Metric Spaces Admitting Low-distortion Embeddings into All n-dimensional Banach Spaces

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Abstract. For a fixed $K \gg 1$ and $n \in \mathbb{N}$, $n \gg 1$, we study metric spaces which admit embeddings with distortion $\leq K$ into each n-dimensional Banach space. Classical examples include spaces embeddable into log n-dimensional Euclidean spaces, and equilateral spaces.

We prove that good embeddability properties are preserved under the operation of metric composition of metric spaces. In particular, we prove that n-point ultrametrics can be embedded with uniformly bounded distortions into arbitrary Banach spaces of dimension log n.

The main result of the paper is a new example of a family of finite metric spaces which are not metric compositions of classical examples and which do embed with uniformly bounded distortion into any Banach space of dimension n. This partially answers a question of G. Schechtman.

1 Introduction

This paper is devoted to the following problem suggested by Gideon Schechtman during the Workshop in Analysis and Probability at Texas A&M University, College Station, Texas, July 2013.

Problem 1.1 Fix a constant $K \gg 1$ and $n \in \mathbb{N}$ satisfying $n \gg 1$. Characterize all metric spaces admitting embeddings with distortion $\leq K$ into each n-dimensional Banach space.

The *distortion* of an (injective) embedding $f \colon X \to Y$ of a metric space (X, d_X) into a metric space (Y, d_Y) is defined by

$$\operatorname{dist}(f) = \sup_{\substack{x, y \in X \\ x \neq y}} \frac{d_Y\big(f(x), f(y)\big)}{d_X(x, y)} \cdot \sup_{\substack{x, y \in X \\ x \neq y}} \frac{d_X(x, y)}{d_Y\big(f(x), f(y)\big)}.$$

If $\operatorname{dist}(f) \leq K$, we say that f is a *K-embedding*, and that a metric space (X, d_X) is *K-embeddable* into a metric space (Y, d_Y).

Problem 1.1 can be viewed as a part of modern Ramsey theory which seeks to characterize types of structures which can be found inside arbitrary structures that are sufficiently large. In the category of metric spaces in relation to their embeddability

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into Banach spaces, this work was initiated in [BFM86]. See [BLMN05, MN13], and references therein for important Ramsey-type results in the category of metric spaces.

Since Problem 1.1 is vague and somewhat unrealistic, Schechtman also suggested a more specific Problem 1.2 (see below). Before stating it we need to list known examples of metric spaces embeddable into each n-dimensional Banach space:

(A) Metric spaces admitting low-distortion embeddings into $\log n$-dimensional Euclidean spaces. This class of examples is obtained as a corollary of the fundamental Dvoretzky theorem (see Theorem 2.1) which implies that for every $M > 1$ there exists $s(M) > 0$ such that for each $n \in \mathbb{N}$ the space ℓ_2^k with $k \le s(M) \ln n$ can be linearly embedded into any n-dimensional Banach space with distortion $\le M$. It follows that any metric space which embeds with distortion $\le K/M$ into such ℓ_2^k can be embedded with distortion $\le K$ into any n-dimensional Banach space. See Section 2.1 for more details.

(B) A metric space is called *equilateral* if the distances between all pairs of distinct points in it are equal to the same positive number. An equilateral space is also called an *equilateral set*. Equilateral spaces of size $\le a^n$, where a depends on K, form another class of metric spaces satisfying the conditions of Problem 1.1. In Section 2.2 we describe the known estimates for the distortion of embeddings of equilateral spaces and give a simple direct proof of the estimates that we will use for our results.

(C) Metric spaces from the classes mentioned in (A) and (B) can be combined using a general construction, called *metric composition* (see Definition 2.8), which was introduced in [BLMN05]. In Section 2.3 we present a detailed study of embeddability properties of metric compositions of metric spaces. We prove that for a suitable choice of parameters, the metric composition of metric spaces which embed well into a given Banach space E, also embeds well into E (Theorem 2.9, Corollary 2.13). Applying this general construction to examples described in (A) and (B) we get more examples of metric spaces satisfying the conditions of Problem 1.1. In particular ultrametrics (see Section 2.4) can be obtained as metric compositions of equilateral sets described in (B). Thus, as a corollary of our results, we obtain that ultrametrics of exponential size embed with uniformly bounded distortion into any n-dimensional Banach space. We also provide a direct proof of this fact (see Proposition 2.15 and Corollary 2.16).

