{n}{j+1} \equiv 0 \mod 2$$

or

$$\tfrac{1}{2^s}\tbinom{n+1}{j+1} \equiv 0 \mod 2. \tag{5.8}$$

By (3.5) we see that $\nu\binom{n+1}{j+1} = \nu\binom{n}{j} = s$, which contradicts (5.8).

If n is odd then using again the formula $\binom{n}{j} + \binom{n}{j+1} = \binom{n+1}{j+1}$ we have from (5.7)

$$\frac{1}{2^s}\binom{n}{j} + (j-1)\cdot\frac{1}{2^s}\binom{n+1}{j+1} \equiv 0 \mod 2. \tag{5.9}$$

Since $\nu\binom{n}{j} = s$ we obtain a contradiction when j is odd. If j is even note that $\binom{n+1}{j+1} = \binom{n}{j}\cdot\frac{n+1}{j+1}$ implies

$$\nu\binom{n+1}{j+1} = \nu\binom{n}{j} + \nu(n+1) - \nu(j+1) > s,$$

so the left hand side of (5.9) must be odd and again we obtain a contradiction, which ends the proof of Theorem 1.3.

Before beginning the proof of (1.4) consider a complex k-plane bundle E over a finite CW complex X, such that $c_1(E) = 0$. Let f classify E and recall the standard Postnikov obstruction scheme defined via the diagram

$$\begin{array}{ccccc}
 & & \mathrm{BSU}(k-1) & & \\
 & & \downarrow & & \\
 & & B_2 & & \\
 & & \downarrow & & \\
 & & B_1 & \xrightarrow{b_1} & K(\mathbb{Z}/2, 2k+1) \\
 & & \downarrow{\scriptstyle p_1} & & \\
X & \xrightarrow{f} & \mathrm{BSU}(k) & \xrightarrow{c_k} & K(\mathbb{Z}, 2k)
\end{array} \tag{5.10}$$

in which B_1 is the fibre of c_k and b_1 is the transgression of the fundamental class of the fibre of $\mathrm{BSU}(k-1) \to B_1$, and B_2 is the fibre of b_1. If $c_k(E) = 0$, so that f has a lift f_1 to B_1, then the second obstruction to lifting f to $\mathrm{BSU}(k-1)$ is defined; it is the element $b(E)$ of $H^{2k+1}(X)/\operatorname{Sq}^2 H^{2k-1}(X)$ represented by $f_1^* b_1$. If the dimension of X is no more than $2k+1$ then E admits a nonvanishing section, *i.e.*, f lifts to $\mathrm{BSU}(k-1)$, if and only if $c_k(E) = 0$ and $b(E) = 0$, because $\mathrm{BSU}(k-1) \to B_2$ is a $(2k+1)$-equivalence. In this context we have the following proposition (*cf.* Chapter VII of [7]).

Proposition 5.11 *Let $k \geq 3$ be odd. Consider a stable secondary cohomology operation φ associated with the Adem relation $\operatorname{Sq}^2\operatorname{Sq}^2 = 0$ valid on integral classes. If E is a k-plane bundle over X with $c_1(E) = c_k(E) = 0$ then $\varphi\big(c_{k-1}(E)\big)$ is defined in $H^{2k+1}(X)/\operatorname{Sq}^2 H^{2k-1}(X)$ and it coincides with the obstruction $b(E)$.*

The corollary below is a consequence of (5.11) and the preceding comments.

Corollary 5.12 *If $k \geq 3$ is odd and E is a complex k-plane bundle over a finite CW complex of dimension at most $2k+1$, and if $c_1(E) = c_{k-1}(E) = c_k(E) = 0$, then E admits a nonvanishing section.*

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Proof of (5.11) The operation φ is defined on classes in integral cohomology which go to zero under Sq^2 and takes values in mod 2 cohomology modulo the image of Sq^2. Since

$$\begin{aligned}\mathrm{Sq}^2\, c_{k-1} &= c_1 c_{k-1} + c_k\\ &= c_k\end{aligned}$$

in $H^*\big(\mathrm{BSU}(k)\big)$ we have (*cf.* (5.10))

(5.13) $$\mathrm{Sq}^2\, p_1^* c_{k-1} = 0$$

so $\varphi(p_1^* c_{k-1})$ is defined.

