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ERGODIC ACTIONS OF COMPACT GROUPS ON OPERATOR ALGEBRAS II: CLASSIFICATION OF FULL MULTIPLICITY ERGODIC ACTIONS

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Introduction. In the first paper of this series [**17**], we set up some general machinery for studying ergodic actions of compact groups on von Neumann algebras, namely, those actions $\alpha: G \rightarrow \text{Aut}\,\mathcal{M}$ for which $\mathcal{M}^G = \mathbf{C}$. In particular we obtained a characterisation of the full multiplicity ergodic actions:

THEOREM A. *If α is an ergodic action of G on $\mathcal{M}$, then the following conditions are equivalent*:

(1) *Each spectral subspace $\mathcal{M}_\pi$ has multiplicity* $\dim \pi$ *for π in $\hat{G}$.*

(2) *Each π in $\hat{G}$ admits a unitary eigenmatrix in $\mathcal{M}$.*

(3) *The W^* crossed product is a* (*Type* I) *factor.*

(4) *The C^* crossed product of the C^* algebra of norm continuity is isomorphic to the algebra of compact operators on a Hilbert space.*

This is a slightly simplified formulation of Theorem 15 in [**17**], to which we refer for further details. Our aim in the present paper is to obtain a classification of such actions, in direct analogy to that given in the case of compact Abelian groups in [**1**] and subsequently in [**13**]. As will become apparent below, our results can be regarded as classifying coactions of compact groups on Type I factors. Adrian Ocneanu (in unpublished work) has initiated the study of such coactions on the hyperfinite Type II factors.

Briefly our programme is as follows:

1. to define cocycles for the dual of a compact group.

2. to define cocycle representations for the group dual.

3. to show that the usual formalism applies for the crossed product by a cocycle representation.

4. to classify full multiplicity ergodic actions in terms of cocycles.

5. to show that any coaction of a compact group on a Type I factor is implemented by a cocycle representation of the group dual.

6. to show how cocycles and cocycle representations may be normalised to facilitate subsequent definitions.

7. to introduce a C^* algebra for each dual cocycle ω which is universal (in an appropriate sense) for ω-representations of the group dual. (Thus

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these algebras may be regarded as generalisations of the irrational rotation algebras, if they exist.) This algebra may be identified with the C^* algebra of norm continuity (generated by the spectral subspaces) for the corresponding ergodic actions (or equivalently on a simple C^* algebra).

8. to associate to each dual cocycle a pair consisting of a dual bicharacter and a perturbed comultiplication. The cocycle is essentially determined up to equivalence by the pair and a non-degeneracy criterion is given in terms of the bicharacter for the corresponding action to be on a factor.

Finally we mention some open problems concerning ergodic actions arising from the present paper.

A. Compute $H^2(\hat{G})$, the set of equivalence classes of dual cocycles, for an arbitrary non-Abelian compact group G and in particular find the non-degenerate cocycles. This computation is made in [**18**] for the groups $SU(2)$ and $SO(3)$; and we have been able to extend the techniques introduced there to show that all full multiplicity ergodic actions of the groups $SU(2) \times \mathbf{T}$, $SU(2) \times SU(2)$ and $SU(3)$ are induced from actions of the maximal torus (see [**19**]). Furthermore it is known (as an off-shoot of the classification of finite groups) that a finite group has a non-degenerate dual cocycle only if it is solvable: such groups are said to be of "central type"; see [**8**].

B. Extend the full multiplicity classification to general ergodic actions starting from the foundations laid in [**17**].

A preliminary version of the first seven sections of this paper originally formed part of the second chapter of my doctoral dissertation [**16**]. I would like to thank my advisor Jonathan Rosenberg for his encouragement then, as well as Vaughan Jones, who instigated this research. Section 8 was inspired by Section 10 of V. G. Drinfeld's report [**5**], which was kindly made available by Pierre Cartier. I would also like to acknowledge the support during the final stages of this work of George Elliott, University of Toronto, and the Miller Institute, University of California, Berkeley. I would finally like to point out that Magnus Landstad [**11**] simultaneously developed similar ideas to some of those in this paper: he was kind enough to show me unpublished notes as well as mentioning the statement of Lemma 24 to me.

1. Cocycles. Let G be a compact group and $\rho:G \rightarrow \mathscr{B}(L^2(G))$ the right regular representation of $\mathscr{R}(G)$, generating the (right) von Neumann algebra $\mathscr{R}(G) = \rho(G)''$ of G. On G we have a canonical comultiplication

$$\delta_G:\mathscr{R}(G) \rightarrow \mathscr{R}(G) \otimes \mathscr{R}(G),$$

namely the *-isomorphism extending $\delta_G(\rho(g)) = \rho(g) \otimes \rho(g)$. (If $\mathscr{M}$ and $\mathscr{N}$ are von Neumann algebras, their von Neumann tensor product will be denoted by $\mathscr{M} \otimes \mathscr{N}$.)

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A *cocycle* of $\hat{G}$ (or *multiplier*) is a unitary ω in $\mathscr{R}(G) \otimes \mathscr{R}(G)$ satisfying the cocycle identity:

$$(\delta_G \otimes \iota(\omega))(\omega \otimes I) = (\iota \otimes \delta_G(\omega))(I \otimes \omega).$$

(Note that when G is Abelian, this reduces to the usual definition of $Z^2(\hat{G}, \mathbf{T})$, since the Fourier transform furnishes an isomorphism between $\mathscr{R}(G)$ and $l^\infty(\hat{G})$.) Two cocycles ω, ω' are called *cohomologous* or *equivalent* if and only if we can find a unitary v in $\mathscr{R}(G)$ such that

$$\omega' = \delta_G(v^*)\omega(v \otimes v)$$

and the set of equivalence classes is denoted by $H^2(\hat{G})$. Given a cocycle ω, we may construct another cocycle $\tilde{\omega}$ via

$$\tilde{\omega} = \sigma \cdot (\alpha \otimes \alpha)\omega^*$$

where σ is the "flip" on $\mathscr{R}(G) \otimes \mathscr{R}(G)$ (i.e., $\sigma(x \otimes y) = y \otimes x$) and α is the involutive *-antiautomorphism of $\mathscr{R}(G)$ induced by $\alpha(\rho(g)) = \rho(g^{-1})$ for $g \in G$. As we shall see in Section 6 (Lemma 13), $\tilde{\omega}$ is always cohomologous to ω. Our work on normalisation there will also have as a consequence the fact that any cocycle is equivalent to a cocycle ω for which $\tilde{\omega} = \omega$.

2. Representations and coactions. We start by recalling some elementary definitions and properties of coactions from [**12**]. By a coaction of G on the von Neumann algebra $\mathscr{M}$, we mean a *-isomorphism δ of $\mathscr{M}$ into $\mathscr{M} \otimes \mathscr{R}(G)$ satisfying the comultiplication identity

$$(\iota \otimes \delta_G)\delta = (\delta \otimes \iota)\delta.$$

The *crossed product* of $\mathscr{M}$ by $\hat{G}$ with respect to δ is the von Neumann algebra generated by $\delta(\mathscr{M})$ and $\mathbf{C} \otimes L^\infty(G)$ in $\mathscr{M} \otimes \mathscr{B}(L^2(G))$: we denote it by $\mathscr{M} \rtimes_\delta \hat{G}$. The *dual action* of G on $\mathscr{M} \rtimes_\delta \hat{G}$ is defined to be the restriction of $\iota \otimes \mathrm{Ad}\,\lambda$ where λ is the left regular representation of G on $L^2(G)$. We write $\hat{\delta}_g$ for $\iota \otimes \mathrm{Ad}\,\lambda(g)\,|_{\mathscr{M} \rtimes \hat{G}}$. In this realisation of the crossed product, the fixed point algebra of the dual action is equal to $\delta(\mathscr{M})$ (cf. Theorem B (2)). We shall say that a coaction δ is *implemented* if there is a unitary W in $\mathscr{M} \otimes \mathscr{R}(G)$ for which

$$(*) \quad \delta(x) = W(x \otimes I)W^*$$

for all x in $\mathscr{M}$.

LEMMA 1. (1) *Let $\mathscr{M}$ be a factor. Then the equation* (*) *defines a coaction on $\mathscr{M}$ if and only if $f(W)$ lies in $\mathscr{R}(G) \otimes \mathscr{R}(G)$, where*

$$f(W) = \iota \otimes \delta_G(W^*)(W \otimes I)\iota \otimes \sigma(W \otimes I).$$

(2) *For any algebra $\mathscr{M}$, if $f(W)$ lies in $\mathscr{R}(G) \otimes \mathscr{R}(G)$, then $f(W) = I \otimes \omega$ where ω is a cocycle of $\hat{G}$.*

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Proof. (*) defines a coaction if and only if

$$(\delta \otimes \iota)\delta(x) = (\iota \otimes \delta_G)\delta(x)$$

for all x in $\mathcal{M}$. We now compute both sides of this equation:

$$(\delta \otimes \iota)\delta(x) = \mathrm{Ad}[\,(W \otimes I)\iota \otimes \sigma(W \otimes I)\,](x \otimes I \otimes I)$$

$$(\iota \otimes \delta_G)\delta(x) = \mathrm{Ad}[\iota \otimes \delta_G(W)\,](x \otimes I \otimes I).$$

Thus equality holds if and only if $f(W)$ lies in the relative commutant of $\mathcal{M}$, that is $\mathcal{R}(G) \otimes \mathcal{R}(G)$.

Let us check that if $I \otimes \omega = f(W)$, then ω satisfies the cocycle condition. We define automorphisms of $\mathcal{R}(G) \otimes \mathcal{R}(G) \otimes \mathcal{R}(G)$ by

$$\sigma_{12}(a \otimes b \otimes c) = b \otimes a \otimes c$$

$$\sigma_{13}(a \otimes b \otimes c) = c \otimes b \otimes a$$

$$\sigma_{123}(a \otimes b \otimes c) = c \otimes a \otimes b.$$

Then we have on the one hand

$$\begin{aligned}
&I \otimes (\delta_G \otimes \iota(\omega)\omega \otimes I) \\
&= \iota \otimes \delta_G \otimes \iota\{\iota \otimes \delta_G(W^*)(W \otimes I)\iota \otimes \sigma(W \otimes I)\,\}I \otimes \omega \otimes I \\
&= [\iota \otimes (\delta_G \otimes \iota)\delta_G W^*][\iota \otimes \delta_G(W) \otimes I][I \otimes \omega \otimes I] \\
&\qquad\qquad \times [\iota \otimes \sigma_{13}(W \otimes I \otimes I)\,] \\
&= [\iota \otimes (\iota \otimes \delta_G)\delta_G W^*] \\
&\qquad\qquad \times (W \otimes I \otimes I)\iota \otimes \sigma_{12}(W \otimes I \otimes I)\iota \otimes \sigma_{13}(W \otimes I \otimes I)
\end{aligned}$$

since $\iota \otimes \sigma_{13}(W \otimes I \otimes I)$ and $I \otimes \omega \otimes I$ commute and δ_G is a coaction, while on the other hand

$$\begin{aligned}
&I \otimes (\iota \otimes \delta_G(\omega)I \otimes \omega) \\
&= \iota \otimes \iota \otimes \delta_G\{\iota \otimes \delta_G(W^*)(W \otimes I)\iota \otimes \sigma(W \otimes I)\,\}I \otimes I \otimes \omega \\
&= [\iota \otimes (\iota \otimes \delta_G)\delta_G W^*][W \otimes I \otimes I] \\
&\qquad\qquad \times [\iota \otimes \sigma_{123}(\iota \otimes \delta_G(W) \otimes I)\,]I \otimes I \otimes \omega \\
&= [\iota \otimes (\iota \otimes \delta_G)\delta_G W^*][W \otimes I \otimes I] \\
&\qquad \times [\iota \otimes \sigma_{123}(W \otimes I \otimes I(\iota \otimes \sigma(W \otimes I) \otimes \mathrm{I})I \otimes \omega^* \otimes I)\,]I \otimes I \otimes \omega \\
&= [\iota \otimes (\iota \otimes \delta_G)\delta_G W^*][W \otimes I \otimes I] \\
&\qquad\qquad \times [\iota \otimes \sigma_{12}(W \otimes I \otimes I)\,][\iota \otimes \sigma_{13}(W \otimes I \otimes I)\,]
\end{aligned}$$

where we have used the identity $(\iota \otimes \delta_G)\sigma(a \otimes b) = \sigma_{13}(\delta_G(a) \otimes b)$.

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We shall refer to W as an *ω-representation* of the group dual $\hat{G}$ in $\mathscr{M}$, or on $\mathscr{H}$ in case $\mathscr{M} = \mathscr{B}(\mathscr{H})$. (We note incidentally that if W is an ω-representation on $\mathscr{H}$ giving the coaction $\delta(x) = W(x \otimes I)W^*$, then the other ω-representations of $\hat{G}$ on $\mathscr{H}$ correspond exactly to the unitary cocycles for δ on $\mathscr{B}(\mathscr{H})$ as defined on page 89 of [**12**].) If furthermore there is an action α_g of G on $\mathscr{M}$, then we shall say that W is an *equivariant* ω-representation of $\hat{G}$ provided that for all g in G

$$\alpha_g \otimes \iota(W) = (I \otimes \rho(g)^{-1})W.$$

Conversely, given a cocycle ω we may define the *regular ω-representation* W_ω in $L^2(G)$ via

$$W_\omega = W_G \cdot \sigma\omega,$$

where W_G is defined by

$$(W_G f)(s, t) = f(s, ts) \quad (f \in L^2(G \times G)).$$

We recall that $\delta_G(x) = W_G^*(x \otimes I)W_G$ for x in $\mathscr{R}(G)$, together with the identities

$$\begin{aligned}\iota \otimes \delta_G(W_G) &= (W_G \otimes I)\iota \otimes \sigma(W_G \otimes I)\\ &= \iota \otimes \sigma(W_G \otimes I)(W_G \otimes I)\end{aligned}$$

which may be found on page 19 of [**12**]. We must verify that W_ω is indeed an ω-representation: for this we need a preliminary result.

LEMMA 2. *If ω is a cocycle, then so is $\sigma\omega$. (We shall call $\sigma\omega$ the inverse cocycle to ω.)*

Proof. In fact

$$\delta_G \otimes \iota(\sigma\omega) = \sigma_{13}(\iota \otimes \delta_G(\omega))$$

$$\iota \otimes \delta_G(\sigma\omega) = \sigma_{13}(\delta_G \otimes \iota(\omega))$$

$$\sigma\omega \otimes I = \sigma_{13}(I \otimes \omega)$$

$$I \otimes \sigma\omega = \sigma_{13}(\omega \otimes I)$$

so the cocycle identity for $\sigma\omega$ follows by applying σ_{13} to the identity for ω.

LEMMA 3. *W_ω is an ω-representation and is equivariant with respect to the action $g \mapsto$ Ad $\lambda(g)$ of G on $\mathscr{B}(L^2(G))$.*

Proof. Letting $W = W_\omega$, we have

$$\iota \otimes \delta_G(W^*)(W \otimes I)\iota \otimes \sigma(W \otimes I)$$

$$= \iota \otimes \delta_G(\sigma\omega^* W_G^*)(W_G \otimes I)(\sigma\omega \otimes I)\iota \otimes \sigma(W_G \otimes I)\iota \otimes \sigma(\sigma\omega \otimes I)$$

$$= \iota \otimes \delta_G(\sigma\omega^*)\iota \otimes \sigma(W_G^* \otimes I)(\sigma\omega \otimes I)\iota \otimes \sigma(W_G \otimes I)\iota \otimes \sigma(\sigma\omega \otimes I)$$

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$$= \iota \otimes \delta_G(\sigma\omega^*)\iota \otimes \sigma\{ (W_G^* \otimes I)(\iota \otimes \sigma(\sigma\omega \otimes I))(W_G \otimes I) \}$$
$$\iota \otimes \sigma(\sigma\omega \otimes I)$$
$$= \iota \otimes \delta_G(\sigma\omega^*)\iota \otimes \sigma(\delta_G \otimes \iota(\sigma\omega)\sigma\omega \otimes I))$$
$$= \iota \otimes \delta_G(\sigma\omega^*)\iota \otimes \sigma(\iota \otimes \delta_G(\sigma\omega)I \otimes \sigma\omega) \quad \text{(by Lemma 2)}$$
$$= \iota \otimes \delta_G(\sigma\omega^*)\iota \otimes \delta_G(\sigma\omega)(I \otimes \omega) \quad (\text{since } \sigma \cdot \delta_G = \delta_G)$$
$$= I \otimes \omega$$

as claimed.

To establish the equivariance of W_ω, we must show that for all g

$$(*) \qquad \text{Ad}\, \lambda(g) \otimes \iota(W_\omega) = (I \otimes \rho(g)^{-1})W_\omega.$$

Substituting in the expression for W_ω and cancelling $\sigma\omega$ from both sides, we see that we are reduced to verifying that (*) holds when ω equals I and $W_\omega = W_G$. This of course may be verified directly, but we prefer to give a more conceptual proof. In fact W_G lies in $L^\infty(G) \otimes \mathscr{R}(G)$ so may be regarded as a bounded function from G to $\mathscr{R}(G)$; it corresponds to the function $\widetilde{W}_G(x) = \rho(x)$. Moreover the restriction of Ad $\lambda(g)$ to $L^\infty(G)$ gives the action of G by left translation. Hence

$$(\text{Ad}\, \lambda(g))\widetilde{W}_G(x) = \widetilde{W}_G(g^{-1}x) = \rho(g)^{-1}\widetilde{W}_G(x)$$

which clearly implies (*) for the case $\omega = I$.

3. Crossed products by implemented coactions. Although we shall construct a C^* algebra later which is universal for ω-representations of $\hat{G}$, it is possible to define such notions as commutant, enveloping von Neumann algebra, equivalence, quasi-equivalence, intertwining operators, etc. in a fairly straightforward manner without referring to the C^* algebra. For example, we define the commutant and bicommutant of an ω-representation W on $\mathscr{H}$ by

$$(W)' = \{x \in \mathscr{B}(\mathscr{H}) : W(x \otimes I)W^* = x \otimes I\}$$
$$(W)'' = ((W)')'$$

respectively. The first definition should be compared with the definition of the fixed point algebra of a coaction δ of G on $\mathscr{M}$, namely

$$\mathscr{M}^\delta = \{x \in \mathscr{M} : \delta(x) = x \otimes I\}.$$

We shall suggestively denote the bicommutant of the regular ω-representation by $\pi_\omega(\hat{G})''$ in accordance with the customary notation in the Abelian case. The following theorem is a direct analogue of the well-known Abelian result.

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THEOREM 1. *Let δ be a coaction of G on $\mathscr{M}$, which is implemented by an ω-representation of $\hat{G}$ in $\mathscr{M}$. Then*

$$\mathscr{M} \rtimes \hat{G} \cong \mathscr{M} \otimes \pi_{\sigma\omega}(\hat{G})''.$$

Under this isomorphism the action of the group is given by $\hat{\delta}_g = \iota \otimes \mathrm{Ad}\,\lambda(g)$ *and has fixed point algebra* $\mathscr{M}$.