Problem 1.2 (Schechtman)   Can one suggest examples satisfying the condition of Problem 1.1 which are completely different from the ones mentioned in (A)–(C)?

The main goal of this paper is to give an answer to this problem, that is, to present an example of a family of metric spaces that satisfy the condition of Problem 1.1, but do not belong to any of the classes (A)–(C).

In Section 3 we show an example of a family of graphs, that we call *weighted diamonds W_n*, which we prove do not arise from any of the discussed above examples, and which embed with uniformly bounded distortion into any Banach space of dimension at least $\exp(c(\log \log |W_n|)^2)$ for a suitably chosen $c > 0$. The family $\{W_n\}$ was first constructed in [Ost14], as an example of a family of topologically complicated series-parallel graphs that embed with uniformly bounded distortion into ℓ_2 (of infinite dimension). In the present paper we show that W_n's are snowflaked versions of

standard diamonds D_n's. However D_n's are not uniformly doubling so the embeddability results for W_n proved in [Ost14] and in the present paper do not follow from Assouad's theorem (see Remark 3.11). Our method of proof of the embeddability of W_n's uses a mixture of δ-net arguments and some "linear" manipulations.

In Section 4 we show a more general construction of "hierarchically built weighted graphs" which can have topologically very complicated structures and which embed with uniformly bounded distortion into any Banach space of specified dimension (see Theorem 4.3).

2 Low-dimensional Euclidean Subsets, Equilateral Spaces, and Their Combinations

2.1 Low-dimensional Euclidean Subsets

We recall the following improvement of the Dvoretzky theorem [Dvo61], which is due to Milman [Mil71] (we state it in a somewhat unconventional way).

Theorem 2.1 *For each $M \in (1, \infty)$ there exists $s(M) \in (0, \infty)$ such that any n-dimensional Banach space X contains a k-dimensional subspace X_k with $k \geq s(M) \ln n$, such that X_k is linearly isomorphic with ℓ_2^k with distortion $\leq M$.*

We assume that $s(M)$ is chosen in the optimal way. Let $M \leq K$. Theorem 2.1 immediately implies that any metric space which admits an embedding into ℓ_2^k, where $k = \lceil s(M) \ln n \rceil$, with distortion $\leq \frac{K}{M}$, also admits an embedding with distortion $\leq K$ into any n-dimensional Banach space.

This relates our study with the following major open problem of the theory of metric embeddings: *Find an intrinsic characterization of those separable metric spaces (X, d_X) that admit bilipschitz embeddings into ℓ_2^n for some $n \in \mathbb{N}$.* See [Sem99, LP01, Hei03] for a discussion of this problem. One of the most important results on this problem is the Assouad theorem, which we mention below (Theorem 3.9). However, it should be mentioned that in the present context we are interested in the version of the problem for which the dimension is specified. See very interesting recent results related to this problem in [NN12, DS13], and related comments in [Hei03, Remark 3.16].

2.2 Equilateral Spaces

It follows from standard volumetric estimates that the maximal δ-separated set for $\delta > 0$ contained in the unit ball of any n-dimensional Banach space has cardinality at least δ^{-n}. It is also easy to see that a bijection between any δ-separated set in the unit ball and an equilateral space of the same cardinality has distortion $\leq (2/\delta)$. Therefore an equilateral space of size $\leq (\frac{K}{2})^n$ admits an embedding with distortion $\leq K$ into an arbitrary n-dimensional Banach space.

A standard volumetric estimate also gives an upper bound on the cardinality of an equilateral space which can be embedded with distortion at most K in an n-dimensional Euclidean space.

Lemma 2.2 Let $\{a_i\}_{i=1}^N$ be an equilateral space which is K-embeddable into an n-dimensional Banach space. Then $N \leq (2K+1)^n$.

