EXTENSIONS OF CERTAIN MAPS TO AUTOMORPHISMS OF m

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Introduction. In this paper we consider Banach space automorphisms of m, the space of bounded sequences, which map c, the space of convergent sequences, into itself. In particular, we consider the problem of determining which maps from c_0, the space of sequences converging to 0, to c can be extended to such automorphisms.

The origin of this note lies in an incorrect conjecture of mine. If the automorphism $T: m \rightarrow m$ is given by a matrix, that is, a sequence of elements of l^1, and if T is conservative, that is, $T(c) \subset c$, then $T(c) = c$. That is, T restricted to c is an automorphism of c. We had hoped this would hold even if T were not a matrix. We can see, for example, that if the conservative automorphism T is bounded on the unit cube of m by 1 and ρ, where $\rho > \frac{1}{2}$, then $T(c) = c$. However, in general it is possible for a conservative automorphism of m to map c properly into c.

For expository convenience we develop our results first for extensions of operators on c_0. It is clear that if $T: c_0 \rightarrow c_0$ can be extended to an automorphism $T': m \rightarrow m$, then T must be one-to-one and of closed range. If $T: c_0 \rightarrow c_0$ is one-to-one, of closed range, and of finite deficiency, then $T^c: m \rightarrow m$, the unique matrix extension of T, has the same properties. By adding to T^c appropriately chosen operators with kernels including c_0 we can produce an automorphism $T': m \rightarrow m$ which extends T. This is our principal construction.

The characteristic difficulty of this extension lies in the fact that operators from m to m which are 0 on c_0 have notoriously awkward kernels. For example, if T is such an operator, then any infinite subset of the integers has in turn an infinite subset X such that $m(X)$ is in that kernel. Therefore, when we add to T^c an operator which is 0 on c_0 we must leave T^c unchanged on a large subspace of m.

By extending the well-known isomorphism between c_0 and c to an automorphism of m, we can pass from operators on c_0 to maps from c_0 to c. It is clear that if $T: c_0 \rightarrow c$ can be extended to a conservative automorphism $T': m \rightarrow m$, then T must be one-to-one, of closed range, and of non-zero deficiency. We show that if $T: c_0 \rightarrow c$ is one-to-one, of closed range, and of finite non-zero deficiency p, then there is a conservative automorphism

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T': $m \to m$ extending T. We observe that $T'(c)$ is of deficiency $p - 1$ in c, thereby disposing of our conjecture.

We have not been able to prove the existence of a conservative extension in the case $p = \infty$; however, our method of extension works in all the particular examples of $p = \infty$ that we have been able to construct so far.

Since this paper was written, Lindenstrauss and Rosenthal [**3**] have obtained results which imply the existence of such an extension (cf. the remark following Theorem 2.3).

This paper is essentially self-contained. We use without proof only well-known results on Banach spaces in general. As a source we cite Day [**2**]. A neat exposition of matrix mappings of m and c can be found in [**4**].

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Notation and preliminaries. By m we denote the Banach space of complex-valued sequences with the supremum norm; by c we denote the subspace of m consisting of convergent sequences; by c_0 we denote the subspace of c consisting of sequences with limit 0.

We denote the positive integers by ω. If X is a subset of ω, we denote the space of bounded sequences with support X by $m(X)$.

We denote by 1 the constant sequence with 1 in each place and we denote by δ_k the sequence which is 0 at each entry except k, at which place it is 1.

We shall according to the situation write sequences functionally, $x(n)$, or indicially, x_n. However, whenever convenient we will suppress arguments entirely and simply write x.

When we refer to a map from one space to another we mean a continuous linear map unless we say otherwise. We will often, for notational convenience, restrict maps by merely writing down new domains and ranges without bothering to change the map designation.

If T: $X \to Y$ is a map of closed range, we will define the index of T to be its deficiency minus its nullity. We will never encounter the case of infinite deficiency and infinite nullity simultaneously.

One particular map from m to m with which we will be concerned is the projection operator defined as follows.