Proof. Let

$$\delta(x) = W(x \otimes I)W^* \quad (x \in \mathscr{M})$$

with W an ω-representation of $\hat{G}$ in $\mathscr{M}$. Thus $\mathscr{M} \rtimes \hat{G}$ is generated by $\delta(\mathscr{M})\mathbf{C} \otimes L^\infty(G)$. Thus $\mathscr{M} \otimes \mathbf{C}$ and $W^*(\mathbf{C} \otimes L^\infty(G))W$ generate an algebra isomorphic to $\mathscr{M} \rtimes \hat{G}$. Let

$$\mathscr{H} = \mathscr{H}_1 \otimes L^2(G), \text{ where } \mathscr{M} \subseteq \mathscr{B}(\mathscr{H}_1).$$

We compute commutants in $\mathscr{B}(\mathscr{H})$: $\leftrightarrow$ denotes "commutes with." Firstly we note that

$$(\mathbf{C} \otimes L^\infty(G))' = \{x \in \mathscr{B}(\mathscr{H}) : x \otimes I \leftrightarrow I \otimes W_G\}$$

since $L^\infty(G) = \{y \in \mathscr{B}(L^2(G)) : y \otimes I \leftrightarrow W_G\}$.
So

$$(W^*(\mathbf{C} \otimes L^\infty(G))W)'$$
$$= \{x \in \mathscr{B}(\mathscr{H}) : x \otimes I \leftrightarrow (W^* \otimes I)(I \otimes W_G)(W \otimes I)\}.$$

Hence

$$(\mathscr{M} \vee (W^*(\mathbf{C} \otimes L^\infty(G))W))'$$
$$= (\mathscr{M} \otimes \mathbf{C})' \cap (W^*(\mathbf{C} \otimes L^\infty(G))W)'$$
$$= \mathscr{M}' \otimes \mathscr{B}(L^2(G)) \cap (W^*(\mathbf{C} \otimes L^\infty(G))W)'.$$

But if $y \in \mathscr{M}' \otimes \mathscr{B}(L^2(G))$,

$$y \otimes I \leftrightarrow (W^* \otimes I)(I \otimes W_G)(W \otimes I) = (I \otimes W_G\omega)(\iota \otimes \sigma(W^* \otimes I))$$

if and only if

$$y \otimes I \leftrightarrow I \otimes W_G\omega = I \otimes W_{\sigma\omega}$$

if and only if

$$y \in \mathscr{M}' \otimes \pi_{\sigma\omega}(\hat{G})'.$$

Therefore

$$\mathscr{M} \vee (W^*(\mathbf{C} \otimes L^\infty(G))W) = (\mathscr{M}' \otimes \pi_{\sigma\omega}(\hat{G})')' = \mathscr{M} \otimes \pi_\omega(\hat{G})''.$$

Thus the automorphism $a \mapsto W^*aW$ of $\mathscr{M} \otimes \mathscr{B}(L^2(G))$ takes $\delta(\mathscr{M})$ to $\mathscr{M}$, $\mathscr{M} \rtimes \hat{G}$ to $\mathscr{M} \otimes \pi_{\sigma\omega}(\hat{G})''$, and fixes $I \otimes \lambda(g)$, so all the assertions of the theorem follow.

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4. Classification of full multiplicity ergodic actions. Our purpose here is to establish a bijection between (equivalence classes of) full multiplicity ergodic actions of a compact group G and $H^2(\hat{G})$. Before doing so we recall some standard duality results on coactions.

THEOREM B. (see pages 9, 12 and 25 of [**12**]) *Let δ be a coaction of G on $\mathcal{M}$, with dual action $\hat{\delta}$ of G on $\mathcal{M} \rtimes \hat{G}$. Then we have*

(1) $(\mathcal{M} \rtimes_\delta \hat{G})^{\hat{\delta}} = \delta(\mathcal{M})$.

(2) $(\mathcal{M} \rtimes \hat{G}) \rtimes G \cong \mathcal{M} \otimes \mathscr{B}(L^2(G))$.

Conversely, an action α of G on $\mathcal{M}$ arises as the dual of a coaction if and only if one of the following equivalent conditions is satisfied.

(a) *There is an equivariant *-isomorphism of $L^\infty(G)$ into $\mathcal{M}$.*

(b) *There is an ordinary equivariant representation V of $\hat{G}$ in $\mathcal{M}$.*

In this case the action α is dual to the coaction on $\mathcal{M}^\alpha$ defined by

$$\delta(x) = V^*(x \otimes I)V.$$

(By an *ordinary* representation, we mean of course a trivial-cocycle representation.)

We are now in a position to state our principal result.

THEOREM 2. *For each cocycle ω of $\hat{G}$, $\alpha_g = \mathrm{Ad}\,\lambda(g)$ defines a full multiplicity ergodic action of G on $\pi_\omega(\hat{G})''$. Moreover every full multiplicity ergodic action arises in this way and the cocycle ω is uniquely determined up to equivalence by the existence of an equivariant ω-representation in the algebra.*

Thus there is a natural bijection between equivalence classes of full multiplicity ergodic actions and $H^2(\hat{G})$.

In Section 7, once we have introduced the G-algebra $L^1_\omega(\hat{G})$ and established some of its elementary properties, we will be able to give the outline of a different (but equivalent) proof of Theorem 2 in terms of quasi-equivalence of equivariant representations of $L^1_\omega(\hat{G})$ and the GNS construction. The present proof, however, uses the minimum of machinery and also has as spin-off the implementation theorem of the next section. We shall need three lemmas for the proof of Theorem 2.

LEMMA 4. *$g \mapsto \mathrm{Ad}\,\lambda(g)$ defines a full multiplicity ergodic action on $\pi_\omega(\hat{G})''$ and W_ω defines an equivariant ω-representation in $\pi_\omega(\hat{G})''$.*

Proof. Let us take a cocycle ω and consider the regular ω-representation of $\hat{G}$ on $L^2(G)$. Thus we obtain a coaction of G on $\mathscr{B}(L^2(G))$ with crossed product $\mathscr{B}(L^2(G)) \otimes \pi_\omega(\hat{G})''$ by Theorem 1. In this realisation, the dual action of G is given by $\iota \otimes \mathrm{Ad}\,\lambda(g)$ and has fixed point algebra $\mathscr{B}(L^2(G))$, so leaves $\pi_\omega(\hat{G})''$ invariant and acts ergodically on it. To check that the restriction of the action of G to $\pi_\omega(\hat{G})''$ is of full multiplicity, it suffices to

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check that the crossed product by this action is a factor by Theorem 15 of [**17**]; but this is immediate from Theorem B (2). Finally we already verified in Lemma 3 that W_ω was an equivariant ω-representation in $\pi_\omega(\hat{G})''$.

We note that the invariance of $\pi_\omega(\hat{G})''$ under Ad $\lambda(g)$ also follows straight from the equivariance of W_ω. For the definition of the commutant $(W_\omega)'$ and the equivariance of W_ω together show that the commutant is invariant under Ad $\lambda(g)$, and hence so is the bicommutant $\pi_\omega(\hat{G})''$.

LEMMA 5. *If $\alpha:G \to \mathrm{Aut}(\mathscr{M})$ is any full multiplicity ergodic action of G, then there is an equivariant ω-representation of $\hat{G}$ in $\mathscr{M}$ for some cocycle ω of $\hat{G}$, unique up to equivalence.*

Proof. By Theorem 15 of [**17**], we know that for each π in $\hat{G}$ there is a unitary eigenmatrix M_π in $\mathscr{M} \otimes \mathrm{End}\ V_\pi$ such that $\alpha_g(M_\pi) = M_\pi \pi(g)$. Bearing in mind the Plancherel decomposition

$$\mathscr{R}(G) \cong \bigoplus_{\pi \in \hat{G}} \mathrm{End}\ V_\pi,$$

we may 'patch' the eigenmatrices M_π together to form a unitary $M = (M_\pi)$ in $\mathscr{M} \otimes \mathscr{R}(G)$ such that

$$\alpha_g(M) = M\rho(g).$$

If we now let $U = M^*$, it is easily verified that $\iota \otimes \delta_G(U^*)(U \otimes I)\iota \otimes \sigma(U \otimes I)$ is fixed by α_g, so lies in $\mathscr{R}(G) \otimes \mathscr{R}(G)$. It now follows from Lemma 1 (2) that U is the required equivariant cocycle representation.

To establish the uniqueness of ω, we observe that a unitary U in $\mathscr{M} \otimes \mathscr{R}(G)$ satisfying $\alpha_g(U) = \rho(g)^{-1}U$ is unique up to right multiplication by a unitary v in $\mathscr{R}(G)$; and on taking Uv in place of U, ω is replaced by $\delta_G(v^*)\omega(v \otimes v)$.

LEMMA 6. *Suppose that $\alpha:G \to \mathrm{Aut}(\mathscr{M})$ is an ergodic action and that the action $\alpha_g \otimes \iota$ on $\mathscr{M} \otimes \mathscr{B}(\mathscr{H})$ is dual. Then*

(1) *the action $\alpha \otimes \iota$ is dual to a coaction on $\mathscr{B}(\mathscr{H})$ implemented by a $\sigma\omega$-representation for some cocycle ω of $\hat{G}$.*

(2) *$\mathscr{M} \cong \pi_\omega(\hat{G})''$ as a G-algebra.*

Proof. Since the action on $\mathscr{M}$ is ergodic and stably dual, it follows from Theorem 15 of [**17**] that it is of full multiplicity. So by Lemma 5 we can find an equivariant ω-representation U of $\hat{G}$ in $\mathscr{M}$ where ω is unique up to equivalence. Since $\alpha \otimes \iota$ is dual, Theorem B (b) implies that there is an ordinary equivariant representation V of $\hat{G}$ in $\mathscr{M}$ and the coaction δ on $\mathscr{B}(\mathscr{H})$ is implemented by V^*,

$$\delta(x) = V^*(x \otimes I)V.$$

But then $W = V^*U$ is fixed by α and therefore lies in $\mathscr{B}(\mathscr{H}) \otimes \mathscr{R}(G)$. Thus $V = UW^*$ where U and W are cocycle representations in the commuting algebras $\mathscr{M}$ and $\mathscr{B}(\mathscr{H})$ respectively. Hence if $x \in \mathscr{B}(\mathscr{H})$, we have

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$$\delta(x) = V^*(x \otimes I)V = WU^*(x \otimes I)UW^* = W(x \otimes I)W^*$$

so that δ is implemented by W. Moreover since V is an ordinary representation of $\hat{G}$ and U is an ω-representation, we have the relations

$$\iota \otimes \delta_G(V) = (V \otimes I)\iota \otimes \sigma(V \otimes I),$$

$$\iota \otimes \delta_G(U^*)(U \otimes I)\iota \otimes \sigma(U \otimes I) = I \otimes \omega.$$

Substituting $V = UW^*$ in the first relation, we obtain

$$\iota \otimes \delta_G(U)\iota \otimes \delta_G(W^*) = (U \otimes I)\iota \otimes \sigma(U \otimes I)\iota \otimes \sigma(W^* \otimes I)(W^* \otimes I)$$

and the second relation shows that

$$\iota \otimes \delta_G(W^*)(W \otimes I)\iota \otimes \sigma(W \otimes I) = I \otimes \sigma\omega$$

as required.

Proof of Theorem 2. The first assertion is an immediate consequence of Lemma 4, while the second follows from Lemmas 5 and 6, bearing in mind that a full multiplicity ergodic action is stably dual by Theorem 15 of [**17**].

We remark that stable duality in Lemma 6 could be checked directly in Theorem 2 without appealing to Theorem 15 of [**17**]. Indeed taking the action $\alpha \otimes \iota$ on $\mathcal{M} \otimes \mathcal{B}(L^2(G))$ we see that the product $UW^*_{\sigma\omega}$ (formed in the obvious way) defines an equivariant ordinary representation of $\hat{G}$ in $\mathcal{M} \otimes \mathcal{B}(L^2(G))$. Finally we note the following interesting consequence of Theorem 2, which begs an obvious question.

COROLLARY OF THEOREM 2. *For any cocycle ω, the von Neumann algebra generated by $\pi_\omega(\hat{G})''$ and $\lambda(G)$ is $\mathcal{B}(L^2(G))$. Furthermore the von Neumann algebra generated by $\pi_\omega(\hat{G})'$ and $\lambda(G)$ is $\mathcal{B}(L^2(G))$ and* Ad λ *defines an ergodic action on $\pi_\omega(\hat{G})'$.*

Proof. Let $\mathcal{L}(G) = \lambda(G)''$ be the von Neumann algebra generated by the left regular representation of G in $\mathcal{B}(L^2(G))$. Thus $\mathcal{L}(G)$ and $\mathcal{R}(G)$ are each other's commutants in $\mathcal{B}(L^2(G))$. Since Ad λ implements the action of G on $\pi_\omega(\hat{G})''$, the von Neumann algebra $\mathcal{M} = \pi_\omega(\hat{G})'' \vee \mathcal{L}(G)$ is a homomorphic image of the crossed product $\pi_\omega(\hat{G})'' \rtimes G$ and therefore is a Type I factor. So its commutant $\mathcal{M}' = \pi_\omega(\hat{G})' \cap \mathcal{R}(G)$ is also a Type I factor. But

$$\mathcal{R}(G) \cong \bigoplus_{p \in \hat{G}} \operatorname{End}(V_\pi),$$

so that $\dim(\mathcal{M}') \leqq (\dim \pi)^2$ for all $\pi \in \hat{G}$. In particular taking the trivial representation of G we conclude that $\mathcal{M}' = \mathbf{C}$. Thus $\mathcal{M} = \mathcal{B}(L^2(G))$ as required. We note that the invariance of $\pi_\omega(\hat{G})''$ under Ad λ forces the invariance of its commutant; but then

$$(\pi_\omega(\hat{G})')^{\mathrm{Ad}\lambda} = \pi_\omega(\hat{G})' \cap \mathcal{R}(G) = \mathbf{C},$$

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so that the action is indeed ergodic. Finally

$$(\pi_\omega(\hat{G})' \vee \mathscr{L}(G))' = \pi_\omega(\hat{G})'' \cap \mathscr{R}(G) = \mathbf{C}$$

so that $\pi_\omega(\hat{G}) \vee \mathscr{L}(G) = \mathscr{B}(L^2(G))$ as required.

Since $\pi_\omega(\hat{G})''$ reduces to $L^\infty(G)$ when ω is trivial, the first statement above may be regarded as a generalisation of the Stone-von Neumann Theorem. The corollary implies some kind of symmetry between $\pi_\omega(\hat{G})''$ and its commutant: this will be fully explained by the duality theorem (Theorem 9) of Section 7 which identifies the commutant of $\pi_\omega(\hat{G})''$ with $\pi_{\sigma\omega}(\hat{G})''$. In order to prove this result we will need the L^1 algebras of Section 7 combined with the theory of Hilbert algebras [**3**]. (Even in the case of the trivial cocycle where one has to establish the self-commutation property $L^\infty(G)' = L^\infty(G)$, one still needs an application, albeit simple, of this theory.)

5. The implementation theorem. The circle of ideas encountered in the above proof can be recycled to give a proof of the following result.

THEOREM 3. *Any coaction of a compact group on a Type* I *factor is implemented by a cocycle representation of the group dual.*

Proof. Let $\mathscr{M}_1 = \mathscr{B}(\mathscr{H}) \rtimes_\delta \hat{G}$ with dual action $\hat{\delta}$ and let $\mathscr{M}$ be the relative commutant of $\delta(\mathscr{B}(\mathscr{H}))$ in $\mathscr{M}_1$. Thus $\mathscr{M}_1 = \mathscr{M} \otimes \delta(\mathscr{B}(\mathscr{H}))$ and, since the fixed point algebra of $\hat{\delta}$ is precisely $\delta(\mathscr{B}(\mathscr{H}))$, $\alpha = \hat{\delta}|_{\mathscr{M}}$ is an ergodic action on $\mathscr{M}$ and $\hat{\delta} = \alpha \otimes \iota$. Thus the hypotheses of Lemma 6 are fulfilled and the proof of Theorem 3 follows.

We observe that if the factor in the statement of Theorem 3 is Type I_n with n finite, then the result is true for more trivial reasons since we may use the Plancherel decomposition

$$\mathscr{R}(G) \cong \bigoplus_{\pi \in \hat{G}} \mathrm{End}(V_\pi)$$

to break up a coaction $\delta: M_n(\mathbf{C}) \to M_n(\mathbf{C}) \otimes \mathscr{R}(G)$ into unital *-isomorphisms

$$\delta_\pi: M_n(\mathbf{C}) \to \mathrm{End}(V_\pi),$$

each of which is evidently unitarily implemented.

6. Normalisation of cocycles and cocycle representations. In preparation for the subsequent sections it will be necessary to do some preliminary work on the normalisation of cocycles and cocycle representations. We shall start by briefly reminding the reader of what this amounts to in the Abelian case. If G is Abelian, then a cocycle ω is said to be *normalised* provided that

$$\omega(\xi, \xi^{-1}) = 1 \quad \text{for all } \xi \in \hat{G}$$

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and an ω-representation U_ξ is said to be *normalised* provided that

$$U_\xi^{-1} = U_{\xi^{-1}} \quad \text{for all } \xi \in \hat{G} \text{ and } U_1 = I.$$

It is a straightforward exercise to show that any cocycle is cohomologous to a normalised cocycle and that a cocycle is normalised if and only if its regular representation is normalised, or equivalently any representation corresponding to that cocycle is normalised. Such a representation satisfies

$$U_\xi U_\eta = U_{\xi\eta}\omega(\xi, \eta)$$

so that we have the relations

$$(*) \quad \omega(1, 1)\omega(\xi^{-1}, \xi) = U_{\xi^{-1}}U_\xi \quad \text{and} \quad \omega(1, 1)I = U_1.$$

Now it follows trivially from the cocycle identity that for all ξ one has

$$\omega(\xi, 1) = \omega(1, 1) = \omega(1, \xi), \quad \omega(\xi, \xi^{-1}) = \omega(\xi^{-1}, \xi).$$

Thus when ω is normalised the constant $\omega(1, 1)$ is equal to 1. Another consequence of normalisation is that ω is *alternating* in that

$$\overline{\omega(\xi, \eta)} = \omega(\eta^{-1}, \xi^{-1}).$$

We finally remark that it is almost invariably easier to understand conditions on cocycles by passing from their inhomogeneous form to the corresponding homogeneous cocycle

$$f(a, b, c) = \omega(a^{-1}b, b^{-1}c).$$

The normalisation conditions simply say that if a and c are equal, then f takes the value 1.

In the general non-commutative case, there is no immediate way of defining an analogue of "$\omega(\xi, \xi^{-1})$," since there is no homomorphism of $\mathscr{R}(G) \otimes \mathscr{R}(G)$ into $\mathscr{R}(G)$ corresponding to evaluation on the diagonal in the Abelian case. There are, however, two ways out of this difficulty: the first is to use the analogue of equation (*) to give the definition; the second is to use left multiplication by the projection $\delta_G(e_1)$ (where e_1 is the central projection in $\mathscr{R}(G)$ corresponding to the trivial representation) as a substitute for evaluating ω at (ξ, ξ^{-1}). In fact we shall resort to both these devices in our discussion below, which is unfortunately not nearly as straightforward as in the Abelian case.