Proof Let $\{a_i\}_{i=1}^N$ be equilateral with distances equal to 1, and $\varphi: \{a_i\}_{i=1}^N \to X$ be a K-embedding. We may assume that there is $h > 0$ such that

$$h \leq \|\varphi(a_i) - \varphi(a_j)\| \leq Kh.$$

Therefore $N\left(\frac{h}{2}\right)^n < \left(Kh + \frac{h}{2}\right)^n$ and $N \leq (2K+1)^n$. ∎

The most precise, known, estimates for the distortion of an embedding of exponential size equilateral spaces in any Banach space can be obtained using the following result of Arias-de-Reyna, Ball, and Villa [ABV98] (an earlier version of this result was proved by Bourgain, see [FL94, Theorem 4.3], using Milman's [Mil85] quotient of subspace theorem).

Theorem 2.3 ([ABV98]) Let $\varepsilon > 0$, X be an n-dimensional Banach space, B its closed unit ball, and μ, the Lebesgue measure on B normalized so that $\mu(B) = 1$. If $t = \sqrt{2}\,(1 - \varepsilon)$, then

$$\mu \otimes \mu\{(x, y) \in B \times B : \|x - y\| \leq t\} \leq \left(1 - \varepsilon^2(2 - \varepsilon)^2\right)^{\frac{n}{2}}.$$

Theorem 2.3 implies that the set L of points x in B for which

$$\mu\{y \in B : \|x - y\| \leq t\} \leq 2\left(1 - \varepsilon^2(2 - \varepsilon)^2\right)^{\frac{n}{2}},$$

satisfies $\mu(L) \geq \frac{1}{2}$. Choosing t-separated points in the set L, one by one, we get a t-separated set of cardinality at least

$$\frac{1}{4\left(1 - \varepsilon^2(2 - \varepsilon)^2\right)^{\frac{n}{2}}}.$$

This implies the following corollary.

Corollary 2.4 For every $s > \sqrt{2}$ there exists $C(s)$ $\left(= \ln(s^2/2\sqrt{s^2 - 1})\right)$ so that an equilateral set of size $\frac{1}{4}\exp(C(s)n)$ embeds in any n-dimensional Banach space X with distortion $\leq s$.

It is a long standing open problem whether $\sqrt{2}$ in Corollary 2.4 can be replaced by 1.

Problem 2.5 Does there exist a function $C:(0,1) \to (0,\infty)$ such that for each $\varepsilon \in (0,1)$ and each $n \in \mathbb{N}$, an equilateral space of size $\exp(C(\varepsilon)n)$ embeds in any n-dimensional Banach space X with distortion $\leq 1 + \varepsilon$?

The answer is known to be positive for Banach space X with a 1-subsymmetric basis [BBK89], for uniformly convex Banach spaces [ABV98] (the function $C(\varepsilon)$ depends on the modulus of convexity), and in some other cases, see [BB91, BPS95].

In the sequel we will use bilipschitz embeddings of equilateral sets into unit spheres of finite-dimensional Banach spaces. For completeness we include a simple proof with the specific constants that we will use.

Lemma 2.6 *There exists a constant $\delta \geq \frac{1}{16}$, so that for every $m \geq 1$ the unit sphere of every m-dimensional Banach space X contains $2 \cdot 4^{m-1}$ elements of mutual distance at least δ.*

Proof Let $\delta > 0$ and T be the maximal δ-separated set on S_X. Then (2δ)-balls centered at T cover the set of points between the spheres of radius $(1+\delta)$ and $(1-\delta)$. Therefore the cardinality $|T|$ of T satisfies

$$|T| \geq \frac{(1+\delta)^m - (1-\delta)^m}{(2\delta)^m}.$$

So we need δ such that the following inequality holds for all $m \geq 1$:

$$\frac{1}{2}(8\delta)^m \leq (1+\delta)^m - (1-\delta)^m.$$

Let $\delta = \frac{1}{16}$. Since the right-hand side is $\geq 2m\delta$, the conclusion follows for $m \geq 2$. For $m = 1$ the conclusion is obvious. ∎

Corollary 2.7 *If we replace $2 \cdot 4^{m-1}$ by 2^m in the statement of Lemma 2.6, we can take $\delta = 1/8$.*

2.3 Metric Compositions

The following general construction of combining metric spaces was introduced by Bartal, Linial, Mendel and Naor for their study of metric Ramsey-type phenomena [BLMN05].