Let $X \subset \omega$. Then E^X: $m \to m$ is defined by

$$\begin{aligned} E^X z(k) &= 0, & k &\notin X, \\ E^X z(k) &= z(k), & k &\in X. \end{aligned}$$

We will abbreviate E^X, where $X = \{k | \, k > n\}$, as $E^{(n,\infty)}$.

If a map T: $m \to m$ has the property that $T(c) \subset c$, we will conform to the usage of summability theory and call the map conservative.

By an automorphism of m we mean a one-to-one, onto, continuous, hence bicontinuous, map of m to m.

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By m^* we denote the conjugate space of m and by l^1 we denote the subspace of m^* consisting of absolutely summable sequences.

It is clear that any map T: $m \to m$ is equivalent to a uniformly bounded sequence of elements of m^*. Each f in m^* has an l^1 part, f^c, defined by

$$f^c(x) = f(x) - \lim_{n\to\infty} f(E^{(n,\infty)}x) = \sum_{i=1}^{\infty} f(E^i x).$$

We can easily check that there exists a sequence $a \in l^1$ such that

$$f^c(x) = \sum a(i)x(i).$$

Hence, to any map T: $m \to m$, where $T = f_1, f_2, \ldots, f_n, \ldots$, we may associate a matrix part T^c: $m \to m$ defined by

$$(T^c x)(k) = f_k{}^c(x) = \sum_i a_k(i)x(i).$$

Hence we may represent T^c uniquely by a matrix $\{a_{ij}\}$, where

$$\sup_i \sum_j |a_{ij}| = ||T^c|| < \infty.$$

It is also clear that any such matrix represents such a T^c.

Conversely, we observe that if T: $c_0 \to m$ is given, T may be represented by exactly one matrix $\{a_{ij}\}$ satisfying

$$\sup_i \sum_j |a_{ij}| = ||T|| < \infty.$$

Hence we see that the operator T: $c_0 \to m$ has a unique natural matrix extension T^c: $m \to m$.

If $T(c_0) \subset c_0$, then we may identify T^c with T^{**} by considering T as T: $c_0 \to c_0$.

An example. We begin by presenting an example of a conservative automorphism of m which does not map c onto c. The construction here makes clear the ideas of the more complicated construction in Theorem 1.4.

We will extend the right-hand shift operator T: $c_0 \to c_0$, of nullity 0 and deficiency 1, to a conservative automorphism of m. In this particular case, T^c: $m \to m$ is itself conservative.

Example 1. We write the positive integers as the union of a countable collection of disjoint infinite sets and call them $S_1, S_2, \ldots$. We let $\lim_{S_i} \in m^*$ be a functional of norm 1 such that

$$\lim_{S_i} x = \lim_{\substack{n\to\infty;\\ n\in S_i}} x(n),$$

where this limit exists.

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We now define the transformation $T\colon m \to m$ by

$$Tx(1) = 2 \lim_{S_1} x,$$

$$Tx(k) = x(k-1) + 2 \lim_{S_r} x, \quad \text{where } k > 1 \text{ and } k - 1 \in S_{r-1}.$$

We now show that $T\colon m \to m$ is an automorphism. We first show that T is one-to-one. Indeed, if $Tx(1) = 0$, then $\lim_{S_1} x = 0$. However, a glance at the construction shows us that $Tx(n+1) = 0$ for all $n \in S_1$ implies that x is constant on S_1 and since $\lim_{S_1} x = 0$, we see that $x(n) = 0$ for $n \in S_1$. Thus we see that $\lim_{S_2} x = 0$. Proceeding as before, we establish that $x \equiv 0$.

We now show that T is onto. Let $y \in m$ be given. We may suppose $||y|| \leqq 1$. We now produce $x \in m$ such that $Tx = y$. We define $\tilde{y} \in m$ by

$$\tilde{y}(n) = y(n+1) \quad \text{for } n \geqq 1.$$

We define $k \in m$ by

$$k(1) = \tfrac{1}{2}y(1) - \lim_{S_1} \tilde{y}, \qquad k(n) = -\tfrac{1}{2}k(n-1) - \lim_{S_n} \tilde{y} \quad \text{for } n \geqq 2.$$