After this preliminary discussion, we are ready to make our basic definitions. A cocycle ω is said to be *normalised* if and only if $\delta_G(e_1)\omega = \delta_G(e_1)$ and an ω-representation W is said to be *normalised* if and only if

$$\alpha \otimes \iota(W) = W^* \quad \text{and} \quad W(I \otimes e_1) = I \otimes e_1.$$

It is a trivial exercise to verify that the regular (ordinary) representation W_G is normalised and it is also easily verified that if ω is normalised then it commutes with $\delta_G(e_1)$.

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THEOREM 4. *If ω is a cocycle, then the following conditions are equivalent*:

(1) ω *is normalised*

(2) W_ω *is normalised*

(3) *any* (*and hence every*) ω-*representation is normalised.*

The proof of Theorem 4 will be accomplished through a series of lemmas.

LEMMA 7. *Let $e_1 \in \mathscr{R}(G)$ be the central projection corresponding to the trivial representation of G, and let ω be a cocycle. Then*

$$(I \otimes e_1)\omega = \lambda(I \otimes e_1), \quad (e_1 \otimes I)\omega = \lambda(e_1 \otimes I)$$

for some complex number $\lambda = \lambda(\omega)$ of modulus 1.

Proof. Since e_1 is minimal and central in $\mathscr{R}(G)$, we have $\omega(I \otimes e_1) = v_1 \otimes e_1$ for a unitary v_1 in $\mathscr{R}(G)$. We multiply the cocycle identity

$$(\delta_G \otimes \iota(\omega))(\omega \otimes I) = (\iota \otimes \delta_G(\omega))(I \otimes \omega)$$

by $I \otimes e_1 \otimes e_1$ and after a little manipulation obtain

$$(\delta_G \otimes \iota(v_1 \otimes e_1))(v_1 \otimes e_1 \otimes e_1) = (\iota \otimes \delta_G(v_1 \otimes e_1))(I \otimes e_1 v_1 \otimes e_1)$$

so that (on cancelling $v_1 \otimes I \otimes I$ and recalling that $\delta_G(x)(I \otimes e_1) = x \otimes e_1$) we obtain

$$v_1 \otimes e_1 = \delta_G(v_1)(I \otimes e_1) = I \otimes (v_1 e_1).$$

From this we deduce that $\omega(I \otimes e_1)$ has the stated form. A similar argument can be applied to $\omega(e_1 \otimes I)$, or one can simply apply σ to the corresponding relation for $\sigma\omega$. Thus we have

$$\omega(I \otimes e_1) = \lambda(I \otimes e_1), \quad \omega(e_1 \otimes I) = \lambda'(E_1 \otimes I).$$

Multiplying both these expressions by $e_1 \otimes e_1$, we find that $\lambda = \lambda'$ as desired.

So far we have two "normalisable" quantities associated with ω, namely $\lambda(\omega)$ and $x(\omega) = \delta_G(e_1)\omega$. We next introduce a third quantity $u(\omega)$ such that the pair (u, λ) determines and is determined by x.

LEMMA 8. *Let ω be any cocycle. Then*

$$\iota \otimes (W_\omega)W_\omega = I \otimes u,$$

where $u = u(\omega)$ is an α-invariant unitary in $\mathscr{R}(G)$, and

$$W_\omega(I \otimes e_1) = \lambda(\omega) \otimes e_1.$$

Proof. We recall that W_G lies in $L^\infty(G) \otimes \mathscr{R}(G)$ and satisfies

$$\beta_g(W_G) = I \otimes \rho(g)^{-1} W_G$$

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where β_g denotes the action by left translation on $L^\infty(G)$, implemented by Ad $\lambda(g) \otimes \iota$ where λ is the left regular representation of G on $L^2(G)$. From this relation we obtain

$$\beta_g(W_\omega) = I \otimes \rho(g)^{-1} W_\omega.$$

Let us now consider $\iota \otimes \alpha(W_\omega)W_\omega$. This is a unitary in $\mathscr{B}(L^2(G)) \otimes \mathscr{R}(G)$ fixed by the action β. On the other hand we know from Theorem 2 that W_ω, and hence this unitary, lies in the algebra $\mathscr{M} \otimes \mathscr{R}(G)$ where $\mathscr{M} = \pi_\omega(\hat{G})''$. Moreover β restricts to an ergodic action on $\mathscr{M}$. Hence $\iota \otimes \alpha(W_\omega)W_\omega$ lies in $\mathbf{C} \otimes \mathscr{R}(G)$:

$$\iota \otimes \alpha(W_\omega)W_\omega = I \otimes u$$

where u is a unitary in $\mathscr{R}(G)$. The defining equation for u immediately implies that $\alpha u = u$. The last assertion of Lemma 8 follows from Lemma 7 and the easily verified identity $W_G(I \otimes e_1) = I \otimes e_1$.

Our next task is to find equations relating $x(\omega)$, $\lambda(\omega)$, and $u(\omega)$. Prior to doing so, we state some properties of the map $a \mapsto \delta_G(e_1)a$ on $\mathscr{R}(G) \otimes \mathscr{R}(G)$. To prove these, we shall need to use some properties of the Fourier algebra of G which can be found in [**12**]. For the benefit of the reader we include a brief outline in our special case. By definition the Fourier algebra $A(G)$ of G is the unital Banach *-algebra whose underlying Banach space is the predual $\mathscr{R}(G)_*$ of $\mathscr{R}(G)$ with multiplication given by

$$\langle \phi \circ \psi, x \rangle = \langle \phi \otimes \psi, \delta_G(x) \rangle$$

and involution given by $\phi^\dagger = \alpha\phi^*$. (If $\mathscr{M}$ is a von Neumann algebra, the notations $\mathscr{M}_*$, $\mathscr{M}^u$, and $\mathscr{M}^\natural$ will be used to denote the predual, unitary group and centre of $\mathscr{M}$ respectively. We recall that $\mathscr{M}_*$ is naturally a bimodule for $\mathscr{M}$, with the bimodule structure specified by $\langle a \cdot \phi \cdot b, x \rangle = \langle \phi, bxa \rangle$ for $a, b, x \in \mathscr{M}$ and $\phi \in \mathscr{M}_*$.) There is a faithful *-isomorphism of $A(G)$ into the Abelian C^* algebra $C(G)$ given by $\phi \mapsto \tilde{\phi}$ where $\tilde{\phi}(g) = \phi(\rho(g))$. Under this identification, it is clear that

$$\phi(e_1) = \int \tilde{\phi}(g)dg$$

and hence we obtain the formula

$$\phi \circ \psi(e_1) = \langle \phi \otimes \psi, \delta_G(e_1) \rangle = \int \tilde{\phi}(g)\tilde{\psi}(g)dg.$$

(We shall give a generalisation of this formula in the next section.) Finally we note that, by Theorem A.1 (b) of [**12**] or Theorem 8 in the next section, ordinary representations W of $\hat{G}$ correspond to *-representations π of $A(G)$ via the formula

$$\pi(\phi) = (\mathrm{id} \otimes \phi)W.$$

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LEMMA 9.

(1) $\delta_G(e_1)(I \otimes x) = \delta_G(e_1)(\alpha x \otimes I)$.

(2) $\delta_G(e_1)(I \otimes x) = 0$ *if and only if* $x = 0$.

Thus for each a *in* $\mathscr{R}(G) \otimes \mathscr{R}(G)$ *there is a unique* x *in* $\mathscr{R}(G)$ *such that*

$$\delta_G(e_1)a = \delta_G(e_1)(I \otimes x).$$

Proof. Since $\delta_G(e_1) = \delta_G(e_1 g) = \delta_G(e_1)\rho(g) \otimes \rho(g)$, we find that the identity in (1) is satisfied when $x = \rho(g)$. It follows in general by linearity and continuity.

To prove (2), we note that if $\delta_G(e_1)(I \otimes x) = 0$, then for all ϕ, ψ in $\mathscr{R}(G)_*$, we have

$$\langle \delta_G(e_1)(I \otimes x), \phi \otimes \psi \rangle = 0.$$

Thus

$$\int \phi(g)(x \cdot \psi)(g) dg = 0.$$

Hence $x \cdot \psi = 0$, and hence $x = 0$ as required.

LEMMA 10.

(1) $x(\omega)(I \otimes e_1) = \lambda(\omega) e_1 \otimes e_1$.

(2) $\delta_G(e_1)(I \otimes u(\omega)) = \lambda(\omega) x(\omega)$.

Thus the pair $(u(\omega), \lambda(\omega))$ *uniquely determines and is uniquely determined by* $x(\omega)$, *and* ω *is normalised if and only if* $u(\omega) = I$ *and* $\lambda(\omega) = 1$.

Proof. (1) follows immediately from Lemma 7 and the identity $\delta_G(e_1)(I \otimes e_1) = e_1 \otimes e_1$. To prove (2), we note that

$$\iota \otimes \alpha(W_G \omega) W_G = (I \otimes u)\omega^*$$

so that multiplying both sides on the right by $\delta_G(e_1) = W_G^*(e_1 \otimes I) W_G$, we obtain

$$\iota \otimes \alpha(W_G \omega)(e_1 \otimes I) W_G = (I \otimes u) x(\omega)^*.$$

So using Lemma 7 and the fact that W_G is normalised, we may rewrite this equation as

$$\lambda(\omega) W_G^*(e_1 \otimes I) W_G = (I \otimes u) x(\omega)^*,$$

which clearly implies (2).

In order to extend Lemma 8 to any ω-representation, we shall use a generalisation to the non-Abelian setting of a well-known cohomological device (cf. **[10]**). In the Abelian case one knows that if π is the regular representation of $\hat{G}$, π_ω the regular ω-representation, and μ any other ω-representation of $\hat{G}$, then

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(*) $\mu \otimes \pi \cong \dim \mu \cdot \pi_\omega$.

We now describe how to define the tensor product of an ordinary representation U of $\hat{G}$ and an ω-representation W of $\hat{G}$; cf. pages 132-134 of [**12**]. In fact if $U \in \mathcal{M}_1 \otimes \mathcal{R}(G)$ and $W \in \mathcal{M}_2 \otimes R$, then, with an obvious notation,

$$V = U_{13} \cdot W_{23} \in \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{R}(G)$$

is an ω-representation of $\hat{G}$ in $\mathcal{M}_1 \otimes \mathcal{M}_2$ which we shall denote by $U \circledast W$. (Note that already in the proof of Theorem 2 we used a somewhat related construction; namely that if W_1 was an ω-representation and W_2 a $\sigma\omega$-representation, then $V = W_1 \circledast W_2^*$ was an ordinary representation of $\hat{G}$.) Likewise we can define $W \circledast U$ in an obvious way. The following result is a non-commutative analogue of (*).

LEMMA 11. *If W is an ω-representation on $\mathcal{H}$ and W_G is the regular representation on $L^2(G)$, then*

$$W_G \circledast W \cong W_\omega \circledast U_0,$$

where $U_0 = I$ is the trivial representation on $\mathcal{H}$.

Proof. To prove that the two representations are unitarily equivalent, we must produce a unitary y in $\mathcal{B}(L^2(G)) \otimes \mathcal{B}(\mathcal{H})$ such that

$$(y \otimes I)(W_G)_{13} W_{23} = (W_G)_{13} \omega_{31} (y \otimes I).$$

In fact we claim that

$$y = W_{21}^* \in \mathcal{R}(G) \otimes \mathcal{B}(\mathcal{H}) \subseteq \mathcal{B}(L^2(G)) \otimes \mathcal{B}(\mathcal{H})$$

does the trick. This is an immediate consequence of the equation

$$\iota \otimes \delta_G(W^*)(W \otimes I)\iota \otimes \sigma(W \otimes I) = I \otimes \omega$$

and the fact that $\delta_G(x) = W_G^*(x \otimes I)W_G$.

LEMMA 12. *If W is an ω-representation of $\hat{G}$, then*

$$\iota \otimes \alpha(W)W = I \otimes u(\omega) \quad \text{and} \quad W(I \otimes e_1) = \lambda(\omega) \otimes e_1.$$

Proof. Let W be an ω-representation of $\hat{G}$ and consider $V = W_G \circledast W = (W_G)_{13} W_{23}$. Then since W_G is normalised, we obtain

$$\iota \otimes \alpha(V)V = \iota \otimes \alpha(W_{23})\iota \otimes \alpha((W_G)_{13})(W_G)_{13} W_{23} = \iota \otimes \alpha(W_{23}) W_{23}$$

and

$$V(I \otimes e_1) = (W_G)_{13}(I \otimes e_1) W_{23}(I \otimes e_1) = W(I \otimes e_1).$$

On the other hand we know from Lemma 11 that

$$V = (y \otimes I)(W_\omega)_{13}(y^* \otimes I) \quad \text{with } y = W_{21}^*.$$

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This expression for V leads to

$$\iota \otimes \alpha(V)V = (y \otimes I)(\iota \otimes \alpha(W_\omega)_{13}(W_\omega)_{13})(y^* \otimes I)$$
$$= (y \otimes I)(I \otimes u(\omega))(y^* \otimes I) = I \otimes I \otimes u(\omega)$$

and

$$V(I \otimes e_1) = (y \otimes I)(W_\omega)_{13}(I \otimes e_1)(y^* \otimes I)$$
$$= (y \otimes I)(\lambda(\omega) \otimes e_1)(y^* \otimes I) = \lambda(\omega) \otimes e_1.$$

Thus we obtain $\iota \otimes \alpha(W)W = I \otimes u(\omega)$ and $W(I \otimes e_1) = \lambda(\omega) \otimes e_1$ as required.

In particular this lemma implies that any ordinary representation is normalised. This is also a consequence of Theorem A.1 (b) of [**12**]; for if W is an ordinary representation of $\hat{G}$, then they show that $\phi \mapsto (\mathrm{id} \otimes \phi)W$ defines a *-representation of the Fourier algebra $\mathscr{R}(G)_*$. Since the involution on the Fourier algebra is prescribed by $\phi \mapsto \alpha\phi^*$, this implies that $\iota \otimes \alpha(W) = W^*$.

Proof of Theorem 4. The equivalence of conditions (1), (2) and (3) follows immediately from Lemmas 8, 10 and 12.

We already know that given a cocycle ω, $\sigma\omega$ is also a cocycle; moreover, since $\sigma^2 = \iota$, $\omega \mapsto \sigma\omega$ is an involution. Now if α is the *-antiautomorphism of $\mathscr{R}(G)$ extending inversion on G, we shall write α instead of $\alpha \otimes \ldots \otimes \alpha$ on $\mathscr{R}(G) \otimes \ldots \otimes \mathscr{R}(G)$. Using α we can define another involution on cocycles, namely $\omega \mapsto \alpha\omega^*$. It is an easy exercise to check that $\alpha\omega^*$ does indeed satisfy the cocycle identity; this also follows from the following stronger result.

LEMMA 13. *$\alpha\sigma\omega^* = \delta_G(v^*)\omega(v \otimes v)$ where $v = u(\omega)^*$.*

Proof. We have with $W = W_\omega$

$$I \otimes \omega = \iota \otimes \delta_G(W^*)(W \otimes I)\iota \otimes \sigma(W \otimes I)$$

so that $\iota \otimes \delta_G(W)I \otimes \omega = W_{12}W_{13}$. Applying $\iota \otimes \sigma\alpha$ to both sides yields

$$(I \otimes \alpha\sigma\omega)\iota \otimes \delta_G(\iota \otimes \alpha(W)) = \iota \otimes \alpha(W_{13}W_{12})$$
$$= \iota \otimes \alpha(W_{13})\iota \otimes \alpha(W_{12}).$$

(Note that the order need only be reversed on the left hand side.) But

$$\iota \otimes \alpha(W) = (I \otimes u)W^*,$$

so we get

$$(I \otimes \alpha\sigma\omega)(I \otimes \delta_G(u))(\iota \otimes \delta_G(W)) = (I \otimes u \otimes u)W_{13}^*W_{12}^*.$$

So finally $\alpha\sigma\omega^* = \delta_G(u)\omega(u^* \otimes u^*)$ as required.

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We are now in a position to state the remaining results to be proved on normalisation of cocycles. Prior to this we introduce the notation ω^v as a shorthand for $\delta_G(v^*)\omega(v \otimes v)$.

THEOREM 5. *Every cocycle is equivalent to a normalised cocycle.*

THEOREM 6. *Let ω be a normalised cocycle. Then*
(1) $\sigma\omega = \alpha\omega^*$.
(2) $\omega(e_1 \otimes I) = e_1 \otimes I, \quad \omega(I \otimes e_1) = I \otimes e_1$.
(3) *ω^v is normalised if and only if v satisfies $\alpha v = v^*$ and $ve_1 = e_1$: such a $v \in \mathcal{R}(G)$ will be called normalised.*

The proofs of both these theorems hinge on the following simple lemma.

LEMMA 14. *Let χ_0 denote the trivial character of G extended to $\mathcal{R}(G)$, so that $ae_1 = \chi_0(a)e_1$. Then if ω is a cocycle of $\hat{G}$, we have*

(i) $x(\omega^v) = \overline{\chi_0(v)}x(\omega)(v \otimes v)$.

(ii) $W_{\omega^v} = (v^* \otimes I)W_\omega(v \otimes v)$.

(iii) $u(\omega^v) = (\alpha v)u(\omega)v$.

(iv) $\lambda(\omega^v) = \chi_0(v)\lambda(\omega)$.

Proof.

(i) $x(\omega) = \delta_G(e_1 v^*)\omega(v \otimes v) = \chi_0(v^*)x(\omega)(v \otimes v)$.

(ii) $W_{\omega^v} = W_G\delta_G(v^*)\omega(v \otimes v) = (v^* \otimes I)W_G\omega(v \otimes v)$
$= (v^* \otimes I)W_\omega(v \otimes v)$.

(iii)
$$\begin{aligned} I \otimes u(\omega^v) &= \iota \otimes \alpha(W_\omega)W_\omega \\ &= \iota \otimes \alpha(\,(v^* \otimes I)W_\omega(v \otimes v)\,)(v^* \otimes I)W_\omega(v \otimes v) \\ &= (v^* \otimes \alpha v)(\iota \otimes \alpha(W_\omega)W_\omega)(v \otimes v) = I \otimes (\alpha v)u(\omega)v. \end{aligned}$$

(iv)
$$\begin{aligned} \lambda(\omega^v) \otimes e_1 &= (I \otimes e_1)\delta_G(v^*)\omega(v \otimes v) = (v^* \otimes I)(I \otimes e_1)\omega(v \otimes v) \\ &= \lambda(\omega) \otimes \chi_0(v)e_1. \end{aligned}$$

Proof of Theorem 5. To produce the normalisation of ω we shall first arrange that λ becomes 1, and then that u becomes I. Indeed by Lemma 14, if $\zeta = \overline{\lambda(\omega)}$ then $\lambda(\omega^\zeta) = 1$. Now we choose v with $\chi_0(v) = 1$, $\alpha v = v$ such that $v^2 = u(\omega^\zeta)$. This is possible provided that $\chi_0(u(\omega^\zeta)\,) = 1$, since we can always take a unitary square root in $\mathcal{R}(G)^\alpha$ and then adjust on the e_1 component to arrange that $\chi_0(v) = 1$. But from Lemma 14 applied to ω^ζ, we have

$$\delta_G(e_1)(I \otimes u(\omega^\zeta)\,) = x(\omega^\zeta),$$

so that on multiplying both sides by $I \otimes e_1$, we obtain

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$$e_1 \otimes \chi_0(u(\omega^\zeta))e_1 = e_1 \otimes e_1.$$

Thus $\chi_0(u(\omega^\zeta)) = 1$ as required.