Definition 2.8 (Metric composition, [BLMN05]) Let M be a finite metric space. Suppose that there is a collection of disjoint finite metric spaces N_x associated with the elements x of M. Let $\mathcal{N} = \{N_x\}_{x \in M}$. For $\beta \geq 1/2$, the β-composition of M and $\mathcal{N}$, denoted by $M_\beta[\mathcal{N}]$, is a metric space on the disjoint union $\dot{\bigcup}_x N_x$. Distances in $M_\beta[\mathcal{N}]$ are denoted d_β and defined as follows. Let $x, y \in M$ and $u \in N_x, v \in N_y$. Then

$$d_\beta(u, v) = \begin{cases} d_{N_x}(u, v) & x = y, \\ \beta\gamma d_M(x, y) & x \neq y, \end{cases}$$

where $\gamma = \frac{\max_{z \in M} \mathrm{diam}(N_z)}{\min_{x \neq y \in M} d_M(x, y)}$. It is easily checked that the choice of the factor $\beta\gamma$ guarantees that d_β is indeed a metric.

We prove that the metric composition preserves embeddability properties of metric spaces in the following sense.

Theorem 2.9 *Let $C, D \geq 1$. Let E be a Banach space and M be a finite metric space which is C-embeddable in E. Let $\mathcal{N} = \{N_x\}_{x \in M}$ be a family of finite metric spaces which are D-embeddable in E. Let $\beta > 2(C+1)$. Then $M_\beta[\mathcal{N}]$ is A-embeddable in E, where $A = \max\left(D, C + \frac{2C(C+1)}{\beta - 2(C+1)}\right)$.*

In particular, if $\varepsilon > 0$, $D \leq C$, and $\beta > 2\left(\frac{C+\varepsilon}{\varepsilon}\right)(C+1)$, then $M_\beta[\mathcal{N}]$ is $(C+\varepsilon)$-embeddable in E.

Proof   Without loss of generality we can assume that

$$(2.1) \qquad \min_{x \neq y \in M} d_M(x, y) = 1.$$

Let $\gamma = \max_{x \in M}(\operatorname{diam} N_x)$. We denote the metric in $M_\beta[\mathcal{N}]$ by d_β. For all $y_1, y_2 \in M_\beta[\mathcal{N}]$ there exist $x_1, x_2 \in M$ so that $y_1 \in N_{x_1}, y_2 \in N_{x_2}$. We have

$$d_\beta(y_1, y_2) = \begin{cases} d_{N_x}(y_1, y_2) & \text{if } x_1 = x_2 = x, \\ \beta \gamma d_M(x_1, x_2) & \text{if } x_1 \neq x_2. \end{cases}$$

By assumption, there exists $\Psi \colon M \to E$ so that for all $x_1, x_2 \in M$,

$$\frac{1}{C} d_M(x_1, x_2) \leq \|\Psi(x_1) - \Psi(x_2)\| \leq d_M(x_1, x_2).$$

Also, for all $x \in M$ there exist $\Phi_x \colon N_x \to E$, so that for all $y, y_1, y_2 \in N_x$

$$(2.2) \qquad \frac{1}{D} d_{N_x}(y_1, y_2) \leq \|\Phi_x(y_1) - \Phi_x(y_2)\| \leq d_{N_x}(y_1, y_2),$$

and $\|\Phi_x(y)\| \leq \operatorname{diam}(N_x)$.

Let $\lambda = \beta - 2$, and define $\Phi \colon M_\beta[\mathcal{N}] \to E$ by $\Phi(y) = \Phi_x(y) + \lambda \gamma \Psi(x)$, if $y \in N_x$ for some $x \in M$.

We claim that Φ is an A-embedding, where $A = \max\left(D, C + \frac{2C(C+1)}{\beta - 2(C+1)} \right)$.

First we consider $y_1, y_2 \in M_\beta[\mathcal{N}]$ so that there exists $x \in M$ with $y_1, y_2 \in N_x$. Then $d_\beta(y_1, y_2) = d_{N_x}(y_1, y_2)$ and

$$\Phi(y_1) - \Phi(y_2) = \Phi_x(y_1) + \lambda \gamma \Psi(x) - (\Phi_x(y_2) + \lambda \gamma \Psi(x)) = \Phi_x(y_1) - \Phi_x(y_2).$$

Thus by (2.2) we have

$$(2.3) \qquad \frac{1}{D} d_\beta(y_1, y_2) \leq \|\Phi(y_1) - \Phi(y_2)\| \leq d_\beta(y_1, y_2).$$