Consider the diagram of fibrations

(5.14) $$\begin{array}{ccccc} K(\mathbb{Z}, 2k-1) & \xrightarrow{r} & K(\mathbb{Z}/2, 2k-1) & & \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j} & & \\ B_1 & \xrightarrow{g} & B & & \\ \downarrow{\scriptstyle p_1} & & \downarrow & & \\ \mathrm{BSU}(k) & \xrightarrow{c_{k-1}} & K(\mathbb{Z}, 2k-2) & \xrightarrow{\mathrm{Sq}^2} & K(\mathbb{Z}/2, 2k) \end{array}$$

in which B is the fibre of Sq^2 and g is a lift, which exists by (5.13), j and j_1 are inclusions of fibres, and r is induced by g. By the Serre exact sequence there is a class $b \in H^{2k+1}(B)$ such that $j^* b = \mathrm{Sq}^2\, \iota$, where ι is the fundamental class of $K(\mathbb{Z}/2, 2k-1)$, and

(5.15) $$g^* b \in \varphi(p_1^* c_{k-1})$$

by definition of φ. From (5.14) and naturality of transgression

$$\tau r^* \iota = \mathrm{Sq}^2\, c_{k-1} = c_k$$

in $H^*\big(\mathrm{BSU}(k)\big)$, so $r^*\iota$ must be the generator of $H^{2k-1}\big(K(\mathbb{Z}, 2k-1)\big) \cong \mathbb{Z}/2$. Then

$$j_1^* g^* b = r^* j^* b = \mathrm{Sq}^2\, r^* \iota \neq 0,$$

and thus

(5.16) $$g^* b = b_1,$$

because the Serre exact sequence easily shows that $H^{2k+1}(B_1) \cong \mathbb{Z}/2$ with generator b_1. Thus by (5.15) and (5.16)

(5.17) $$\varphi(p_1^* c_{k-1}) = b_1$$

with zero indeterminacy, because $H^{2k-1}(B_1) = 0$ by the Serre exact sequence. Proposition 5.11 now follows from (5.17) by naturality of φ.

Proof of Corollary 1.4 To show that $n\lambda$ does not admit a trivial complex subbundle of dimension $n - j + 1$ we apply (1.3), noting that by (3.5)

$$\begin{aligned} \nu\tbinom{n}{j} &= \alpha(j) + \alpha(n - 2^{\nu(n)} + 2^{\nu(n)} - j) - \alpha(n) \\ &= \alpha(j) + \alpha(n - 2^{\nu(n)}) + \alpha(2^{\nu(n)} - j) - \alpha(n) \\ &= 1 + \alpha(n) - 1 + \alpha\big(j(2^s - 1)\big) - \alpha(n) \\ &= \alpha(2^s - 1) \\ &= s, \end{aligned}$$

since $j = 2^{\nu(n)-s}$.

To prove that $n\lambda$ does admit a trivial subbundle of dimension $n - j$ note first that it admits one of dimension $n - j - 1$ because

$$\mathrm{BU}(j+1) \to \mathrm{BU}(n)$$

is a $(2j+3)$-equivalence and $L^{j+1}(2^s)$ is a CW complex of dimension $2j+3$ (*cf.* (5.1)). Write

$$n\lambda = E \oplus (n - j - 1) \tag{5.18}$$

where $n - j - 1$ denotes a trivial complex bundle and E is a complex subbundle of $n\lambda$ of dimension $j + 1$. The Chern classes of E coincide with those of $n\lambda$ by (5.18) and it is easy to verify that $c_1(E) = c_j(E) = c_{j+1}(E) = 0$, so that E admits a nonvanishing section by (5.12). This implies that $n\lambda$ admits a trivial complex subbundle of dimension $n - j$, again by (5.18), and that the proof of (1.4) is finished.

Appendix

Proof of Lemma 3.6 We shall deduce (3.6) from the following lemma, which we owe to the referee.