It is now easy to see that ω^{v_1} is normalised where v_1 is the α-invariant unitary $v\zeta$.

Proof of Theorem 6. (1), (2) and (3) follow from Lemmas 13, 7 and 14 respectively.

In our future discussion we shall restrict our attention to normalised cocycles and cocycle representations. We define the *centraliser* of a normalised cocycle by

$$C(\omega) = \{u \in \mathscr{R}(G)^u : \omega^u = \omega\}.$$

Note that by (iii) of Theorem 6, $C(\omega)$ is a subgroup of

$$\mathscr{G} = \{u \in \mathscr{R}(G)^u : \chi_0(u) = 1, \alpha u = u^*\},$$

a group containing $\rho(G)$. In fact the group $\mathscr{G}$ is quite large. For example if h is self-adjoint with $\alpha h = -h$, then $\exp(ih)$ is in $\mathscr{G}$. Also one has a sort of "polar decomposition". Let u be any unitary in $\mathscr{R}(G)$ with $\chi_0(u) = 1$. Pick an α-invariant unitary square root of $(\alpha u)u$ with $\chi_0(v) = 1$, and set $w = uv^{-1}$. Then it is easily verified that $u = vw$ with w in $\mathscr{G}$ and v normalised. In Section 8 we will obtain another interpretation of the centraliser in terms of one-dimensional representations of a certain algebra associated with ω.

7. The L^1 and C^* algebras for ω-representations of $\hat{G}$. Just as in the Abelian case $C^*_\omega(\hat{G})$ is defined as the enveloping algebra of $L^1_\omega(\hat{G})$, the algebra of integrable functions on $\hat{G}$ with multiplication given by convolution "twisted" by the cocycle ω, so in the general case $C^*_\omega(\hat{G})$ can be defined as the enveloping C^* algebra of the Fourier algebra $\mathscr{R}(G)_*$ with multiplication perturbed by ω. (We have already discussed the case of the trivial cocycle in the last section where we obtained a correspondence between ordinary representations and *-representations of the Fourier algebra $A(G)$.)

We are now ready for our basic definitions. We define $L^1_\omega(\hat{G})$ to be $\mathscr{R}(G)_*$, the predual of $\mathscr{R}(G)$, with its usual Banach space structure but with multiplication given by

$$\langle \phi \circ \psi, x\rangle = \langle \phi \otimes \psi, \delta_G(x)\omega\rangle \quad (\phi, \psi \in \mathscr{R}(G)_*, x \in \mathscr{R}(G))$$

and its usual (Fourier algebra) involution $\phi^+ = \alpha\phi^*$.

THEOREM 7. *$L^1_\omega(\hat{G})$ is a unital Banach *-algebra.*

Proof. (a) $L^1_\omega(\hat{G})$ is an algebra. The only non-trivial thing to check here is associativity. In fact, say $\phi, \psi, \theta \in L^1_\omega(\hat{G})$. Then we have for $x \in \mathscr{R}(G)$

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$$\begin{aligned}\langle (\phi \circ \psi) \circ \theta, x \rangle &= \langle (\phi \circ \psi) \otimes \theta, \delta_G(x)\omega \rangle \\ &= \langle \phi \otimes \psi \otimes \theta, (\delta_G \otimes \iota(\delta_G(x)\omega))(\omega \otimes I) \rangle \\ &= \langle \phi \otimes \psi \otimes \theta, (\delta_G \otimes \iota)(\delta_G(x))(\delta_G \otimes \iota(\omega))(\omega \otimes I) \rangle \\ &= \langle \phi \otimes \psi \otimes \theta, (\iota \otimes \delta_G)(\delta_G(x))(\iota \otimes \delta_G(\omega))(I \otimes \omega) \rangle \\ &= \langle \phi \otimes (\psi \circ \theta), \delta_G(x)\omega \rangle \\ &= \langle \phi \circ (\psi \circ \theta), x \rangle.\end{aligned}$$

(b) $L^1_\omega(\hat{G})$ is a Banach algebra. We must check that the norm is sub-multiplicative, that is

$$\|\phi \circ \psi\| \leqq \|\phi\| \, \|\psi\| \quad \text{for } \phi, \psi \in L^1_\omega(\hat{G}).$$

But

$$\begin{aligned}\|\phi \circ \psi\| &= \sup_{\|x\| \leqq 1} |\langle \phi \circ \psi, x \rangle| = \sup_{\|x\| \leqq 1} |\langle \phi \otimes \psi, \delta_G(x)\omega \rangle| \\ &\leqq \sup_{\|x\| \leqq 1} \|\phi \otimes \psi\| \, \|\delta_G(x)\omega\| = \|\phi\| \, \|\psi\|\end{aligned}$$

since δ_G is a *-isomorphism and ω is a unitary.

(c) $\phi \mapsto \phi^+$ is an involution on the algebra $L^1_\omega(\hat{G})$. It suffices to verify that

$$(\phi \circ \psi)^+ = \psi^+ \circ \phi^+.$$

But from the definitions we have on the one hand

$$\langle (\phi \circ \psi)^+, x \rangle = \overline{\langle \phi \otimes \psi, \delta_G(\alpha x^*)\omega \rangle}$$

while on the other hand

$$\langle \psi^+ \circ \phi^+, x \rangle = \overline{\langle \phi \otimes \psi, \delta_G(ax^*)\sigma\alpha\omega^* \rangle}.$$

Since ω is normalised, Theorem 6 (1) implies that $\sigma\alpha\omega^* = \omega$, so we obtain

$$\langle (\phi \circ \psi)^+, x \rangle = \langle \psi^+ \circ \phi^+, x \rangle$$

as claimed.

(d) The unit ϵ of the Fourier algebra is a unit for $L^1_\omega(\hat{G})$ too. In fact, the unit ϵ of $A(G)$ is specified by

$$xe_1 = \langle \epsilon, x \rangle e_1 \quad (x \in \mathscr{R}(G))$$

so that in the notation of the preceding section $\epsilon = \chi_0$. But then if $x \in \mathscr{R}(G)$, we have

$$\begin{aligned}\langle \epsilon \circ \phi, x \rangle &= \langle \epsilon \otimes \phi, \delta_G(x)\omega \rangle \\ &= \langle \epsilon \otimes \phi, \delta_G(x)\omega(e_1 \otimes I) \rangle\end{aligned}$$

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$$= \langle \epsilon \otimes \phi, \delta_G(x)(e_1 \otimes I) \rangle$$
$$= \langle \epsilon \otimes \phi, e_1 \otimes x \rangle$$
$$= \langle \phi, x \rangle$$

where we have used Theorem 6 (2). Hence ϵ is a self-adjoint left unit and hence a unit.

It should be remarked that if ω^v is a normalised cocycle cohomologous to ω, then the map $\phi \mapsto v^*\phi$ defines an isometric *-isomorphism of $L^1_\omega(\hat{G})$ into $L^1_{\omega^v}(\hat{G})$. We define $C^*_\omega(\hat{G})$ to be the enveloping C^* algebra of $L^1_\omega(\hat{G})$ in the sense of Dixmier [**4**]. Before establishing that $C^*_\omega(\hat{G})$ has the usual properties well-known in the Abelian case (see [**10**] and [**6**]), we must study the link between *-representations of $L^1_\omega(\hat{G})$ and ω-representations of $\hat{G}$.

THEOREM 8. *If ω is a normalised cocycle of $\hat{G}$, then the formula*

$$(*) \qquad \langle \pi(\phi), \xi \rangle = \langle W, \xi \otimes \phi \rangle \quad (\phi \in L^1_\omega(\hat{G}), \xi \in \mathscr{B}(\mathscr{H})_*)$$

*establishes a one-one correspondence between unital *-representations of $L^1_\omega(\hat{G})$ and (normalised) ω-representations of $\hat{G}$ in $\mathscr{H}$.*

The proof of this theorem will be achieved in three steps. In the first, we show in a straightforward manner how a *-representation of $L^1_\omega(\hat{G})$ is determined by an ω-representation of $\hat{G}$. The second step establishes some general facts about the action of G on $L^1_\omega(\hat{G})$ and the link (via the GNS construction) between the regular ω-representation and the canonical trace on $L^1_\omega(\hat{G})$. Finally we use this information to show that every unital *-representation of $L^1_\omega(\hat{G})$ arises from an ω-representation of $\hat{G}$ as in (*). The last part of this programme is inspired by the proof given in [**12**], pages 130-131, but afterwards we indicate how an alternative proof can be given on slightly more traditional element-by-element lines using unitary eigenmatrices.

Proof. Step I. Suppose that W is a normalised ω-representation of $\hat{G}$ in $\mathscr{H}$. Then $\xi \mapsto \langle W, \xi \otimes \phi \rangle$ is in $(\mathscr{B}(\mathscr{H})_*)^* = \mathscr{B}(\mathscr{H})$. Thus we have a unique $\pi(\phi)$ for which (*) holds. Moreover the relation (*) immediately implies that $\|\pi(\phi)\| \leqq \|\phi\|$ and that π is linear. So it remains to verify that

$$\pi(\phi \circ \psi) = \pi(\phi)\pi(\psi), \ \pi(\phi^+) = \pi(\phi)^* \text{ and } \pi(\epsilon) = I.$$

(i) $\pi(\phi \circ \psi) = \pi(\phi)\pi(\psi)$.

$$\langle \pi(\phi \circ \psi), \xi \rangle = \langle W, \xi \otimes (\phi \circ \psi) \rangle$$
$$= \langle \iota \otimes \delta_G(W)(I \otimes \omega), \xi \otimes \phi \otimes \psi \rangle$$
$$= \langle (W \otimes I)(\iota \otimes \sigma(W \otimes I)), \xi \otimes \phi \otimes \psi \rangle$$

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$$= \langle W \otimes I, (\iota \otimes \sigma(W \otimes I))(\xi \otimes \phi \otimes \psi) \rangle$$
$$= \langle \pi(\phi) \otimes I, W(\xi \otimes \psi) \rangle$$
$$= \langle W, (\xi\pi(\phi) \otimes \psi) \rangle$$
$$= \langle \pi(\psi), \xi\pi(\phi) \rangle$$
$$= \langle \pi(\phi)\pi(\psi), \xi \rangle.$$

(ii) $\pi(\phi^+) = \pi(\phi)^*$.

$$\langle \pi(\phi^+), \xi \rangle = \langle W, \xi \otimes \phi^+ \rangle = \langle W, \xi \otimes \alpha\phi^* \rangle$$
$$= \overline{\langle \iota \otimes \alpha(W^*), \xi^* \otimes \phi \rangle}$$
$$\langle \pi(\phi)^*, \xi \rangle = \overline{\langle \pi(\phi), \xi^* \rangle} = \overline{\langle W, \xi^* \otimes \phi \rangle}$$
$$= \overline{\langle \iota \otimes \alpha(W^*), \xi^* \otimes \phi \rangle}$$

(iii) $\pi(\epsilon) = I$. We have $W(I \otimes e_1) = (I \otimes I)$ by Theorem 4 (3), since ω is normalised. Therefore

$$\langle \pi(\epsilon), \xi \rangle = \langle W, \xi \otimes \epsilon \rangle = \langle W(I \otimes e_1), \xi \otimes \epsilon \rangle$$
$$= \langle I \otimes e_1, \xi \otimes \epsilon \rangle = \xi(I).$$

So $\pi(\epsilon) = I$ as asserted.

Proof. Step II. We start by defining an action of G on $\mathscr{R}(G)_*$ by $\alpha_g(\phi) = \phi \cdot \rho(g)^{-1}$. Thus the action of G is specified by

$$(*) \quad \langle \alpha_g(\phi), x \rangle = \langle \phi, \rho(g^{-1})x \rangle \quad (x \in \mathscr{R}(G))$$

and therefore corresponds to left translation on G in the realisation of $\mathscr{R}(G)_*$ as the Fourier algebra, that is as continuous functions on G,

$$\alpha_g(\tilde{\phi})(x) = \tilde{\phi}(g^{-1}x).$$

This makes it clear that the only elements of $\mathscr{R}(G)_*$ fixed by G are scalar multiples of ϵ. Furthermore, it is immediate from (*) that α_g defines an isomorphism of G into the group of *-automorphisms of $L^1_\omega(\hat{G})$, endowed with the topology of pointwise norm convergence, and this action is ergodic. Thus there is a unique G-invariant state on $L^1_\omega(\hat{G})$. Of course much more is true, as we now verify directly.

LEMMA 15. *The unique G-invariant state on* $L^1_\omega(\hat{G})$ *is given by*

$$\mathrm{Tr}(\phi) = \langle \phi, e_1 \rangle = \int \tilde{\phi}(g)dg.$$

Moreover the expressions $\mathrm{Tr}(\phi \circ \psi)$ *and* $\mathrm{Tr}(\phi^+ \circ \psi)$ *are independent of* ω *and are given by the formulas*

$$\mathrm{Tr}(\phi \circ \psi) = \int \tilde{\phi}(g)\tilde{\psi}(g)dg$$

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$$\mathrm{Tr}(\phi^+ \circ \psi) = \int \overline{\tilde{\phi}(g)}\psi(g)dg.$$

In particular Tr *is a faithful trace on* $L^1_\omega(\hat{G})$.

Proof. The unique G-invariant state on $L^1_\omega(\hat{G})$ is specified by

$$\mathrm{Tr}(\phi)\epsilon = \int \alpha_g(\phi)dg$$

so that using the realisation of $\mathscr{R}(G)_*$ as functions on G and evaluating at 1, we find

$$\mathrm{Tr}(\phi) = \int \tilde{\phi}(g^{-1})dg = \int \tilde{\phi}(g)dg = \langle \phi, e_1 \rangle.$$

Since ω is normalised, $\delta_G(e_1)\omega = \delta_G(e_1)$. Thus for $\phi, \psi \in \mathscr{R}(G)_*$ we have

$$\begin{aligned}\mathrm{Tr}(\phi \circ \psi) &= \langle \phi \circ \psi, e_1 \rangle \\ &= \langle \phi \otimes \psi, \delta_G(e_1)\omega \rangle \\ &= \langle \phi \otimes \psi, \delta_G(e_1) \rangle\end{aligned}$$

and similarly

$$\mathrm{Tr}(\phi^+ \circ \psi) = \langle \phi^+ \otimes \psi, \delta_G(e_1) \rangle.$$

The two expressions on the left hand side are thus independent of ω and can therefore be computed in $A(G)$ to give the desired results.

Our present aim is to relate the G-algebras $L^1_\omega(\hat{G})$ and $\pi_\omega(\hat{G})''$. To do so we need to set up part of the dictionary translating properties of ω-representations into properties of the corresponding representations of $L^1_\omega(\hat{G})$. We first state three lemmas, the proofs of which are almost immediate consequences of the defining equation (*) in the statement of Theorem 8.

LEMMA 16 (Intertwining Operators). *Let* $W_i \in \mathscr{B}(\mathscr{H}_i) \otimes \mathscr{R}(G)$ $(i = 1, 2)$ *be two ω-representations of* $\hat{G}$ *with corresponding representations* π_i *of* $L^1_\omega(\hat{G})$ *in* $\mathscr{B}(\mathscr{H}_i)$. *Then the set of maps intertwining* π_1 *and* π_2 *is equal to the set of maps intertwining* W_1 *and* W_2, *that is*

$$\{x \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2) : \pi_2(\phi)x = x\pi_1(\phi)\} = \{x \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2) : W_2(x \otimes I) = (x \otimes I)W_1\}.$$

In particular, taking $W_1 = W_2$, we have the following correspondence.

LEMMA 17 (Commutants and Bicommutants). *Let W be an ω-representation of* $\hat{G}$ *on* $\mathscr{H}$ *with* π *the corresponding representation of* $L^1_\omega(\hat{G})$. *Then*

$$(W)' = \pi(L^1_\omega(\hat{G}))' \quad \textit{and} \quad (W)'' = \pi(L^1_\omega(\hat{G}))''.$$

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LEMMA 18 (Equivariance). *Let W be an ω-representation of $\hat{G}$ in $\mathcal{M}$ and let π be the corresponding representation of $L^1_\omega(\hat{G})$. Suppose furthermore that G acts on $\mathcal{M}$. Then π is an equivariant representation of $L^1_\omega(\hat{G})$ if and only if W is equivariant.*

We next investigate the regular ω-representation and its relation to the canonical trace on $L^1_\omega(\hat{G})$. Let π_ω be the representation of $L^1_\omega(\hat{G})$ obtained by applying the GNS construction to the canonical trace. By Lemma 15, we see that, using the map $\phi \mapsto \tilde{\phi}$, the representation π_ω may be realised on the Hilbert space $L^2(G)$ in a natural way.

LEMMA 19 (The regular ω-representation and the canonical trace). *The representation π_ω of $L^1_\omega(\hat{G})$ on $L^2(G)$ corresponds via (*) to the regular ω-representation of $\hat{G}$ on $L^2(G)$ and is equivariant with respect to the action* Ad $\lambda(g)$ *of G on $\mathcal{B}(L^2(G))$. Moreover the trace* Tr *is the matrix coefficient (or vector state) defined by the constant function* 1 *in $L^2(G)$ and thus π_ω is faithful on $L^1_\omega(\hat{G})$.*

Proof. Let π be the representation of $L^1_\omega(\hat{G})$ on $L^2(G)$ corresponding to W_ω via (*). The assertion of the lemma will clearly follow if we can show that for $\phi \in \mathcal{R}(G)$.

(**) $\pi(\phi)\tilde{\epsilon} = \tilde{\phi}$.

Let $\xi \in \mathcal{B}(\mathcal{H})_*$ be defined by

$$\langle T, \xi \rangle = (T\tilde{\epsilon}|f) \quad (T \in \mathcal{B}(L^2(G)))$$

where $f \in L^2(G)$. Then $e_1\xi = \xi$ and

$$\begin{aligned}(\pi(\phi)\tilde{\epsilon}|f) &= \langle \xi, \pi(\phi) \rangle \\ &= \langle \xi \otimes \phi, W_G\sigma\omega \rangle \\ &= \langle e_1\xi \otimes \phi, W_G\sigma\omega \rangle \\ &= \langle \xi \otimes \phi, W_G\sigma\omega(e_1 \otimes I) \rangle \\ &= \langle \xi \otimes \phi, W_G(e_1 \otimes I) \rangle \\ &= \langle \xi \otimes \phi, W_G \rangle\end{aligned}$$

since ω is normalised. The right hand side can be computed using the regular representation of the Fourier algebra $A(G)$ and yields

$$(\pi(\phi)\tilde{\epsilon}|f) = (\tilde{\phi}|f)$$

so that (**) follows.