Next we consider $y_1, y_2 \in M_\beta[\mathcal{N}]$ so that $y_1 \in N_{x_1}, y_2 \in N_{x_2}$, for some $x_1, x_2 \in M$, with $x_1 \neq x_2$. Then $d_\beta(y_1, y_2) = \beta \gamma d_M(x_1, x_2)$, and

$$\begin{aligned} \|\Phi(y_1) - \Phi(y_2)\| &= \|\Phi_{x_1}(y_1) + \lambda \gamma \Psi(x_1) - (\Phi_{x_2}(y_2) + \lambda \gamma \Psi(x_2))\| \\ &\leq \|\Phi_{x_1}(y_1)\| + \|\Phi_{x_2}(y_2)\| + \lambda \gamma \|\Psi(x_1) - \Psi(x_2)\| \\ &\leq 2\gamma + \lambda \gamma d_M(x_1, x_2) \stackrel{\text{by } (2.1)}{\leq} (2\gamma + \lambda \gamma) d_M(x_1, x_2) \\ &= \beta \gamma d_M(x_1, x_2) = d_\beta(y_1, y_2). \end{aligned}$$

On the other hand,

$$
\begin{aligned}
\|\Phi(y_1) - \Phi(y_2)\| &= \|\Phi_{x_1}(y_1) + \lambda\gamma\Psi(x_1) - (\Phi_{x_2}(y_2) + \lambda\gamma\Psi(x_2))\| \\
&\geq \lambda\gamma\|\Psi(x_1) - \Psi(x_2)\| - \|\Phi_{x_1}(y_1)\| - \|\Phi_{x_2}(y_2)\| \\
&\geq \frac{1}{C}\lambda\gamma d_M(x_1, x_2) - 2\gamma \overset{\text{by (2.1)}}{\geq} \left(\frac{1}{C}\lambda - 2\right)\gamma d_M(x_1, x_2) \\
&= \left(\frac{1}{C}(\beta - 2) - 2\right)\gamma d_M(x_1, x_2) \\
&= \frac{\beta - 2C - 2}{C\beta}\beta\gamma d_M(x_1, x_2) \\
&= \frac{\beta - 2C - 2}{C\beta}d_\beta(y_1, y_2).
\end{aligned}
$$

This and (2.3) imply that Φ is an A-embedding of $M_\beta[\mathcal{N}]$ into E.

Note that if $\beta > 2\left(\frac{C+\varepsilon}{\varepsilon}\right)(C+1)$, then

$$
\frac{C\beta}{\beta - 2C - 2} = C + \frac{2C(C+1)}{\beta - 2C - 2} < C + \frac{2C(C+1)}{2(1 + \frac{C}{\varepsilon})(C+1) - 2(C+1)} = C + \varepsilon.
$$

Thus, if $D \leq C$, then $A \leq C + \varepsilon$.   ■

Definition 2.10 ([BLMN05])   Given a class $\mathcal{M}$ of finite metric spaces and $\beta \geq 1$, we define its closure $\text{comp}_\beta(\mathcal{M})$ under $\geq \beta$-compositions as the smallest class $\mathcal{C}$ of metric spaces that contains all spaces in $\mathcal{M}$ and satisfies the following condition: let $M \in \mathcal{M}$, $\beta' \geq \beta$, and associate with every $x \in M$ a metric space N_x that is isometric to a space in $\mathcal{C}$. Then the space $M_{\beta'}[\mathcal{N}]$ is also in $\mathcal{C}$.

Note that $\text{comp}_\beta(\mathcal{M})$ can be described as a union of smaller classes which have increasing complexity. More precisely

$$
(2.4) \qquad\qquad\qquad \text{comp}_\beta(\mathcal{M}) = \bigcup_{m=0}^{\infty} \mathcal{C}_m,
$$

where $\mathcal{C}_0 = \mathcal{M}$, and for $m \in \mathbb{N}$, $\mathcal{C}_m$ is the class of metric spaces of the form $M_{\beta'}[\mathcal{N}]$, where $M \in \mathcal{M}$, $\beta' \geq \beta$, and $\mathcal{N} = \{N_x\}_{x \in M}$, where for every $x \in M$ a metric space N_x is isometric to a space in $\bigcup_{i=0}^{m-1} \mathcal{C}_i$.