It is clear that $||k|| \leqq 2$. Finally, we define the desired $x \in m$ by

$$x(n) = \tilde{y}(n) + k(j), \quad \text{where } n \in S_j.$$

It is easy to check that $Tx = y$. Indeed, we note that if $n \in S_{r+1}$, then

$$x(n) = \tilde{y}(n) + k(r+1) = \tilde{y}(n) - \tfrac{1}{2}k(r) - \lim_{S_{r+1}} \tilde{y},$$

and so $\lim_{S_{r+1}} x = -\tfrac{1}{2}k(r)$. Thus we see that if $n \in S_r$, then

$$Tx(n+1) = x(n) + 2 \lim_{S_{r+1}} x = y(n+1) + k(r) + 2 \lim_{S_{r+1}} x = y(n+1).$$

A similar calculation shows that $\lim_{S_1} x = \tfrac{1}{2}y(1)$, and so $Tx(1) = y(1)$.

It is clear that $T(c) \subset c$. It is also clear that if we define $x_0 \in m$ by $x_0(k) = 1/(-2)^r$ for $k \in S_r$, then $Tx_0 = (-1, 0, 0, \ldots)$. Hence $T(c) \subset c$, $T(c) \neq c$.

Hence T is an automorphism of m such that $T(c) \subset c$, $T(c) \neq c$. It is clear that $T\colon c_0 \to c_0$ is the right-hand shift. This completes our example.

1. Maps from c_0 to c_0 which extend to automorphisms of m.

LEMMA 1.1. *Let $T\colon c_0 \to c_0$ have matrix extension $T^c\colon m \to m$. Then T has closed range if and only if T^c has closed range. If T has closed range, is of deficiency p and nullity $r < \infty$, then T^c is also of deficiency p and nullity r, and $T^c(m \backslash c_0) \subset (m \backslash c_0)$.*

Proof. These are all routine verifications observing that $T^{**} = T^c$.

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LEMMA 1.2. *Let T: $c_0 \to c_0$ be one-to-one and of deficiency $p < \infty$. Then there are p vectors of the form $\delta_{n1}, \ldots, \delta_{n_p}$ such that*

$$T^c(m) \oplus \delta_{n1} \oplus \ldots \oplus \delta_{n_p} = m.$$

That is, $\delta_{n1} \oplus \ldots \oplus \delta_{n_p}$ is a complementary subspace in m of $T^c(m)$.

Moreover, we may define $f_{ni} \in [m(\omega\backslash(n_1, \ldots, n_p))]^*$ by $f_{ni}(y) = x(n_i)$, where $x \in m$ is defined by

(1) *$x(k) = y(k)$ for $k \in \omega\backslash(n_1, \ldots, n_p)$,*

(2) *$x \in T^c(m)$.*

That is, abusing the notation, for $x \in T^c m$ we may express $x(n_i)$ as a continuous linear functional of $\{x(n)\}$, where $n \neq n_1, \ldots, n_p$.

Proof. We write $x \in m$ as $\sum_i \alpha_i \delta_i$ with coordinate-wise convergence. If each δ_i with non-zero α_i is in $T(m)$, we choose $y \in m$ to be a coordinate-wise limit point of the sequence of bounded subsets of m given by $\{(T^c)^{-1}\sum_{i=1}^n \alpha_i\delta_i\}$. Then $T^c y = x$. Therefore if $T^c(m) \neq m$, there is a δ_{n1} such that $\delta_{n1} \notin T^c(m)$. If $T^c(m) \oplus \delta_{n1} \neq m$, then there is a δ_{n2} so that $\delta_{n2} \notin T^c(m) \oplus \delta_{n1}$. We continue in this way and extract p such spike vectors which clearly span the complementary subspace.

To see that evaluation at n_j is a continuous linear functional on $E^{\omega\backslash(n_1,\ldots,n_p)}T^c(m)$, we observe that $\delta_{n_j} \notin T^c(m)$ but $T^c(m)$ is closed. Hence there exists $\rho > 0$ such that if $x \in T^c(m)$ and if $|x(n)| \leqq 1$, for $n \neq n_1, \ldots, n_p$, then $|x(n_j)| \leqq \rho$. This completes the proof of Lemma 1.2.