Before proceeding to the third step in the proof of Theorem 8, we indicate how the preceding results may be used to give an alternative proof of Theorem 2. In fact we have already seen in Lemma 4 that, given a full multiplicity ergodic action of G on $\mathcal{M}$, there is a cocycle ω, unique up to

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equivalence, and an equivariant representation π of $L^1_\omega(\hat{G})$ in $\mathcal{M}$. Since the unique trace on $\mathcal{M}$ necessarily pulls back to the canonical trace on $L^1_\omega(\hat{G})$, this representation is faithful. The fact that every irreducible representation of G has the same multiplicity in both $L^1_\omega(\hat{G})$ and $\mathcal{M}$ shows that $\pi(L^1_\omega(\hat{G}))$ is ultraweakly dense in $\mathcal{M}$. But then $\mathcal{M}$ may be equivariantly identified with $\pi_\omega(\hat{G})''$ using the GNS construction. Another way of phrasing these observations is that every equivariant representation of $L^1_\omega(\hat{G})$ is quasi-equivalent to π_ω; this is also reflected in the fact that

$$C^*_\omega(\hat{G}) \rtimes G \cong \mathcal{K}(L^2(G))$$

(cf. Theorem 10).

Proof. Step III. We now suppose that we have a unital *-representation π of $L^1_\omega(\hat{G})$ on $\mathcal{H}$. We wish to produce a unitary solution W to (*). Since $\mathcal{R}(G)$ is isomorphic to a direct sum of matrix algebras, the bilinear form on $\mathcal{B}(\mathcal{H})_* \times \mathcal{R}(G)_*$ given by

$$(\xi, \phi) \mapsto \langle \pi(\phi), \xi \rangle$$

may be represented by an element W of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{R}(G) = (\mathcal{B}(\mathcal{H})_* \otimes \mathcal{R}(G)_*)^*$,

$$\langle W, \xi \otimes \phi \rangle = \langle \pi(\phi), \xi \rangle \quad (\phi \in \mathcal{R}(G)_*, \xi \in \mathcal{B}(\mathcal{H})_*).$$

The relations $\pi(\phi \circ \phi) = \pi(\phi)\pi(\psi)$, $\pi(\phi^+) = \pi(\phi)^*$ and $\pi(\epsilon) = I$ imply the formulas

$$(\iota \otimes \delta_G(W))(I \otimes \omega) = (W \otimes I)\iota \otimes \sigma(W \otimes I),$$

$$\iota \otimes \alpha(W) = W^*,$$

$$W(I \otimes e_1) = I \otimes e_1.$$

It only remains to check the unitarity of W.

LEMMA 20. $WW^* = W^*W = I$.

Proof 1 (cf. pages 130-131, [**12**]). Let $x, y \in \mathcal{H}$ and $\phi, \psi \in \mathcal{R}(G)$. We shall compute $\omega_{\tilde{\phi},\tilde{\psi}}$ as an element of $\mathcal{R}(G)_*$, where $\tilde{\phi}$ and $\tilde{\psi}$ are regarded as elements of $L^2(G)$. Indeed

$$\begin{aligned}\langle \omega_{\tilde{\phi},\tilde{\psi}}, \rho(g) \rangle &= \int \tilde{\phi}(hg)\overline{\tilde{\psi}(h)}dh \\ &= \int \overline{\tilde{\psi}(h)}\widetilde{\alpha_h^{-1}\phi}(g)dh \\ &= \left\langle \int \overline{\tilde{\psi}(h)}\alpha_h^{-1}(\phi)dh, \rho(g) \right\rangle.\end{aligned}$$

Thus

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$$\omega_{\tilde{\phi},\tilde{\psi}} = \int \overline{\tilde{\psi}(h)}\alpha_h^{-1}(\phi)dh,$$

and similarly

$$\omega_{\tilde{\phi},\tilde{\psi}} = \int \tilde{\phi}(h)\alpha_h^{-1}(\psi^*)dh.$$

After this preparation, we find

$$\begin{aligned}(W(x \otimes \tilde{\phi}) \,|\, y \otimes \tilde{\psi}) &= \langle W, \omega_{x,y} \otimes \omega_{\tilde{\phi},\tilde{\psi}}\rangle \\ &= \int (\pi(\alpha_g^{-1}\phi)x|\, y)\overline{\tilde{\psi}(g)}dg.\end{aligned}$$

Thus $(W(x \otimes \tilde{\phi}))(g) = \pi(\alpha_g^{-1}\phi)x$ almost everywhere. Hence we obtain

$$\begin{aligned}(W(x \otimes \tilde{\phi}) \,|W(y \otimes \tilde{\psi})) &= \int (\pi(\alpha_g^{-1}\phi)x|\pi(\alpha_g^{-1}\psi)y)dg \\ &= \int \langle \pi(\alpha_g^{-1}(\psi^+ \circ \phi)), \omega_{x,y}\rangle dg \\ &= \mathrm{Tr}(\psi^+ \circ \phi)(x|\, y) \\ &= (\tilde{\phi}|\tilde{\psi})(x|\, y).\end{aligned}$$

Thus $W^*W = I$. Similarly we may use the second formula for $\omega_{\tilde{\phi},\tilde{\psi}}$ to show that

$$W^*(x \otimes \tilde{\phi})(g) = \pi(\alpha_g^{-1}\psi^*)^*x$$

almost everywhere and hence deduce that $WW^* = I$.

Proof 2. Let $\sigma:G \rightarrow \mathrm{End}(V_\sigma)$ be an irreducible unitary representation of G. We define an element Ψ_σ of $\mathscr{R}(G)_* \otimes \mathrm{End}(V_\sigma)$ by

$$\tilde{\Psi}_\sigma(g) = \sigma(g) \quad (g \in G).$$

Thus $\alpha_g\Psi_\sigma = \sigma(g)^{-1}\Psi_\sigma$ in $\mathscr{R}(G)$. Next we observe that in $L^1_\omega(\hat{G}) \otimes \mathrm{End}(V_\sigma)$, the element $\Psi_\sigma^+ \circ_\omega \Psi_\sigma$ is fixed by the action α of G. So it lies in $\mathrm{End}(V_\sigma)$ and its coefficients can therefore be calculated using the trace on $L^1_\omega(\hat{G})$. By Lemma 15, the result is independent of ω, so that the calculation can be done in $A(G)$ and implies that

$$\Psi_\sigma^+ \circ_\omega \Psi_\sigma = I.$$

Since $L^1_\omega(\hat{G})$ has a faithful trace, it is finite and therefore we see that Ψ_σ is unitary.

On the other hand

$$\langle \Psi_\sigma, \rho(g) \rangle = \sigma(g)$$

so that, under the isomorphism

$$\mathscr{B}(\mathscr{H}) \otimes \mathscr{R}(G) \cong \bigoplus_{\sigma\in\hat{G}} \mathscr{B}(\mathscr{H}) \otimes \mathrm{End}(V_\sigma),$$

we have

$$W = (W(\sigma))_{\sigma\in\hat{G}}$$

where

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$$W(\sigma) = (\mathrm{id} \otimes \Psi_\sigma)(W) = \pi(\Psi_\sigma)$$

(a formula which incidentally could have been used as a direct way of defining W). Thus the unitarity of W follows from the unitarity of the eigenmatrices Ψ_σ.

The proof of this lemma completes the proof of Theorem 8.

With the above information on $L^1_\omega(\hat{G})$ at our disposal, we can now answer some of the questions that arose earlier concerning the connection between ω and its inverse cocycle $\sigma\omega$. We start with a simple observation.

LEMMA 21. *If* $\phi, \psi \in \mathcal{R}(G)_*$, *then* $\phi \circ_\omega \psi = \psi \circ_{\sigma\omega} \phi$. *Thus* $L^1_\omega(\hat{G})$ *is naturally the opposite Banach *-algebra of* $L^1_{\sigma\omega}(\hat{G})$. *Moreover the following conditions on the cocycle* ω *are equivalent.*

(1) $\sigma\omega = \omega$, *that is* ω *is symmetric.*

(2) ω *is trivial.*

(3) $L^1_\omega(\hat{G})$ *is Abelian.*

(4) $\pi_\omega(\hat{G})''$ *is Abelian.*

(*Conditions* (1) *and* (2) *are therefore equivalent even for unnormalised cocycles of* $\hat{G}$.)

Proof. The first two assertions follow from the identities

$$\begin{aligned}\langle \phi \circ_\omega \psi, x\rangle &= \langle \phi \otimes \psi, \delta_G(x)\omega\rangle = \langle \psi \otimes \phi, \delta_G(x)\sigma\omega\rangle \\ &= \langle \psi \circ_{\sigma\omega} \phi, x\rangle.\end{aligned}$$

These immediately imply the equivalence of (1) and (3). Conditions (3) and (4) are equivalent since π_ω is faithful by Lemma 19. The equivalence of (2) and (4) follows from Theorem 2 and the fact that $L^\infty(G)$ is the only Abelian algebra on which G acts ergodically with full multiplicity.

THEOREM 9. (Duality) $\pi_\omega(\hat{G})''$ *and* $\pi_{\sigma\omega}(\hat{G})''$ *are each other's commutant in* $\mathcal{B}(L^2(G))$.

Proof 1. We shall use the notations and results of [**3**], pages 69-71. The inner product

$$(\phi|\psi) = \mathrm{Tr}(\psi^+ \circ \phi)$$

makes $L^1_\omega(\hat{G})$ into a Hilbert algebra, the associated Hilbert space of which may be identified with $L^2(G)$ by the map $\phi \mapsto \tilde{\phi}$. Under this identification the map $J:\phi \mapsto \phi^+$ becomes complex conjugation of functions and the left regular representation ("application canonique") of $L^1_\omega(\hat{G})$ is then just π_ω. On the other hand, we have

$$J\pi_\omega(\phi)J(\tilde{\psi}) = (\phi \widetilde{\circ_\omega} \psi^+)^+ = \psi \widetilde{\circ_\omega} \phi^+ = \phi^+ \widetilde{\circ_{\sigma\omega}} \psi = \pi_{\sigma\omega}(\phi^+)\tilde{\psi}$$

using the first identity of Lemma 21. Thus $J\pi_\omega(\phi)J = \pi_{\sigma\omega}(\phi^+)$, so that $\phi \mapsto \pi_{\sigma\omega}(\phi)$ is identified with the right regular representation of $L^1_\omega(\hat{G})$. Theorem 9 is therefore a consequence of the commutation theorem for Hilbert algebras.

Proof 2. We first verify directly that $\pi_\omega(\hat{G})''$ and $\pi_{\sigma\omega}(\hat{G})''$ commute with each other. This is immediate, however, from the formulas

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$$\pi_\omega(\phi)\tilde{\psi} = \phi \,\widetilde{\circ_\omega}\, \psi,$$
$$\pi_{\sigma\omega}(\phi)\tilde{\psi} = \phi \,\widetilde{\circ_{\sigma\omega}}\, \psi$$
$$= \psi \,\widetilde{\circ_\omega}\, \phi.$$

Thus $\pi_\omega(\hat{G})'' \subseteq \pi_{\sigma\omega}(\hat{G})'$. But by the corollary to Theorem 2, G acts ergodically on both these algebras via Ad λ. Thus the multiplicity of $\pi \in \hat{G}$ in either of these algebras is no greater than dim π by [**7**] or [**17**]. Since the multiplicity bounds are attained in the smaller algebra, the two algebras are indeed equal.

To end this section we gather together some properties of the enveloping C^* algebra $C^*_\omega(\hat{G})$ of $L^1_\omega(\hat{G})$.

THEOREM 10. (1) *The action of G on $L^1_\omega(\hat{G})$ extends to a strongly continuous action on $C^*_\omega(\hat{G})$. It is an ergodic action of full multiplicity.*

(2) *The regular representation π_ω is faithful on $C^*_\omega(\hat{G})$ and permits $C^*_\omega(\hat{G})$ to be identified with the C^* algebra of norm continuity of $\pi_\omega(\hat{G})''$.*

(3) $C^*_\omega(\hat{G}) \rtimes G \cong \mathcal{K}$.

(4) *$C^*_\omega(\hat{G})$ is nuclear.*

Proof. (1) It is well known that a strongly continuous action on an involutive Banach algebra extends to a strongly continuous action on its enveloping C^* algebra. Using the conditional expectation

$$E = \int \alpha_g dg,$$

we see that $C^*_\omega(\hat{G})^G$ is just the norm closure of $L^1_\omega(\hat{G})^G$, so that the action on $C^*_\omega(\hat{G})$ is ergodic. The same reasoning applies to the other spectral subspaces, so that this action has full multiplicity.

(2) The conditional expectation E yields a (unique) G-invariant faithful state on $C^*_\omega(\hat{G})$, which necessarily restricts to the canonical trace Tr on $L^1_\omega(\hat{G})$. Hence, since Tr is a vector state for π_ω, π_ω is faithful on $C^*_\omega(\hat{G})$.

(3) The assertion here follows from Theorem A (4).

(4) The nuclearity of $C^*_\omega(\hat{G})$ can either be deduced by the methods summarised in [**7**] or is a consequence of (3) and the following (well-known) general result.

LEMMA 22. *Let $\alpha:G \to \mathrm{Aut}(A)$ be a strongly continuous action of a compact group on a C^* algebra A. Then if the crossed product $A \rtimes G$ is nuclear, so too is A.*

Proof. We begin by recalling that the crossed product $A \rtimes G$ is isomorphic to

$$(A \otimes \mathcal{K}(L^2(G)))^{\alpha\otimes\mathrm{Ad}\lambda}.$$

Now A is nuclear if and only if for every C^* algebra B the surjective homomorphism

$$\theta:B \otimes_{\max} A \to B \otimes_{\min} A$$

is an injection. If we take the trivial action of G on B, then the map θ is

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equivariant. Furthermore the map

$$\theta \otimes \mathrm{id}: B \otimes_{\max} (A \otimes \mathscr{K}(L^2(G))) \to B \otimes_{\min} (A \otimes \mathscr{K}(L^2(G)))$$

is equivariant with respect to the action $\iota \otimes \alpha \otimes \mathrm{Ad}\,\lambda$ and has kernel $\ker(\theta) \otimes \mathscr{K}(L^2(G))$. Passing to fixed point algebras using the conditional expectation associated with this action and recalling our preliminary remark on crossed products, we see that $\ker(\theta) \rtimes G$ may be identified with the kernel of the natural homomorphism

$$B \otimes_{\max} (A \rtimes G) \to B \otimes_{\min} (A \rtimes G).$$

Since $A \rtimes G$ is nuclear, this latter kernel is trivial. Hence $\ker(\theta) = (0)$ and A is nuclear.

This completes the proof of Theorem 10.

We note that the faithfulness of the regular ω-representation can also be established using the cohomological device introduced in Lemma 11. Let us recall how the proof proceeds in the Abelian case [**10**]. If π is the regular representation of $\hat{G}$, π_ω the regular ω-representation of $\hat{G}$, and μ and any other ω-representation, then

$$(*) \quad \mu \otimes \pi \cong \dim \mu \cdot \pi_\omega.$$

One also has the direct integral decomposition of the left hand side

$$(**) \quad \mu \otimes \pi \cong \int_G \mu \circ \alpha_g dg.$$

This implies that for ϕ in $L^1_\omega(\hat{G})$

$$\|\mu(\phi)\| \leqq \sup_{g \in G} \|\mu(\alpha_g(\phi))\| = \|\mu \otimes \pi(\phi)\| = \|\pi_\omega(\phi)\|.$$

Since the C^* norm of $L^1_\omega(\hat{G})$ is given by $\sup_\mu \|\mu(\phi)\|$, it follows that this norm is equal to $\|\pi_\omega(\phi)\|$ and hence that π_ω is faithful on $C^*_\omega(\hat{G})$.

These arguments will carry over to the general case provided that we establish the validity of (**), since (*) is already known by Lemma 11.

LEMMA 23. *Let W be an ω-representation of $\hat{G}$ in $\mathscr{M}$ and let μ be the corresponding representation of $L^1_\omega(\hat{G})$. If $\tilde{\mu}$ $(= \mu \otimes \pi)$ is the representation of $L^1_\omega(\hat{G})$ in $\mathscr{M} \otimes L^\infty(G)$ corresponding to $W \circledast W_G$, then for $\phi \in L^1_\omega(\hat{G})$ we have*

$$\tilde{\mu}(\phi)(g) = \mu(\alpha_g^{-1}\phi)$$

where $\mathscr{M} \otimes L^\infty(G)$ has been identified with $\mathscr{M}$-valued functions on G.

Proof. Let $f \in L^1(G) = L^\infty(G)_*$. We shall once again use the fact that $W_G \in L^\infty(G) \otimes \mathscr{R}(G)$ is represented by the $\mathscr{R}(G)$-valued function $g \mapsto \rho(g)$ (cf. Lemma 3). Then for ξ in $\mathscr{M}_*$ we have

$$\begin{aligned} \int \langle \tilde{\mu}(\phi)(g), \xi \rangle f(g) dg &= \langle \tilde{\mu}(\phi), \xi \otimes f \rangle \\ &= \langle W \circledast W_G, \xi \otimes f \otimes \phi \rangle \\ &= \int \langle W, \xi \otimes \phi \cdot \rho(g) \rangle f(g) dg \end{aligned}$$

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$$= \int \langle \mu(\alpha_g^{-1}(\phi)), \xi \rangle f(g)dg$$

from which the desired formula follows.

8. Bicharacters and factoriality. We preface our general discussion by briefly recalling what is known in the Abelian case (see **[9]**, **[10]**, **[6]** and **[15]**). Firstly there is an isomorphism of $H^2(\hat{G}, \mathbf{T})$ into the group of alternating bicharacters on $\hat{G}$ given by $[\omega] \mapsto \beta$ where $\omega \in Z^2(\hat{G}, \mathbf{T})$ and

$$\beta(x, y) = \omega(x, y)\overline{\omega(y, x)} \quad (x, y \in \hat{G}).$$

Thus β satisfies

$$\beta(xy, z) = \beta(x, z)\beta(y, z),$$
$$\beta(x, yz) = \beta(x, y)\beta(x, z) \quad (x, y, z \in \hat{G}),$$
$$\overline{\beta(x, y)} = \beta(y, x).$$

This bicharacter defines a homomorphism Λ of $\hat{G}$ into G via

$$\Lambda : x \mapsto \beta(x, -).$$

In these circumstances it turns out that the following conditions are equivalent.

(1) $\pi_\omega(\hat{G})''$ is a (finite) factor.

(2) Λ is injective.

(3) the image of Λ is dense in G.

If either of conditions (2) or (3) is satisfied then we say that ω or β is *non-degenerate* or *totally skew*. The above conditions are in turn equivalent to any of the following C^* algebraic conditions.

(4) $C^*_\omega(\hat{G})$ has a unique trace.

(5) $C^*_\omega(\hat{G})$ has trivial centre.

(6) $C^*_\omega(\hat{G})$ is simple.