The operation of composition has the following associativity property.

Proposition 2.11   *Let M be a finite metric space, $\beta_1 \geq 1/2$, $\beta_2 \geq 1$, $\mathcal{N}_1 = \{N_x\}_{x \in M}$ and $\mathcal{N}_2 = \{\tilde{N}_y\}_{y \in M_{\beta_1}(\mathcal{N}_1)}$ be families of finite metric spaces. Then*

$$
\left(M_{\beta_1}(\mathcal{N}_1)\right)_{\beta_2}(\mathcal{N}_2) = M_{\beta_1}(\mathcal{N}_3),
$$

where $\mathcal{N}_3$ is a family of finite metric spaces of the form $(N_x)_{\beta_x}(\{\tilde{N}_y\}_{y \in N_x})$, where $x \in M$ and $\beta_x \geq \beta_2$.

Proof For each $x \in M$, define $N_x^\star$ as the disjoint union of $\tilde{N}_y$ over $y \in N_x$, with the metric inherited from $\left(M_{\beta_1}(\mathcal{N}_1) \right)_{\beta_2}(\mathcal{N}_2)$. Let $\mathcal{N}_3$ be the collection of all $N_x^\star$

$$\mathcal{N}_3 = \{N_x^\star\}_{x \in M} = \{\dot{\bigcup}_{y \in N_x} \tilde{N}_y\}_{x \in M}.$$

Denote the metric in $\left(M_{\beta_1}(\mathcal{N}_1) \right)_{\beta_2}(\mathcal{N}_2)$ by d. For all $z_1, z_2 \in \left(M_{\beta_1}(\mathcal{N}_1) \right)_{\beta_2}(\mathcal{N}_2)$ there exist $y_1, y_2 \in M_{\beta_1}(\mathcal{N}_1)$ so that $z_1 \in \tilde{N}_{y_1}$, $z_2 \in \tilde{N}_{y_2}$, $x_1, x_2 \in M$, and $y_1 \in N_{x_1}$, $y_2 \in N_{x_2}$. We have

$$d(z_1, z_2) = \begin{cases} d_{\tilde{N}_y}(z_1, z_2) & \text{if } z_1, z_2 \in \tilde{N}_y, \\ \beta_2 \gamma_2 d_{N_x}(y_1, y_2) & \text{if } z_1, z_2 \in N_x^\star, y_1 \neq y_2, \\ \beta_2 \gamma_2 \beta_1 \gamma_1 d_M(x_1, x_2) & \text{if } z_1 \in N_{x_1}^\star, z_2 \in N_{x_2}^\star, x_1 \neq x_2, \end{cases}$$

where

$$\gamma_1 = \frac{\max_{x \in M} \operatorname{diam}(N_x)}{\min_{x_1 \neq x_2 \in M} d_M(x_1, x_2)}, \quad \gamma_2 = \frac{\max_{y \in M_{\beta_1}(\mathcal{N}_1)} \operatorname{diam}(\tilde{N}_y)}{\min_{y_1 \neq y_2 \in M_{\beta_1}(\mathcal{N}_1)} d_{M_{\beta_1}(\mathcal{N}_1)}(y_1, y_2)}.$$

We claim that for every $x \in M$, there exists $\beta_x \geq \beta_2$ so that

(2.5) $$N_x^\star = (N_x)_{\beta_x}(\{\tilde{N}_y\}_{y \in N_x}).$$

Indeed, it is enough to observe that for every $x \in M$, $\gamma_x \leq \gamma_2$, where

$$\gamma_x \overset{\text{def}}{=} \frac{\max_{y \in N_x} \operatorname{diam}(\tilde{N}_y)}{\min_{y_1 \neq y_2 \in N_x} d_{N_x}(y_1, y_2)}.$$

To prove that $\left(M_{\beta_1}(\mathcal{N}_1) \right)_{\beta_2}(\mathcal{N}_2) = M_{\beta_1}(\mathcal{N}_3)$, we define

$$\gamma_3 \overset{\text{def}}{=} \frac{\max_{x \in M} \operatorname{diam}(N_x^\star)}{\min_{x_1 \neq x_2 \in M} d_M(x_1, x_2)}.$$