The following proposition results from Cambern's extension of the Banach-Stone Theorem [**1**] and the observation that any homeomorphism of the Stone-Čech compactification of ω must carry ω to ω. However, in our case, a much easier direct proof is now available.

PROPOSITION 1.3. *Let T: $c_0 \to c_0$ be given. Suppose that there is an automorphism T': $m \to m$ extending T. Suppose further that $||T'|| = 1$ and that there is a $\rho > \frac{1}{2}$ such that*

$$\rho \leqq \inf_{||x||=1} ||T'x||.$$

Then $T(c_0) = c_0$.

Proof. To each δ_i we associate an n_i such that $|T'\delta_i(n_i)| \geqq \rho$. Denote this set of n_i by N. It is easy to see that if $||x|| = 1$ and $x(i) = 0$, then $|T'x(n_i)| \leqq 1 - \rho$ since otherwise we could find y such that $||y|| = 1$ and $|T'y(n_i)| > 1$. Hence if $||x|| = 1$ and $|x(i)| = 1$, then $|T'x(n_i)| \geqq 2\rho - 1$. Similarly, it is clear that if $i \neq j$, then $n_i \neq n_j$. Therefore the map

$$E^N T': m \to m(N)$$

is an onto isomorphism. Since T' is an automorphism, we see that $N = \omega$. Now, since $T^c\delta_i = T'\delta_i$ and $||T^c|| \leqq ||T'|| = 1$, it is clear that if $||x|| = 1$

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and $|x(i)| = 1$, then $|T^c x(n_i)| \geqq 2\rho - 1$. Hence T^c is an automorphism and so by Lemma 1.1 we see that $T(c_0) = c_0$.

We now come to our principal result.

THEOREM 1.4. *Let T: $c_0 \to c_0$ be one-to-one and of deficiency $p < \infty$. Then T can be extended to T': $m \to m$, an automorphism of m.*

Proof. We may assume that $||T^c|| = 1$.

Let $R = \{r_1, \ldots, r_p\}$ denote a set of integers such that $\delta_{r_1} \oplus \ldots \oplus \delta_{r_p}$ is a complementary subspace to $T^c(m)$. By Lemma 1.2 we can select $\eta > 0$ so that for each $x \in m$:

$$\text{if } ||E^{\omega\backslash R}T^c x|| < \eta, \text{ then } ||E^R T^c x|| < 1.$$

We note that $E^{\omega\backslash R}T^c$: $m \to m(\omega\backslash R)$ is an isomorphism.

We now see, by the finite dimensionality of $m(R)$, that we can choose an infinite set $K \subset \omega\backslash R$ such that if $x \in m$ satisfies

(1) $||E^{\omega\backslash R}T^c x|| \leqq 1$ and

(2) $E^{\omega\backslash K}T^c x = 0$,

then $||E^R T^c x|| < 1/2p$.

We now select $S_1, \ldots, S_p$, disjoint infinite subsets of K. Then we further break up each S_i into an infinite number of disjoint infinite subsets $S_{i1}, \ldots, S_{in}, \ldots$.

Let $x_{ij} = \chi(S_{ij})$, the characteristic function of S_{ij}. Now we choose $f_{ij} = \lim_{S_{ij}} \in m^*$. That is,

(1) $f_{ij}(x) = 0$ if x has no support in S_{ij},

(2) $f_{ij}(c_0) = 0$,

(3) $f_{ij}(x_{ij}) = ||f_{ij}|| = 1$.

Finally, we choose $\alpha = \max(2, 2/n)$. We then define T_{r_1}: $m \to m$ by

$$T_{r_1}(x) = x + \alpha\left(f_{11}(x)\delta_{r_1} + \sum_{j=2}^{\infty} f_{1j}(x)x_{1,j-1}\right)$$

with the limit taken coordinate-wise.