These conditions are also equivalent to the same conditions applied to the Banach *-algebra $L^1_\omega(\hat{G})$. Below we will show that, suitably interpreted, conditions (1) to (6) and the proofs of their equivalence carry over to the non-Abelian case. Almost all proofs of (6), however, rely crucially on the fact that if ω is non-degenerate, then the action α of G on $C^*_\omega(\hat{G})$ is approximately inner; to verify this we note that α_g is inner if $g \in \Lambda(\hat{G})$. The approximate innerness of the action forces any (non-trivial) ideal J of $C^*_\omega(\hat{G})$ to be automatically G-invariant, so that $J \rtimes G$ would provide a non-trivial ideal in $C^*_\omega(\hat{G}) \rtimes G$. On the other hand Theorem A implies that $C^*_\omega(\hat{G}) \rtimes G$ is simple, so $C^*_\omega(\hat{G})$ must itself be simple as required. In order to obtain an argument that is applicable even in the non-Abelian case, Magnus Landstad realised that instead of relying on the fact that the action was approximately inner, one should just use the condition that $C^*_\omega(\hat{G})$ was primitive. He observed that some ingenious computations of Olesen and Pedersen imply that if a compact group acts on a primitive C^* algebra, then every non-zero ideal in the algebra contains a non-zero invariant ideal. We provide a more conceptual proof of this result in the following lemma.

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LEMMA 24. (Landstad) *If* $\alpha : G \to \mathrm{Aut}(A)$ *is a strongly continuous action of a compact group on a primitive* C^* *algebra* A *such that* $A \rtimes G$ *is simple, then the algebra* A *is simple.*

Proof. Suppose that A is not simple and that I is a non-trivial ideal in A. The ideal I corresponds to a closed subset X of the primitive ideal space $\mathrm{Prim}(A)$ of A, missing the zero ideal. By a result of Glimm (see [**4**]), the group G acts continuously on $\mathrm{Prim}(A)$. Since G is compact, it follows that the saturation $G \cdot X$ of the closed set X is automatically closed. It is also by definition invariant and misses the zero ideal. The corresponding ideal J in A is non-zero, G-invariant and contained in I. Thus $J \rtimes G$ is a non-trivial ideal in $A \rtimes G$, contradicting the simplicity of the crossed product.

COROLLARY. *If* G *is a compact group and* $C^*_\omega(\hat{G})$ *is primitive for some* $\omega \in H^2(\hat{G})$, *then* $C^*_\omega(\hat{G})$ *is simple.*

We are now ready to introduce the formalism of bicharacters. This is closely related to Drinfeld's use of "triangular Hopf algebras" in studying the quantum Yang-Baxter equations (see [**5**], pages 18-19). For our immediate requirements, however, we shall have no need to make this link too explicit and thus will avoid using the language of Hopf-von Neumann (or Kac) algebras, although it should be clear that the general theory sits very naturally within this framework.

Let ω be a normalised cocycle of $\hat{G}$. We define the *bicharacter* β_ω of ω to be the unitary

$$\beta_\omega = (\sigma\omega^*)\omega \quad \text{in } \mathscr{R}(G) \otimes \mathscr{R}(G).$$

Note that if $W \in \mathscr{B}(\mathscr{H}) \otimes \mathscr{R}(G)$ is an ω-representation of $\hat{G}$ in $\mathscr{H}$, then the defining equation

$$\delta_G(W)(I \otimes \omega) = (W \otimes I)\iota \otimes \sigma(W \otimes I)$$

leads to the identity

$$I \otimes \beta_\omega = (W^* \otimes I)\iota \otimes \sigma(W^* \otimes I)(W \otimes I)\iota \otimes \sigma(W \otimes I).$$

In order to display the bicharacter nature of β_ω, it is necessary to introduce a perturbation of the comultiplication δ_G of $\mathscr{R}(G)$. We define the *comultiplication* δ_ω of ω on $\mathscr{R}(G)$ by

$$\delta_\omega : \mathscr{R}(G) \to \mathscr{R}(G) \otimes \mathscr{R}(G), \quad \delta_\omega(x) = \omega^*\delta_G(x)\omega.$$

LEMMA 25. δ_ω *is a comultiplication on* $\mathscr{R}(G)$ *satisfying* $\alpha \cdot \delta_\omega \cdot \alpha = \sigma\delta_\omega$ *and induces the structure of a unital Banach* $*$*-algebra on the predual* $\mathscr{R}(G)_*$.

Proof. (i) Plainly δ_ω is a $*$-isomorphism of $\mathscr{R}(G)$ into $\mathscr{R}(G) \otimes \mathscr{R}(G)$, by definition. We must check that δ_ω is coassociative, that is

$$(*) \quad (\iota \otimes \delta_\omega)\delta_\omega = (\delta_\omega \otimes \iota)\delta_\omega.$$

Let $x \in \mathscr{R}(G)$. Then we have

$$\begin{aligned}(\iota \otimes \delta_\omega)\delta_\omega(x) &= \iota \otimes \delta_\omega(\omega^*\delta_G(x)\omega)\\ &= \mathrm{Ad}[\,(I \otimes \omega^*)\iota \otimes \delta_G(\omega^*)\,](\iota \otimes \delta_G)\delta_G(x)\end{aligned}$$

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and

$$(\delta_\omega \otimes \iota)\delta_\omega(x) = \iota \otimes \delta_\omega(\omega^*\delta_G(x)\omega)$$
$$= \mathrm{Ad}[\,(\omega^* \otimes I)\delta_G \otimes \iota(\omega^*)\,](\delta_G \otimes \iota)\delta_G(x).$$

So the cocycle identity for ω and the coassociativity of δ_G imply the validity of (*).

(ii) To verify the relation $\alpha \cdot \delta_\omega \cdot \alpha = \sigma\delta_\omega$, we need only check it on the elements $\rho(g)$ of $\mathscr{R}(G)$, for the result then follows by linearity and continuity.

In fact,

$$\alpha \cdot \delta_\omega \cdot \alpha(\rho(g)) = \alpha(\omega^*(\rho(g)^* \otimes \rho(g)^*)\omega)$$
$$= \alpha\omega(\rho(g) \otimes \rho(g))\alpha\omega^*$$
$$= \sigma\omega^*(\rho(g) \otimes \rho(g))\sigma\omega$$
$$= \sigma\delta_\omega(\rho(g))$$

since the fact that ω is normalised implies that $\alpha\omega = \sigma\omega^*$ by Theorem 6 (1).

(iii) The comultiplication on $\mathscr{R}(G)$ is specified by

$$\langle \phi \star \psi, x\rangle = \langle \phi \otimes \psi, \delta_\omega(x)\rangle.$$

It is immediate from (*) that this multiplication is associative. The submultiplicativity of the norm and the fact that ϵ is a unit follow by similar reasoning to that used in the proof of Theorem 7. It only remains to check that the involution $\phi \mapsto \phi^+$ still satisfies $(\phi \star \psi)^+ = \psi^+ \star \phi^+$. In fact,

$$\langle (\phi \star \psi)^+, x\rangle = \overline{\langle \phi \otimes \psi, \delta_\omega(\alpha x^*)\rangle}$$
$$= \overline{\langle \phi \otimes \psi, \alpha\sigma\delta_\omega(x^*)\rangle}$$
$$= \langle \psi^+ \otimes \phi^+, \delta_\omega(x)\rangle$$
$$= \langle \psi^+ \star \phi^+, x\rangle$$

using (ii).

We shall denote $\mathscr{R}(G)_*$ by $A_\omega(G)$ when it has this structure as a Banach *-algebra; and when the dependence on ω is clear, we shall simply write (β, δ) for $(\beta_\omega, \delta_\omega)$. We make some preliminary observations about the algebra $A_\omega(G)$. Firstly, we shall see below that the condition $\sigma\delta = \delta$ is no longer necessarily satisfied when G is non-Abelian, so that $A_\omega(G)$ need not in general be commutative. Later we will construct a *-homomorphism Λ of $A_\omega(G)$ into the generally non-commutative von Neumann algebra $\mathscr{R}(G)$; each finite-dimensional irreducible representation of G, and hence of $\mathscr{R}(G)$, will then give rise to a finite-dimensional representation of $A_\omega(G)$. It is easy to see that in general there is a natural correspondence between such representations and certain matrices with coefficients in $\mathscr{R}(G)$. In particular the centraliser of ω

$$C(\omega) = \{x : \delta_G(x^*)\omega(x \otimes x) = \omega\}$$

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is given by the set $\{x : \delta_\omega(x) = x \otimes x\}$ and these correspond to one dimensional representations of $A_\omega(G)$. In fact, such an x defines a character according to the formula $\phi \mapsto \phi(x)$. More generally the representations of $A_\omega(G)$ in $M_n(\mathbf{C})$ correspond to unitaries (x_{ij}) in $M_n(\mathscr{R}(G))$ such that

$$\delta_\omega(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$$

with the representation given by $\phi \mapsto (\phi(x_{ij}))$. In a similar vein, we note that

$$\delta_\omega(x) = W_{\sigma\omega}^*(x \otimes I)W_{\sigma\omega}$$

for x in $\mathscr{R}(G)$, so that the subalgebra $\{x : \delta_\omega(x) = x \otimes I\}$ may be identified with $\pi_{\sigma\omega}(G)' \cap \mathscr{R}(G)$, and is therefore trivial by the corollary to Theorem 2.

The following lemma gathers together the characteristic properties of the pair (β, δ).

LEMMA 26.

(i) $\sigma(\delta(x)) = \beta\delta(x)\beta^*$

(ii) $\beta^* = \sigma\beta$, $\alpha\beta = \beta$, $\iota \otimes \alpha(\beta) = \beta^* = \alpha \otimes \iota(\beta)$

(iii) $(e_1 \otimes I)\beta = e_1 \otimes I$, $(I \otimes e_1)\beta = I \otimes e_1$, $\delta(e_1)\beta = \delta(e_1) = \delta_G(e_1)$

(iv) (*cocycle relations*)

$$(\beta \otimes I)\delta \otimes \iota(\beta) = (I \otimes \beta)\iota \otimes \delta(\beta)$$

(v) (*bicharacter relations*)

$$\delta \otimes \iota(\beta) = \sigma \otimes \iota(I \otimes \beta)(I \otimes \beta) = \beta_{13}\beta_{23},$$
$$\iota \otimes \delta(\beta) = \iota \otimes \sigma(\beta \otimes I)(\beta \otimes I) = \beta_{13}\beta_{12}$$

(vi) (*quantum Yang-Baxter equation*)

$$\beta_{23}\beta_{13}\beta_{12} = \beta_{12}\beta_{13}\beta_{23}.$$

Proof. (i), (ii) and (iii) are for the most part clear from the definitions and the normalisation conditions on ω. To prove the last equations in (ii), one just applies $\iota \otimes \alpha \otimes \iota$ to the equation

$$W_{13}^*W_{12}W_{13} = W_{12}\beta_{23}$$

where W denotes the regular (normalised) ω-representation. The cocycle relations for β with respect to δ follow easily by combining the relations for ω and $\sigma\omega$ with respect to δ_G. To prove the bicharacter relation

$$\iota \otimes \delta_\omega(\beta) = \beta_{13}\beta_{12}$$

we start with the equation

$$\beta_{23} = W_{12}^*W_{13}^*W_{12}W_{13}$$

where W denotes the regular ω-representation. Applying δ_ω to this in the third (tensor) factor and using the relation

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$$\iota \otimes \delta_\omega(W_{13}) = \omega_{34}^* W_{13} W_{14}$$

we obtain

$$\begin{aligned}\iota \otimes \delta_\omega(\beta)_{234} &= W_{12}^*(W_{14}^* W_{13}^* \omega_{34}) W_{12}(\omega_{34}^* W_{13} W_{14}) \\ &= W_{12}^* W_{14}^* W_{13}^* W_{12} W_{13} W_{14} \\ &= W_{12}^* W_{14}^* W_{12}(W_{12}^* W_{13}^* W_{12} W_{13}) W_{14} \\ &= W_{12}^* W_{14}^* W_{12} \beta_{23} W_{14} \\ &= W_{12}^* W_{14}^* W_{12} W_{14} \beta_{23} \\ &= \beta_{24}\beta_{23}.\end{aligned}$$

Thus $\iota \otimes \delta_\omega(\beta) = \beta_{13}\beta_{12}$ as required. The relation for $\delta_\omega \otimes \iota(\beta)$ can be derived in a similar fashion. Finally (vi) follows from (iv) and (v) by cancelling $\delta \otimes \iota(\beta)$ and $\iota \otimes \delta(\beta)$. Equally well (vi) follows from (i) and either of the bicharacter relations.

Our next task is to investigate the equivalence relation on the pairs (β, δ) induced by the equivalence relation we have imposed on cocycles. The following result is immediately verified.

LEMMA 27. *If v is in $\mathcal{G}$, then*

$$\beta_{\omega^v} = \mathrm{Ad}(v^* \otimes v^*) \cdot \beta_\omega,$$

$$\delta_{\omega^v} = \mathrm{Ad}(v^* \otimes v^*) \cdot \delta_\omega \cdot \mathrm{Ad}(v).$$

We shall call two pairs (β_1, δ_1), (β_2, δ_2) *equivalent* if they are related by a unitary v in $\mathcal{G}$ in the above way, that is if

$$\beta_1 = (v^* \otimes v^*)\beta_2(v \otimes v), \quad \delta_1 = \mathrm{Ad}(v^* \otimes v^*) \cdot \delta_2 \cdot \mathrm{Ad}(v).$$

It turns out that if G is a connected compact group, then the conclusion of Lemma 27 can be reversed, i.e., the two notions of equivalence, on pairs and on cocycles, correspond exactly. In general, however, the equivalence relation on the pairs induces a weaker equivalence relation on the corresponding cocycles than that so far defined. To explain this new relation we will have to introduce a subgroup $\mathrm{Aut}_c(G)$ of the automorphism group $\mathrm{Aut}(G)$ of G. This subgroup first made its appearance for finite G in the work of Burnside (**[2]**, Note B) and eventually received a more thorough analysis in **[14]**.

Let $\mathrm{Aut}_c(G)$ be the subgroup of $\mathrm{Aut}(G)$ consisting of those automorphisms of G which act trivially on the dual space $\hat{G}$. Thus $\mathrm{Aut}_c(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ containing the normal subgroup $\mathrm{Inn}(G)$ of inner automorphisms. Elements of $\mathrm{Aut}_c(G)$ can also be characterised by the triviality of their action on central functions on G, or equivalently on the space of conjugacy classes of G. Since $\pi \cdot \alpha$ and π are unitarily equivalent for any $\alpha \in \mathrm{Aut}_c(G)$ and $\pi \in \hat{G}$, it follows from Schur's lemma that there is a unique projective representation of $\mathrm{Aut}_c(G)$ on V_π extending that of $\mathrm{Inn}(G)$. Hence, since

$$\mathcal{R}(G) \cong \oplus_{\pi \in \hat{G}} \mathrm{End}(V_\pi),$$

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there is a unique homomorphism κ of $\mathrm{Aut}_c(G)$ into the projective unitary group

$$PU(\mathscr{R}(G)) = \mathscr{R}(G)^u/(\mathscr{R}(G)^{\natural})^u$$

of $\mathscr{R}(G)$ extending the natural inclusion of $\mathrm{Inn}(G)$, $\mathrm{Ad}(g) \mapsto [\rho(g)]$. (Note that although there is a natural "permutation" representation of $\mathrm{Aut}(G)$ and hence $\mathrm{Aut}_c(G)$ on $L^2(G)$, this will not in general bear any relation to the homomorphism κ.) Our next result identifies the image of κ and permits an internal characterisation of $\mathrm{Aut}_c(G)$ in terms of certain normalising unitaries in $\mathscr{R}(G)$.

LEMMA 28. (1) *Using the homomorphism* κ, $\mathrm{Aut}_c(G)$ *may be identified with the quotient of the group*

$$\mathscr{G}_1 = \{v \in \mathscr{R}(G)^u : \mathrm{Ad}(v)(\rho(G)) = \rho(G)\}$$

by its centre $(\mathscr{R}(G)^{\natural})^u$. *The automorphism* γ *corresponding to* v *is determined by* $v\rho(g)v^* = \rho(\gamma(g))$.

(2) *Any unitary* $u \in \mathscr{R}(G)$ *satisfying* $\mathrm{Ad}(u)(\rho(G)) \subseteq \rho(G)$ *automatically lies in* $\mathscr{G}_1$ *and hence induces an automorphism in* $\mathrm{Aut}_c(G)$.

Proof. To prove (1), we note that if $v \in \mathscr{G}_1$, then $\mathrm{Ad}(v)|_{\rho(G)}$ gives an automorphism of G fixing $\hat{G}$. Thus $v \mapsto \mathrm{Ad}(v)|_{\rho(G)}$, $\mathscr{G}_1/\mathscr{G}_1^{\natural} \to \mathrm{Aut}_c(G)$ yields an inverse to the homomorphism

$$\kappa:\mathrm{Aut}_c(G) \to \mathscr{G}_1/\mathscr{G}_1^{\natural}.$$

To prove (2), we must show that if $u \in \mathscr{R}(G)^u$ satisfies $u\rho(G)u^* \subseteq \rho(G)$, then we actually have equality here.

Suppose first of all that G is a compact Lie group. Let G^0 be the connected component of the identity in G, $\exp(\mathrm{Lie}(G))$, a clopen normal subgroup of G of finite index. Then $u\rho(G^0)u^*$ is connected so lies in $\rho(G^0)$. Since these groups have the same dimension, they must therefore be equal. It follows that $\mathrm{Ad}(u)$ induces an injective map of the quotient $\rho(G)/\rho(G^0)$ into itself. Since this quotient is finite, the induced map must be onto. From this we deduce that $u\rho(G)u^* = \rho(G)$, so the result holds in this case.

To treat the case of a general compact group G, we take a decreasing sequence K_n of closed normal subgroups of G such that G/K_n is a Lie group and $\cap K_n = \{1\}$. (Such a sequence may be obtained by taking

$$K_n = \cap_{i \leqq n} \ker(\pi_i)$$

where $\pi_1, \pi_2, \ldots$ is an enumeration of the inequivalent irreducible representations of G.) The condition $u\rho(G)u^* \subseteq \rho(G)$ implies that under the natural homomorphism

$$\mathscr{R}(G) \to \mathscr{R}(G/K_n), \quad a \mapsto \bar{a},$$

we have $\bar{u}\rho(G/K_n)\bar{u}^* \subseteq \rho(G/K_n)$. Since G/K_n is a Lie group, the previous argument applies and we see that

$$\bar{u}\rho(G/K_n)\bar{u}^* = \rho(G/K_n).$$

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Now let H be the closed subgroup of G defined by $\rho(H) = u\rho(G)u^*$; it is closed because it is a continuous image of G. The condition on $\bar{u}$ above translates into the condition

$$H/H \cap K_n = G/K_n \text{ for all } n.$$

In other words $G = H \cdot K_n$ for each n. The proof of (2) will be complete once we show that $H = G$.

Let $x \in G$. Then $x \in H \cdot K_n$ for each n and so we may write $x = h_n \cdot k_n$ with $h_n \in H$ and $k_n \in K_n$. Since H is compact, we can find a convergent subsequence (h_{n_r}) of (h_n) with $h_{n_r} \to h \in H$ as $r \to \infty$. But then $k_{n_r} \to h^{-1}x$ as $r \to \infty$ and this limit must lie in $\cap K_{n_r}$, since the K_n are closed. This intersection is trivial, so that $x = h$ and x lies in H. Thus $H = G$ as required.