Lemma *Let a, b and c be nonnegative integers such that $b < 2^a$ and $c < 2^a + b$, and let r and s be positive integers. Then*

$$\binom{r\cdot 2^{a+s}+c}{2^a+b} \equiv 0 \mod 2^s.$$

Proof From (3.5) we have

$$\nu\binom{r\cdot 2^{a+s}+c}{2^a+b} = 1 + \alpha(b) + \alpha(r \cdot 2^{a+s} + c - 2^a - b) - \alpha(r) - \alpha(c). \tag{A}$$

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Since

$$r \cdot 2^{a+s} + c - 2^a - b = (r-1)2^{a+s} + 2^{a+1}(2^{s-1} - 1) + c + 2^a - b,$$

and since $0 < c + 2^a - b < 2^{a+1}$, we see that

$$\begin{aligned}\alpha(r \cdot 2^{a+s} + c - 2^a - b) &= \alpha(r-1) + \alpha(2^{s-1} - 1) + \alpha(c + 2^a - b)\\ &= \alpha(r-1) + s - 1 + \alpha(c + 2^a - b)\end{aligned}$$

so that (A) becomes

$$\begin{aligned}\nu\binom{r \cdot 2^{a+s}+c}{2^a+b} &= \alpha(b) - \alpha(c) + 1 + \alpha(r-1) - \alpha(r) + s - 1 + \alpha(c + 2^a - b)\\ \text{(B)} \qquad &= \alpha(b) - \alpha(c) + \nu(r) + s - 1 + \alpha(c + 2^a - b).\end{aligned}$$

Taking $r = s = 1$ in (B) we obtain

$$\alpha(b) - \alpha(c) + \alpha(c + 2^a - b) = \nu\binom{2^{a+1}+c}{2^a+b}$$

so (B) may be re-written as

$$\text{(C)} \qquad \nu\binom{r \cdot 2^{a+s}+c}{2^a+b} = \nu(r) + s - 1 + \nu\binom{2^{a+1}+c}{2^a+b}.$$

If $\binom{2^{a+1}+c}{2^a+b}$ is odd then all powers of 2 in the binary expansion of $2^a + b$ must appear in that of $2^{a+1} + c$, and, therefore, in that of c; in particular $2^a + b \leq c$, contrary to hypotheses. Thus $\binom{2^{a+1}+c}{2^a+b}$ must be even, so

$$\nu\binom{r \cdot 2^{a+s}+c}{2^a+b} \geq s$$

follows from (C), and this proves the Lemma.

To prove Lemma 3.6 write $j = 2^a + b$ with $0 \leq b < 2^a$ and let r be the nonnegative integer such that $n = r \cdot 2^{a+2} + c$ for some $0 \leq c < 2^{a+2}$. Since $\binom{n}{c}$ is odd, either $c < j$ or $2j \leq c$. If $c < j$ we note that r must be positive, because $r = 0$ implies $n = c < j$, so the lemma above applies and shows $\binom{n}{j} \equiv 0 \bmod 4$, a contradiction. This implies $2j \leq c$. Then $2^{a+1} \leq 2^{a+1} + 2b \leq c < 2^{a+2}$, so that $c = 2^{a+1} + c'$ with $0 \leq c' < 2^{a+1}$, and so we have $n = r \cdot 2^{a+2} + 2^{a+1} + c'$, and therefore $\binom{n}{2^{a+1}}$ is odd. Since $2^{a+1} < j$ is impossible, we must have $2j \leq 2^{a+1}$, which means that $2^{a+1} + 2b \leq 2^{a+1}$. Then $b = 0$, so $j = 2^a$. Since j is odd we have that $j = 1$, and that $n = \binom{n}{1} \equiv 2 \bmod 4$, as claimed.

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Centro de Investigación del IPN,
Apartado Postal 14-740,
México
07000 D.F.
email: lastey@math.cinvestav.mx
emicha@math.cinvestav.mx

University of Rochester,
Rochester, NY 14627-0001
USA
email: sgitler@math.rochester.edu

Instituto Tecnológico Autónomo de México,
Rio Hondo No. 1
San Angel
México
01000 D.F.
email: pastor@gauss.rhon.itam.mx