We now show that T_{r_1}: $T^c(m) \to m$ is one-to-one. Suppose that $x \in T^c(m)$ and $T_{r_1}(x) = 0$. It is then clear that on S_{11}, x must be constant, and similarly for each S_{1j} there is a scalar α_j such that $E^{S_{1j}}x = \alpha_j x_{1j}$. Now x must be 0 elsewhere except possibly at r_1. But we suppose that $x(r)_1 = 1$. Then $f_{11}(x) = -\alpha^{-1}$. But we know that x is constant on S_{11} and hence we know that $E^{S_{11}}x = -\alpha^{-1}x_{11}$. Thus we see that $f_{12}(x) = \alpha^{-2}$, and hence that $E^{S_{12}}x = \alpha^{-2}x_{12}$. Proceeding similarly we see that $|x(k)| \leqq \alpha^{-1} < \eta/2$ for $k \neq r_1$. But since $x(r_1) = 1$, this contradicts the choice of η for $x \in T^c(m)$. If $x(r_1) = 0$, the same process shows that $x = 0$.

We now define $T_{r_2}, \ldots, T_{r_p}$ similarly. It is clear that if we define T_R by

$$T_R x = T_{r_p} \ldots T_{r_1} x,$$

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then $T_R T^c$ is one-to-one. Note that each T_{r_i} affects only coordinates not affected by the other T_{r_j}. Indeed,

$$T_R x = x + \alpha \sum_{i=1}^{p} \left(f_{i1}(x)\delta_{r_i} + \sum_{j=2}^{\infty} f_{ij}(x) x_{i,j-1} \right).$$

We must now show that $T_R T^c$: $m \to m$ is onto.

To this end we first show that $\delta_{r_i} \in T_R T^c(m)$. If we define y_i by

$$-y_i = \sum_{j=1}^{\infty} (-\alpha)^{-j} x_{ij},$$

we see that $T_R y_i = \delta_{r_i}$. Since Support $y_i \subset \omega \backslash R$ and $E^{\omega\backslash R} T^c(m) = m(\omega\backslash R)$, we see that there exists an x_i such that

$$T_R E^{\omega\backslash R} T^c x_i = \delta_{r_i}.$$

Now we note that $||y_i|| < 1$ and Support $y_i \subset K$. Hence by our selection of K we see that

$$||E^R T^c x_i|| < 1/2p.$$

Hence, $||T_R T^c x_i - \delta_{r_i}|| < 1/2p$. Moreover, Support $T_R T^c x_i \subset R$. Hence we can solve a system of p linear equations to find x_i' such that $T_R T^c x_i' = \delta_{r_i}$.

We now show that $E^{\omega\backslash R} T_R T^c(m) = m(\omega\backslash R)$. Let $y \in m$ be given. We may assume that $||y|| = 1$. We define $\tilde{y} \in m$ by

$$\tilde{y} = \sum_{i=1}^{p} \left[f_{i2}(y) x_{i2} + \sum_{j=3}^{\infty} (f_{ij}(y) - \alpha^{-1} \beta_{i,j-1}(y)) x_{ij} \right],$$

where β_{ij} is defined as the coefficient of x_{ij} in the above formula, and hence is defined inductively.

Since $||f_{ij}|| = 1$ and $\alpha^{-1} \leqq \frac{1}{2}$, we see that $|\beta_{ij}| \leqq 2$, and hence that $||\tilde{y}|| \leqq 2$.

We then verify that $E^{\omega\backslash R} T_R(y - \tilde{y}) = E^{\omega\backslash R} y$. We also note that $E^{\omega\backslash R} T_R = E^{\omega\backslash R} T_R E^{\omega\backslash R}$. Hence, since $E^{\omega\backslash R} T^c(m) = m(\omega\backslash R)$, we see that there is an x such that $E^{\omega\backslash R} T^c x = E^{\omega\backslash R}(y - \tilde{y})$. Hence, $E^{\omega\backslash R} T_R T^c(x) = E^{\omega\backslash R} y$.

This result, combined with our earlier observations that $\delta_{r_i} \in T_R T^c(m)$, shows us that $T_R T^c(m) = m$. Since T_R is the identity on c_0, we see that $T_R T^c$ is an extension of T: $c_0 \to c_0$.

This completes the proof of Theorem 1.4.

2. Maps from c_0 to c which extend to automorphisms of m. This paper was originally motivated by a desire to find conservative automorphisms of m that mapped c properly into c. With the results we have established in § 1 it is easy now to attack this question.