The next result gives some general information on the structure of $\mathrm{Aut}_c(G)/\mathrm{Inn}(G)$. Note that Lemma 28 identifies $\mathrm{Aut}_c(G)$ with a closed subgroup of $PU(\mathcal{R}(G))$ (in the ultraweak topology). Since

$$\mathcal{R}(G) \cong \bigoplus_{\pi \in \hat{G}} \mathrm{End}(V_\pi),$$

the latter may be identified with the compact group $\prod_{\pi \in \hat{G}} PU(V_\pi)$. Thus $\mathrm{Aut}_c(G)$ is again a compact group.

LEMMA 29. (1) *If G is connected, then* $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$ *and hence*

$$\mathcal{G}_1 = \rho(G) \cdot (\mathcal{R}(G)^{\models})^u.$$

(2) *If G is a compact Lie group (in particular, finite), then* $\mathrm{Aut}_c(G)/\mathrm{Inn}(G)$ *is a finite solvable group.*

(3) *If G is an arbitrary compact group, then* $\mathrm{Aut}_c(G)/\mathrm{Inn}(G)$ *is a projective limit of finite solvable groups.*

Proof. Before embarking on the proof, let us note that any $\gamma \in \mathrm{Aut}_c(G)$ leaves each closed normal subgroup N of G invariant, and hence induces an automorphism $\bar{\gamma}$ of the quotient group G/N which, as is easily verified, lies in $\mathrm{Aut}_c(G/N)$. Moreover $\gamma|_N$ lies in $\mathrm{Aut}(N)$. These observations will often be used without specific reference in the sequel.

(1) Let us first consider the case when G is a compact connected Lie group. Then $G = G_s \cdot A$ where G_s is a connected semisimple group, A is a central torus in G and $A \cap G_s$ is finite. Thus if $\gamma \in \mathrm{Aut}_c(G)$, we see that $\gamma|_A \in \mathrm{Aut}_c(A)$ and $\gamma|_{G_s} \in \mathrm{Aut}_c(G_s)$. It follows immediately that γ acts trivially on A. We claim that $\gamma|_{G_s}$ is inner. Let T be a maximal torus in G_s with topological generator t. By hypothesis $\gamma(t) = gtg^{-1}$ for some $g \in G_s$. So $\mathrm{Ad}(g)^{-1} \cdot \gamma$ fixes T. By an old root space argument of Gantmacher (see for example Theorem 8.11.2 in **[21]**), $\mathrm{Ad}(g)^{-1} \cdot \gamma$ must be given by $\mathrm{Ad}(s)$ on G_s for some $s \in T$. But then γ is given by $\mathrm{Ad}(gs)$ on G_s, so is inner on G_s and hence on G.

If G is an arbitrary connected compact group, we may take closed normal subgroups K_n exactly as in the proof of Lemma 28. Let $\gamma \in \mathrm{Aut}_c(G)$. Then since G/K_n is a connected Lie group, our work above

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shows that the automorphism $\bar{\gamma}$ induced on G/K_n is inner. Hence for each n we can find $g_n \in G$ such that for all x in G

$$\gamma(x) \equiv g_n x g_n^{-1} \mod K_n.$$

Passing to a subsequence if necessary, we may assume that $g_n \to g$ in G. Then, since $\gamma(x)(g_n x g_n^{-1})^{-1}$ is in K_n for each x, we see that in the limit as $n \to \infty$,

$$\gamma(x)(gxg^{-1})^{-1} \in \cap_n K_n = \{1\}.$$

Thus $\gamma(x) = gxg^{-1}$ for all x in G, and hence γ is inner.

(2) When G is a finite group, the assertion is a consequence of Theorem 2.10 of [**14**] and Schreier's conjecture, which is now known to be true by the classification of finite simple groups. Now let G be any compact Lie group with identity component G^0 and let Γ be the finite group G/G^0. In view of the known result for finite groups, it will suffice to show that the kernel of the natural map

$$\mathrm{Aut}_c(\Gamma)/\mathrm{Inn}(G) \to \mathrm{Aut}_c(\Gamma)/I(\Gamma)$$

is solvable. Let T be a maximal torus in G with normaliser $N_G(T)$ and let N be the subgroup of $N_G(T)$ fixing some given choice of positive roots (or Weyl chamber).

Let us first show that the image of $\mathrm{Aut}_c(G)$ in $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is contained in the image of $A_0(G)$, the subgroup of $\mathrm{Aut}(G)$ fixing G^0. Indeed suppose that $\gamma \in \mathrm{Aut}_c(G)$ and let t be a topological generator of T. By assumption $\gamma(t) = gtg^{-1}$ for some $g \in G$, so that $\gamma' = \mathrm{Ad}(g^{-1}) \cdot \gamma$ fixes T. Let Z be the identity component of the centre of G^0 and let $x \mapsto \bar{x}$ be the quotient map of G^0 onto the semisimple group G^0/Z. Thus $\bar{\gamma}'$ fixes the maximal torus T/Z of G^0/Z. By the result of Gantmacher already used in (1), $\bar{\gamma}' = \mathrm{Ad}(\bar{s})$ for some $s \in T$. Hence

$$\gamma'(x) \equiv sxs^{-1} \mod Z \quad \text{for all } x \in G^0.$$

Thus the automorphism

$$\gamma'' = \mathrm{Ad}(gs)^{-1} \cdot \gamma = \mathrm{Ad}(s)^{-1} \cdot \gamma'$$

fixes T as well as having the property that $\varphi(x) = \gamma''(x)x^{-1}$ lies in Z for every $x \in G^0$. But then φ is a homomorphism of G^0 into Z containing Z in its kernel. The semisimplicity of G^0/Z implies that the Abelianisation of G^0/Z is trivial, so that $\varphi(x) \equiv 1$. This means that γ'' fixes G^0, so that $\mathrm{Ad}(xs)^{-1} \cdot \gamma$ lies in $\mathrm{Aut}_0(G)$ as required.

In view of the inclusion just established, the assertions in (2) will follow from the stronger statement that the kernel of the map

$$\pi_G(\mathrm{Aut}_0(G)) \to \mathrm{Out}(\Gamma)$$

is solvable. (Here π_G denotes the map $\mathrm{Aut}(G) \to \mathrm{Out}(G)$). Now any element of $\mathrm{Aut}_0(G)$ fixes T and therefore leaves $N_G(T)$ and N invariant. The subgroup N has the property that its identity component is T and that the inclusion of N in G induces an isomorphism of N/T onto $\Gamma = G/G^0$, so that in particular G is generated by N and G^0. It follows that the

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restriction map $\mathrm{Aut}(G) \to \mathrm{Aut}(N)$ gives rise to an injection of $\mathrm{Aut}_0(G)$ into $\mathrm{Aut}_0(N)$, the subgroup of $\mathrm{Aut}(N)$ fixing T. It is also easy to see that we get an injection

$$\pi_G(\mathrm{Aut}_0(G)) \to \pi_N(\mathrm{Aut}_0(N))$$

on quotienting out by inner automorphisms. Moreover the map

$$\pi_G(\mathrm{Aut}_0(G)) \to \mathrm{Out}(\Gamma)$$

is the composition of this restriction and the map

$$\pi_N(\mathrm{Aut}_0(N)) \to \mathrm{Out}(\Gamma).$$

So it will be enough to show that the kernel of this latter map is solvable.

By definition this kernel is the image under π_N of the group $\mathscr{K}$ consisting of all automorphisms of N that fix T and induce inner automorphisms of N/T. Let $\mathscr{K}_1$ be the subgroup of $\mathrm{Aut}(N)$ consisting of automorphisms that induce trivial automorphisms of N/T and are implemented on T by an inner automorphism of N. Clearly $\pi_N(\mathscr{K}) = \pi_N(\mathscr{K}_1)$, so we may complete the proof by showing that $\mathscr{K}_1$ is solvable.

For $\gamma \in \mathscr{K}_1$ and $x \in N$, let $f_\gamma(x) = \gamma(x)x^{-1}$. Since γ acts trivially on N/T, it follows that $f_\gamma(x)$ lies in T. Moreover f_γ clearly uniquely determines γ. Let α be the homomorphism of N into $\mathrm{Aut}(T)$ given by $\alpha_x = \mathrm{Ad}(x)|_T$. The f_γ satisfies the cocycle identity

$$f_\gamma(xy) = f_\gamma(x)\alpha_x(f_\gamma(y)) \quad (x, y \in N).$$

The set of such cocycles forms an Abelian group B under pointwise multiplication. Let A be the centre of the subgroup α_N of $\mathrm{Aut}(T)$. Clearly the action of A on T induces an action of A on B via $\delta(f)(x) = \delta(f(x))$, where $f \in B$ and $\delta \in A$. Furthermore the automorphism δ_γ of T given by $\delta_\gamma = \gamma|_T$ must lie in A: for if $x \in N$ and $t \in T$, then

$$\alpha_x(\gamma(t)) = \gamma(\alpha_{\gamma(x)}(t)) = \gamma(\alpha_x(t)),$$

since $\gamma(x) \equiv x \bmod T$. It is also readily verified that

$$f_{\gamma_1\gamma_2} = f_{\gamma_1}\delta_{\gamma_1}(f_{\gamma_2}) \quad \text{for } \gamma_1, \gamma_2 \in \mathscr{K}_1.$$

Thus the map $\gamma \to (f_\gamma, \delta_\gamma)$ defines an injective homomorphism of $\mathscr{K}_1$ into the semidirect product $B \rtimes A$. Since both A and B are Abelian, their semidirect product is solvable. Hence $\mathscr{K}_1$ is solvable as required.

(3) Let G be any compact group and choose closed normal subgroups K_n of G as in Lemma 28. Now $\mathrm{Aut}_c(G)/\mathrm{Inn}(G)$ is compact and there are continuous homomorphisms

$$\theta_n : \mathrm{Aut}_c(G)/\mathrm{Inn}(G) \to \mathrm{Aut}_c(G/K_n)/\mathrm{Inn}(G/K_n)$$

where the target groups are already known to be finite solvable groups by (2). Thus (3) will follow if we can show that $\cap_n \ker(\theta_n) = \{1\}$. This is equivalent to showing that an automorphism in $\mathrm{Aut}_c(G)$ which is inner on each quotient G/K_n is inner on G; this, however, may be proved by the same argument as that used in the second part of the proof of (1).

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We next wish to define an action of $\mathrm{Aut}_c(G)/\mathrm{Inn}(G)$ on $H^2(\hat{G})$. To do so we will define an action of $\mathscr{G}_1$ on cocycles which passes immediately to $\mathrm{Aut}_c(G)$. We shall need a preliminary result.

LEMMA 30. *Let* $a \in \mathscr{R}(G)$ *and* $u \in \mathscr{G}_1$.

(i) $\delta_G(uau^*) = (u \otimes u)\delta_G(a)(u^* \otimes u^*)$, *so that* $\delta_G(u^*)(u \otimes u)$ *commutes with* $\delta_G(\mathscr{R}(G))$.

(ii) $\alpha(uau^*) = u(\alpha a)u^*$, *so that* $(\alpha u)u$ *is central in* $\mathscr{R}(G)$.

(iii) u *can be multiplied by a central unitary to yield an element of* $\mathscr{G} \cap \mathscr{G}_1$.

Proof. Let γ be the automorphism of G determined by u, so that

$$u\rho(g)u^* = \rho(\gamma(g)).$$

This equation shows that (i) and (ii) are satisfied when $a = \rho(g)$ and they follow in general by linearity and continuity. To prove (iii), we simply have to premultiply u by a unitary square root of $((\alpha u)u)^*$ in $(\mathscr{R}(G)')^{\alpha}$.

LEMMA 31. *Let* ω *be a cocycle of* $\hat{G}$ *and* $u \in \mathscr{G}_1$. *Then* $(u \otimes u)\omega(u^* \otimes u^*)$ *is a normalised cocycle of* $\hat{G}$ *the class of which in* $H^2(\hat{G})$ *depends only on the class of* ω.

Proof. The cocycle identity for $\omega_1 = (u \otimes u)\omega(u^* \otimes u^*)$ is an easy consequence of the cocycle identity for ω and Lemma 30 (i). Lemma 30 (i) also shows that

$$\delta_G(e_1) = \delta_G(ue_1u^*) = (u \otimes u)\delta_G(e_1)(u^* \otimes u^*),$$

so that the fact that $\delta_G(e_1)\omega = \delta_G(e_1)$ implies that $\delta_G(e_1)\omega_1 = \delta_G(e_1)$. Thus ω_1 is normalised. Finally if $v \in \mathscr{G}_1$, we have

$$(u \otimes u)[\delta_G(v^*)\omega(v \otimes v)](u^* \otimes u^*)$$
$$= \delta_G(uvu^*)[(u \otimes u)\omega(u^* \otimes u^*)](uvu^* \otimes uvu^*)$$

using Lemma 30 (i) again; and from Lemma 30 (ii) it follows that uvu^* is still in $\mathscr{G}$.

We shall say that two cocycles of $\hat{G}$ are *weakly equivalent* if and only if they lie in the same $\mathrm{Aut}_c(G)$-orbit of $H^2(\hat{G})$. We are now in a position to establish the first main result of this section.

THEOREM 11. *Let* ω_1 *and* ω_2 *be cocycles for* $\hat{G}$. *Then* ω_1 *and* ω_2 *are weakly equivalent if and only if* $(\beta_{\omega_1}, \delta_{\omega_1})$ *and* $(\beta_{\omega_2}, \delta_{\omega_2})$ *are equivalent.*

Proof. The equivalence of the pairs follows from the weak equivalence of the pairs using Lemma 27 and Lemma 30 (i). In fact, if ω is a cocycle for $\hat{G}$ and $\omega' = (u \otimes u)\omega(u^* \otimes u^*)$, then

$$\beta_{\omega'} = \mathrm{Ad}(u \otimes u) \cdot \beta_\omega, \quad \delta_{\omega'} = \mathrm{Ad}(u \otimes u) \cdot \delta_\omega \cdot \mathrm{Ad}(u^*),$$

where, by Lemma 30 (iii), we may assume that $u \in \mathscr{G} \cap \mathscr{G}_1$.

Now suppose that $(\beta_{\omega_1}, \delta_{\omega_1})$ and $(\beta_{\omega_2}, \delta_{\omega_2})$ are equivalent. We must show that ω_1 and ω_2 are weakly equivalent. By Lemma 23, we may assume without loss of generality that

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$$\beta_{\omega_1} = \beta_{\omega_2},$$
$$\delta_{\omega_1} = \delta_{\omega_2}.$$

Let $\omega = \omega_1\omega_2^*$. The two equations above imply that ω is symmetric ($\sigma\omega = \omega$) and commutes with $\delta_G(\mathscr{R}(G))$. The latter condition permits us to combine the cocycle identities for ω_1 and ω_2 and deduce that ω is itself a normalised cocycle. Since it is symmetric it must be trivial by Lemma 21, so that

$$\omega = \delta_G(v^*)(v \otimes v) \quad \text{for some } v \in \mathscr{G}.$$

Since ω commutes with $\delta_G(g)$ for each $g \in G$, we have

$$\delta_G(v^*)(v \otimes v)\rho(g) \otimes \rho(g) = \rho(g) \otimes \rho(g)\delta(v^*)(v \otimes v).$$

Let $u = v\rho(g)v^*$. Then the above equation may be rewritten

$$\delta_G(u) = u \otimes u$$

and this is a necessary and sufficient condition for u to lie in $\rho(G)$. Thus $v\rho(G)v^* \subseteq \rho(G)$, so by Lemma 28 (2) we see that v belongs to $\mathscr{G} \cap \mathscr{G}_1$. Finally we obtain

$$\omega_1 = \delta_G(v^*)(v \otimes v)\omega_2 = \delta_G(v^*)[\,(v \otimes v)\omega_2(v^* \otimes v^*)\,](v \otimes v),$$

which demonstrates the weak equivalence of ω_1 and ω_2.

Our last theorem gives various equivalent criteria for $\pi_\omega(\hat{G})''$ to be a factor, analogous to those stated at the beginning of this section for Abelian groups. We need one final ingredient to state the result, namely an analogue of the map $\Lambda:\hat{G} \to G$. In fact we (dually) define a map

$$\Lambda:A_\omega(G) \to \mathscr{R}(G)$$

by $\Lambda(\phi) = (\phi \otimes \mathrm{id})\beta$ so that Λ is specified by the equation

$$\langle \Lambda(\phi), \xi \rangle = \langle \phi \otimes \xi, \beta \rangle \quad (\phi \in A_\omega(G),\ \xi \in \mathscr{R}(G)_*).$$

In view of the bicharacter relations for β, the following lemma and its proof are very close in spirit to Theorem 8 and its proof.

LEMMA 32. (i) Λ *is a norm continuous* $*$*-homomorphism of* $A_\omega(G)$ *into* $\mathscr{R}(G)$.

(ii) $\ker(\Lambda) = (\mathrm{im}(\Lambda))^\perp$ *in* $\mathscr{R}(G)_*$.

Proof. (i) Let $\xi, \eta \in A_\omega(G)$, $\phi \in \mathscr{R}(G)_*$. Then

$$\begin{aligned} \langle \Lambda(\xi \star \eta), \phi \rangle &= \langle \xi \otimes \eta \otimes \phi, \delta_\omega \otimes \iota(\beta) \rangle \\ &= \langle \xi \otimes \eta \otimes \phi, \beta_{13}\beta_{23} \rangle \\ &= \langle \xi \otimes \phi, \beta(I \otimes \Lambda(\eta)) \rangle \\ &= \langle \Lambda(\xi), \Lambda(\eta)\phi \rangle \\ &= \langle \Lambda(\xi)\Lambda(\eta), \phi \rangle, \end{aligned}$$

so that $\Lambda(\xi \star \eta) = \Lambda(\xi)\Lambda(\eta)$. Furthermore

$$\begin{aligned} \langle \Lambda(\epsilon), \phi \rangle &= \langle \epsilon \otimes \phi, \beta \rangle = \langle \epsilon \otimes \phi, \beta(e_1 \otimes I) \rangle \\ &= \langle \epsilon \otimes \phi, e_1 \otimes I \rangle = \phi(I) \end{aligned}$$

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so that $\Lambda(\epsilon) = I$. The estimate $\|\Lambda(\xi)\| \leqq \|\xi\|$ is immediate from the unitarity of β. Finally we check the formula $\Lambda(\xi^+) = \Lambda(\xi)^*$. In fact

$$\begin{aligned}\langle \Lambda(\xi^+), \phi\rangle &= \langle \xi^+ \otimes \phi, \beta\rangle = \langle \xi^* \otimes \phi, \beta^*\rangle \\ &= \overline{\langle \Lambda(\xi), \phi^*\rangle} = \langle \Lambda(\xi)^*, \phi\rangle\end{aligned}$$

where we have used the identity $\alpha \otimes \iota(\beta) = \beta^*$ of Lemma 26 (ii).

(ii) By definition

$$\ker(\Lambda) = \{\phi \in \mathscr{R}(G)_* : \langle \phi \otimes \xi, \beta\rangle = 0 \text{ for all } \xi \in \mathscr{R}(G)_*\},$$

$$(\mathrm{im}(\Lambda))^{\perp} = \{\xi \in \mathscr{R}(G)_* : \langle \phi \otimes \xi, \beta\rangle = 0 \text{ for all } \phi \in \mathscr{R}(G)_*\}.$$

But the relations $\alpha\beta = \beta$, $\sigma\beta = \beta^*$ then permit $(\mathrm{im}(\Lambda))^{\perp}$ to be identified with $(\ker(\Lambda))^+$. Since $(\ker(\Lambda))^+ = \ker(\Lambda)$ from (i), the result follows.