We first show that if the automorphism is given by a matrix or is close to an isometry, then it maps c onto c. These results are very easy and the first, at least, is well known in summability.

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LEMMA 2.1. *Let $T: c_0 \to c$ have matrix extension $T^c: m \to m$. If T^c is an automorphism, then $T(c_0)$ is of deficiency 1 in c. If T^c is a conservative automorphism, then $T^c(c) = c$.*

Proof. Suppose that T^c is an automorphism. We let $T^c = T_1^c + T_2^c$, where $T_1^c(c_0) \subset c_0$ and T_2^c is of 1-dimensional range. Then T_1^c is of index 0 and so by Lemma 1.1 we see that $T_1^c: c_0 \to c$ is of index 1. Since addition of a 1-dimensional operator cannot change the index, we see that $T^c: c_0 \to c$ is of deficiency 1. The second assertion is now immediate.

PROPOSITION 2.2. *Let $T: c_0 \to c$ be given. Suppose that $T': m \to m$ is a conservative automorphism of norm 1 which extends T. Suppose, furthermore, that there is a $\rho > \frac{1}{2}$ such that*

$$\rho \leqq \inf_{||x||=1} ||T'x||.$$

Then $T'(c) = c$.

Proof. Just as in Proposition 1.3, we see that $T^c: m \to m$ is an automorphism and so by Lemma 2.1 we see that $T'(c_0)$ is of deficiency 1 in c and thus $T'(c) = c$.

In order to prove the desired analogue of Theorem 1.4, we extend the isomorphism between c_0 and c to an automorphism of m.

Example 2. Let T be the automorphism of m developed in Example 1. Let R be an automorphism of m which exchanges $T1 = (2, 3, 3, \ldots)$ with $Tx_0 = (-1, 0, 0, \ldots)$ while yet remaining the identity on $T(c_0)$. Let $RT = S$. It is clear that $S: m \to m$ is an automorphism. We see that $Sx = Tx$ for $x \in c_0$ and so $S(c_0) = c_0$. Also, $S(1) = (-1, 0, 0, \ldots)$. Hence S is an automorphism of m such that $S(c) = c_0$.

THEOREM 2.3. *Let $T: c_0 \to c$ be one-to-one, of closed range, and of finite deficiency $p \geqq 1$. Then there exists $T': m \to m$, a conservative automorphism of m extending T.*

Proof. We first extend T to $\tilde{T}: c \to c$, a one-to-one map of deficiency $p - 1$. We then note that $S\tilde{T}S^{-1}|_{c_0}: c_0 \to c_0$ is one-to-one, of closed range, and of deficiency $p - 1$. (The automorphism S which is used is that of Example 2.) We then extend $S\tilde{T}S^{-1}|_{c_0}$ by Theorem 1.4 to T'', an automorphism of m. Then $S^{-1}T''S|_c = \tilde{T}$ and hence $T' = S^{-1}T''S$ is our desired conservative automorphism. This completes the proof of Theorem 2.3.

This theorem provides conservative automorphisms of m which map c properly into c. Indeed if $T: c_0 \to c$ is of deficiency $p \geqq 1$, then $T'(c)$ is of deficiency $p - 1$ in c.

In the case of infinite deficiency, our method works in all the cases we have been able to consider, but we have been unable to prove its validity. The fact that T_R would be an infinite product of T_{r_i} does not cause trouble since

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coordinate-wise convergence suffices. The principal problem is our inability to prove that there must be R such that $E^{\omega\backslash R}T^c$: $m \to m(\omega\backslash R)$ is an isomorphism. This is not an allegation that it cannot be done, only that we cannot see how to do it as of this writing.

Since this paper was written, Lindenstrauss and Rosenthal have obtained strong, general results concerning extensions of operators [**3**]. In particular, they were able to prove Theorem 1.4 without the hypothesis $p < \infty$ while yet using much less than the full strength of their methods. Thus we can now easily see that Theorem 2.3 holds without the restriction $p < \infty$. We have accordingly excised an example which this result makes otiose.

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