In view of this lemma, we get a family of finite-dimensional representations of $A_\omega(G)$ by composing Λ with the representations $\mathscr{R}(G) \to \mathrm{End}(V_\pi)$ ($\pi \in \hat{G}$). Each of these representations is clearly equivalent to the compression of $\mathscr{R}(G)$ by a central projection e_π corresponding to the $\mathrm{End}(V_\pi)$ component of $\mathscr{R}(G)$. In particular, e_1 yields what we shall refer to as the *trivial* representation of $A_\omega(G)$. It is easily verified that $\Lambda(\phi)e_1 = \phi(I)e_1$ so that this is essentially the character $\phi \mapsto \phi(I)$ of $A_\omega(G)$. In general a projection $p \in \mathscr{R}(G)$ will be invariant under Λ provided that

$$\Lambda(\phi)p = p\Lambda(\phi) \text{ for all } \phi \in A_\omega(G).$$

This condition is equivalent to the condition that $I \otimes p$ and β commute; applying the flip σ, we see that this is in turn equivalent to the condition that $p \otimes I$ and β commute. In these circumstances we shall say that the restriction on Λ to p is *trivial* if this restriction is equivalent to a direct sum of copies of the trivial representation, that is $\Lambda(\phi)p = \phi(I)p$ for all ϕ; or equivalently if either $(I \otimes p)\beta = I \otimes p$ or $(p \otimes I)\beta = p \otimes I$.

THEOREM 12. *Let ω be a normalised cocycle of $\hat{G}$.* Then the following conditions are equivalent.

(A_1) *$\pi_\omega(\hat{G})''$ is a (finite) factor.*

(A_2) *$C^*_\omega(\hat{G})$ (or $L^1_\omega(\hat{G})$) has trivial centre.*

(A_3) *$C^*_\omega(\hat{G})$ is simple.*

(B_1) *$C^*_\omega(\hat{G})$ (or $L^1_\omega(\hat{G})$) has a unique normalised trace.*

(B_2) *$(x \otimes I)\beta = x \otimes I$ implies that x is a scalar multiple of e_1.*

(C_1) *The image of Λ is ultraweakly dense in $\mathscr{R}(G)$.*

(C_2) *Λ is injective.*

Proof. We start by establishing that each group of conditions A, B, C is equivalent. Then we prove the easy implications C $\Rightarrow$ B and B $\Rightarrow$ A. Finally we reverse both of these implications.

(1) $A_1 \Leftrightarrow A_2$. We note that the G-finite elements of both $\pi_\omega(\hat{G})''$ and $\pi_\omega(C^*_\omega(\hat{G}))$ are contained in $\pi_\omega(L^1_\omega(\hat{G}))$. Thus the G-finite elements in their

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centres must also lie in $\pi_\omega(L^1_\omega(\hat{G}))$ and hence their centres are the closure of the centre of $\pi_\omega L^1_\omega(\hat{G}))$ in the appropriate topologies. This proves the equivalence of A_1 and A_2.

(2) $A_1 \Leftrightarrow A_3$. If $\pi_\omega(\hat{G})''$ is a factor, then $C^*_\omega(\hat{G})$ is primitive since π_ω will provide a faithful factor representation by Theorem 10(2). So $C^*_\omega(\hat{G})$ must be simple by the corollary to Lemma 24. Conversely if $C^*_\omega(\hat{G})$ is simple, it must have trivial centre.

(3) $B_1 \Leftrightarrow B_2$. To establish this equivalence we shall need the following characterisation of the traces of $L^1_\omega(\hat{G})$.

LEMMA 33. (i) *The set of traces on* $L^1_\omega(\hat{G})$ *may be identified with*

$$L = \{x \in \mathscr{R}(G) : (x \otimes I)\beta = x \otimes I\}$$

where $\phi \mapsto \phi(x)$ *is the trace corresponding to* $x \in L$.

(ii) L *is a ultraweakly closed left ideal of* $\mathscr{R}(G)$, *so has the form* $L = \mathscr{R}(G)p$ *for some unique (self-adjoint) projection* p *in* $\mathscr{R}(G)$.

(iii) p *is the largest projection in* $\mathscr{R}(G)$ *invariant under* Λ *such that the restriction of* Λ *to* p *is trivial. In particular* $e_1 \leqq p$.

Proof. Since $L^1_\omega(\hat{G}) = \mathscr{R}(G)_*$ as a Banach space, its dual space may be identified with $\mathscr{R}(G)$ in the obvious way. So a trace $x \in \mathscr{R}(G)$ must satisfy

$$\langle \phi \circ_\omega \psi, x\rangle = \langle \psi \circ_\omega \phi, x\rangle \quad (\phi, \psi \in \mathscr{R}(G)_*),$$

that is

$$\langle \phi \otimes \psi, \delta_G(x)\omega\rangle = \langle \psi \otimes \phi, \delta_G(x)\omega\rangle = \langle \phi \otimes \psi, \delta_G(x)\sigma\omega\rangle.$$

Thus x is a trace if and only if $\delta_G(x)\omega = \delta_G(x)\sigma\omega$. Now we recall that

$$\delta_G(x) = W^*_G(x \otimes I)W_G.$$

So we find that x is a trace if and only if

$$(x \otimes I)W_{\sigma\omega} = (x \otimes I)W_\omega.$$

Applying $\iota \otimes \alpha$ to this relation and recalling that both W_ω and $W_{\sigma\omega}$ are normalised, we see that this is in turn equivalent to the condition that

$$(x \otimes I)W^*_{\sigma\omega} = (x \otimes I)W^*_\omega.$$

We may cancel W_G from this equation to obtain

$$(x \otimes I)\omega^* = (x \otimes I)\sigma\omega^*$$

which may be rewritten as $(x \otimes I)\beta = x \otimes I$. Thus L is indeed the set of traces. The remaining assertions are now immediate, bearing in mind the remarks after Lemma 32.

The equivalence of B_1 and B_2 is now clear, since $L^1_\omega(\hat{G})$ has a unique normalised trace if and only if $p = e_1$ which occurs precisely when $L = \mathbf{C}e_1$.

(4) $C_1 \Leftrightarrow C_2$. The equivalence of C_1 and C_2 is an immediate consequence of Lemma 32 (ii).

(5) $C \Rightarrow B$. Suppose that $\Lambda(L^1_\omega(\hat{G}))$ is ultraweakly dense in $\mathscr{R}(G)$. Since

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the projection p of Lemma 33 is invariant and the restriction of $\Lambda(L^1_\omega(\hat{G}))$ to p is trivial, it follows that p is actually invariant under $\mathscr{R}(G)$ and the restriction of $\mathscr{R}(G)$ to p is trivial. Hence $p = e_1$, so that $L^1_\omega(\hat{G})$ has a unique normalised trace.

(6) B $\Rightarrow$ A. Suppose that Tr is the unique normalised trace on $L^1_\omega(\hat{G})$. Then if ζ were a non-trivial central element in $L^1_\omega(\hat{G})$, $\phi \mapsto \mathrm{Tr}(\phi \circ \zeta)$ would be a trace on $L^1_\omega(\hat{G})$ which was not a multiple of Tr. Hence the centre of $L^1_\omega(\hat{G})$ must be trivial.

(7) A $\Rightarrow$ B. Let p be the projection generating the left ideal L. Thus

$$(p \otimes I)\beta = p \otimes I = \beta(p \otimes I),$$

so that

$$\omega(p \otimes I) = \sigma\omega(p \otimes I).$$

From this it follows that if $\phi, \psi \in \mathscr{R}(G)_*$, $x \in \mathscr{R}(G)$ then

$$\begin{aligned} \langle p \cdot \phi \otimes \psi, \delta_G(x)\omega \rangle &= \langle \phi \otimes \psi, \delta_G(x)\omega(p \otimes I) \rangle \\ &= \langle \phi \otimes \psi, \delta_G(x)\sigma\omega(p \otimes I) \rangle \\ &= \langle \psi \otimes p \cdot \phi, \delta_G(x)\omega \rangle \end{aligned}$$

so that $(p \cdot \phi) \circ \psi = \psi \circ (p \cdot \phi)$ in $L^1_\omega(\hat{G})$ and $p \cdot \phi$ is central. Hence $p \cdot \phi$ must be some multiple of ϵ for each $\phi \in \mathscr{R}(G)_*$. On the other hand, the natural map

$$\phi \mapsto \tilde{\phi}, \quad \mathscr{R}(G)_* \to L^2(G)$$

carries the left module action of $\mathscr{R}(G)$ on $\mathscr{R}(G)_*$ into the right regular representation of G (and $\mathscr{R}(G)$) on $L^2(G)$, since

$$\rho(g) \tilde{\ } \phi(x) = \langle \rho(g)\phi, \rho(x) \rangle = \tilde{\phi}(xg).$$

From this we conclude that $p = e_1$.

(8) B $\Rightarrow$ C. This is by far the least straightforward implication to prove; we need to have some substitute for the fact that in the Abelian case the map $\Lambda:\hat{G} \to G$ is a group homomorphism. Such a substitute is provided by the existence of a tensor product operation on representations of $\mathscr{R}(G)$ induced by the comultiplication δ_ω. We first verify that the ultraweak closure

$$\mathscr{R}_1 = \Lambda(L^1_\omega(\hat{G}))''$$

of the image of Λ is a triangular sub-Hopf-von Neumann algebra of $\mathscr{R}(G)$ in the following sense.

LEMMA 34.

(i) $\delta_\omega(\mathscr{R}_1) \subseteq \mathscr{R}_1 \otimes \mathscr{R}_1$.

(ii) $\beta \in \mathscr{R}_1 \otimes \mathscr{R}_1$.

(iii) $\alpha(\Lambda(\phi)) = \Lambda(\alpha\phi)$.

Proof. To prove (i), it suffices to show that $\langle \delta_\omega(x), \phi_1 \otimes \phi_2 \rangle = 0$ for $x \in \mathrm{im}(\Lambda)$ if either ϕ_1 or ϕ_2 lies in $\mathscr{R}_1^\perp$. But

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$$\begin{aligned}\langle \delta_\omega(\Lambda(\phi)), \phi_1 \otimes \phi_2\rangle &= \langle \Lambda(\phi), \phi_1 \star \phi_2\rangle \\ &= \langle \phi \otimes (\phi_1 \star \phi_2), \beta\rangle \\ &= \langle \alpha\phi \otimes \phi_1 \star \phi_2, \sigma\beta\rangle \\ &= \langle \Lambda(\phi_1 \star \phi_2), \alpha\phi\rangle \\ &= \langle \Lambda(\phi_1)\Lambda(\phi_2), \alpha\phi\rangle\end{aligned}$$

so the result follows immediately from the equality $\ker(\Lambda) = (\mathrm{im}(\Lambda))^{\perp}$. Similar reasoning can be used to prove (ii). The proof of (iii) is a straightforward consequence of the identity

$$\iota \otimes \alpha(\beta) = \beta^* = \alpha \otimes \iota(\beta).$$

Now the assumption B, in conjunction with Lemma 33 (iii), implies that the restriction of Λ to e_π or V_π (where $\pi \in \hat{G}$) contains no copies of the trivial representation unless π is itself trivial. To prove C, we must show that $\mathscr{R}_1 = \mathscr{R}(G)$.

In order to motivate our proof, let us consider an analogous, but simpler, situation. Let H be a closed subgroup of G such that $V_\pi|_H$ contains the trivial representation of H only if π is trivial. It is of course immediate by Frobenius' Reciprocity that $H = G$ and that the natural inclusion $\mathscr{R}(H) \hookrightarrow \mathscr{R}(G)$ is in fact onto. Let us outline a Hopf-algebraic proof of this last assertion which will generalise to a proof of C. $\mathscr{R}(H)$ is a Hopf subalgebra of $\mathscr{R}(G)$ with respect to the comultiplication δ_G and antipode α. The assertion that $\mathscr{R}(H) = \mathscr{R}(G)$ will follow provided we can show that the irreducible representations V_π of $\mathscr{R}(G)$ stay irreducible and inequivalent when restricted to $\mathscr{R}(H)$. Now the tensor product $V \otimes W$ of two (normal) representations V, W of $\mathscr{R}(G)$ can be defined via the composition

$$\mathscr{R}(G) \overset{\delta_G}{\rightarrow} \mathscr{R}(G) \otimes \mathscr{R}(G) \rightarrow \mathrm{End}(V) \otimes \mathrm{End}(W) \cong \mathrm{End}(V \otimes W)$$

and this obviously gives a compatible definition for $\mathscr{R}(H)$. The assumption on H implies that e_1 is contained in $\mathscr{R}(H)$, since $V_\pi|_{\mathscr{R}(H)}$ does not contain the trivial representation unless π is trivial. Suppose that $p \in \mathrm{End}(V_\pi) \subseteq \mathscr{R}(G)$ is a projection onto an $\mathscr{R}(H)$-invariant subspace of V_π and let us consider $V_\pi \otimes V_{\bar{\pi}}$. This contains the trivial representation of G exactly once and it corresponds to the rank one projection

$$e = \delta_G(e_1)(e_\pi \otimes e_{\bar{\pi}}) \quad \text{in } \mathrm{End}(V_\pi) \otimes \mathrm{End}(V_{\bar{\pi}}).$$

Now $p \otimes e_{\bar{\pi}}$ lies in $\mathrm{End}(V_\pi) \otimes \mathrm{End}(V_{\bar{\pi}})$ and commutes with e since $e_1 \in \mathscr{R}(H)$. Hence the minimality of e forces $(p \otimes e_{\bar{\pi}})e = 0$ or e. Since $\alpha(e_{\bar{\pi}}) = e_\pi$, we see from Lemma 9 that

$$\delta_G(e_1)(p \otimes I) = 0 \text{ or } \delta_G(e_1)(e_\pi \otimes I)$$

and hence $p = 0$ or e_π. Consequently the restriction of V_π to $\mathscr{R}(H)$ is irreducible. Finally if V_π and V_σ are isomorphic as $\mathscr{R}(H)$-modules, then $V_\pi \otimes V_{\bar{\sigma}}$ and $V_\sigma \otimes V_{\bar{\sigma}}$ are isomorphic as $\mathscr{R}(H)$-modules and therefore the

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former contains a copy of the trivial representation of $\mathscr{R}(H)$ and hence of $\mathscr{R}(G)$. Thus V_π and V_σ must be isomorphic as $\mathscr{R}(G)$-modules.

We shall now repeat the above arguments with $(\mathscr{R}(G), \mathscr{R}(H), \delta_G)$ replaced by $(\mathscr{R}(G), \mathscr{R}_1, \delta_\omega)$. We first observe that, since $V_\pi|_{\mathscr{R}_1}$ does not contain the trivial representation unless π is itself trivial, e_1 must lie in $\mathscr{R}_1$ and its image yields the projection onto the trivial subrepresentation of any (normal) representation of $\mathscr{R}_1$. Given two representations V, W of $\mathscr{R}(G)$ we shall define their ω-tensor product $V \otimes_\omega W$ as $V \otimes W$ with the $\mathscr{R}(G)$-module structure induced by the composition

$$\mathscr{R}(G) \xrightarrow{\delta_\omega} \mathscr{R}(G) \otimes \mathscr{R}(G) \rightarrow \mathrm{End}(V) \otimes \mathrm{End}(W) \cong \mathrm{End}(V \otimes W)$$

with a similar (compatible) definition for representations of $\mathscr{R}_1$. (Note that it is necessary to introduce conjugation by β to show that $V \otimes_\omega W$ and $W \otimes_\omega V$ are isomorphic as $\mathscr{R}(G)$-modules; a similar argument applies for $\mathscr{R}_1$-modules in view of Lemma 34 (ii).) The fact that ω is normalised implies that $\delta_\omega(e_1) = \delta_G(e_1)$; so it follows that, since the projection onto the trivial submodule of $V_\pi \otimes_\omega V_\sigma$ is given by the image of $\delta_\omega(e_1)$, the module $V_\pi \otimes_\omega V_\sigma$ contains a copy of the trivial representation of $\mathscr{R}(G)$ if and only if $\pi = \bar{\sigma}$ and then only with multiplicity one.

We now show that $V_\pi|_{\mathscr{R}_1}$ is irreducible. Let $p \in \mathrm{End}(V_\pi) \subseteq \mathscr{R}(G)$ be a projection onto an $\mathscr{R}_1$-invariant subspace of V_π. Then $p \otimes e_{\bar{\pi}} \in \mathrm{End}(V_\pi \otimes V_{\bar{\pi}})$ is $\mathscr{R}_1$-invariant. On the other hand,

$$e = \delta_\omega(e_1)(e_\pi \otimes e_{\bar{\pi}}) = \delta_G(e_1)(e_\pi \otimes e_{\bar{\pi}})$$

is a rank one projection in $\mathrm{End}(V_\pi \otimes V_{\bar{\pi}})$ contained in the image of $\mathscr{R}_1$. Hence it commutes with $p \otimes e_{\bar{\pi}}$ and its minimality forces $e(p \otimes e_{\bar{\pi}}) = 0$ or e. As before Lemma 9 allows us to conclude that $p = 0$ or e_π, so that V_π is indeed irreducible as an $\mathscr{R}_1$-module.

Finally, it is clear that if V_1, V_2, U are $\mathscr{R}(G)$-modules and if V_1 and V_2 are isomorphic as $\mathscr{R}_1$-modules, then $V_1 \otimes_\omega U$ and $V_2 \otimes_\omega U$ are isomorphic as $\mathscr{R}_1$-modules. In particular if V_π and V_σ are isomorphic as $\mathscr{R}_1$-modules, we have as before that $V_\pi \otimes_\omega V_{\bar{\sigma}}$ and are isomorphic as $\mathscr{R}_1$-modules. Since the latter contains a trivial representation of $\mathscr{R}(G)$, it contains a trivial representation of $\mathscr{R}_1$; hence the former contains a trivial representation $\mathscr{R}_1$ and hence of $\mathscr{R}(G)$. But then by our previous remarks V_π and V_σ must be isomorphic as $\mathscr{R}(G)$-modules, as required.

This completes the proof of Theorem 12.

Having established what happens in the factorial case, it is fairly natural to ask what can be said when $\pi_\omega(\hat{G})''$ fails to be a factor. In fact it follows from the corollary to Theorem 7 of **[17]** that the ergodic action on $\pi_\omega(\hat{G})''$ is induced from a full multiplicity action of a closed subgroup H of G on a factor. In particular this means that there is a non-degenerate normalised cocycle ω_0 for $\hat{H}$ in $\mathscr{R}(H) \otimes \mathscr{R}(H) \subseteq \mathscr{R}(G) \otimes \mathscr{R}(G)$ and a unitary $v \in \mathscr{G}$ such that $\omega = \omega_0^v$. It is a fairly easy consequence of Theorem 12 that the subalgebra $\mathscr{R}_1 = \Lambda(A_\omega(G))''$ of $\mathscr{R}(G)$ is then just the algebra $v^*\mathscr{R}(H)v$.

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Thus the apparently simpler case discussed in the preamble to part (7) of the proof of Theorem 12 is in fact very close to the actual case to be dealt with.

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