Observe that

$$\max_{x \in M} \operatorname{diam}(N_x^\star) = \max\left(\beta_2 \gamma_2 \max_{x \in M} \operatorname{diam}(N_x), \max_{y \in M_{\beta_1}(\mathcal{N}_1)} \operatorname{diam}(\tilde{N}_y) \right).$$

If $\max_{y \in M_{\beta_1}(\mathcal{N}_1)} \operatorname{diam}(\tilde{N}_y) > \beta_2 \gamma_2 \max_{x \in M} \operatorname{diam}(N_x)$, then by definition of γ_2,

$$\min_{y_1 \neq y_2 \in M_{\beta_1}(\mathcal{N}_1)} d_{M_{\beta_1}(\mathcal{N}_1)}(y_1, y_2) > \beta_2 \max_{x \in M} \operatorname{diam}(N_x),$$

and thus, since $\beta_2 \geq 1$, $\min_{x \in M} \min_{y_1 \neq y_2 \in N_x} d_{N_x}(y_1, y_2) > \max_{x \in M} \operatorname{diam}(N_x)$, which is impossible. Thus $\max_{x \in M} \operatorname{diam}(N_x^\star) = \beta_2 \gamma_2 \max_{x \in M} \operatorname{diam}(N_x)$. Hence,

$$\frac{\beta_1 \beta_2 \gamma_1 \gamma_2}{\gamma_3} = \beta_1,$$

and the proposition is proved. ∎

As an immediate corollary of Proposition 2.11 and (2.5) we obtain.

Corollary 2.12 *Let $\mathcal{M}$ be a family of finite metric spaces and $\beta \geq 1$. Then*

$$\operatorname{comp}_\beta(\operatorname{comp}_\beta(\mathcal{M})) = \operatorname{comp}_\beta(\mathcal{M}).$$

As a consequence of Theorem 2.9 we obtain the following result.

Corollary 2.13 *Let $C \geq 1$. Let E be a Banach space and $\mathcal{M}$ be a family of finite metric spaces which are C-embeddable in E. Let $\beta > 2(C+1)$. Then every space in $\text{comp}_\beta(\mathcal{M})$ is A-embeddable in E, where $A = C + \frac{2C(C+1)}{\beta - 2(C+1)}$.*

In particular, for any $\varepsilon > 0$, if $\beta > 2\left(\frac{C+\varepsilon}{\varepsilon}\right)(C+1)$, then every space in $\text{comp}_\beta(\mathcal{M})$ is $(C+\varepsilon)$-embeddable in E.

Proof We prove this by induction on the level of complexity of spaces in $\text{comp}_\beta(\mathcal{M})$. If $M \in \mathcal{C}_0 = \mathcal{M}$, then by assumption M is C-embeddable in E.

Suppose that for some $m \in \mathbb{N}$, every space in $\bigcup_{i=0}^{m-1} \mathcal{C}_i$ is A-embeddable in E. Let $X \in \mathcal{C}_m$. Then there exist $M \in \mathcal{M}$, $\beta' \geq \beta$, and $\mathcal{N} = \{N_x\}_{x \in M}$, where for every $x \in M$ a metric space N_x is isometric to a space in $\bigcup_{i=0}^{m-1} \mathcal{C}_i$, so that $X = M_{\beta'}[\mathcal{N}]$. By assumption, M is C-embeddable in E, and by inductive hypothesis, every space in $\mathcal{N}$ is A-embeddable in E. Thus by Theorem 2.9, X is B-embeddable in E, where

$$B = \max\left(A, C + \frac{2C(C+1)}{\beta - 2(C+1)} \right) = A. \qquad \blacksquare$$

2.4 Ultrametrics and Hierarchically Well-separated Trees

An *ultrametric* is a metric space (M, d) such that for every $x, y, z \in M$,

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

These spaces appeared in a natural way in the study of p-adic number fields, see [Sch84]. Currently ultrametrics play an important role in many branches of mathematics, see for example [BLMN05, Hug12, MN13], and references therein. It is known that ultrametrics have very good embedding properties, see [Shk04] and its references. In particular, Shkarin [Shk04] proved that for any finite ultrametric (M, d), there exists $m = m(M, d) \in \mathbb{N}$ such that for any Banach space E with $\dim E \geq m$ there exists an isometric embedding of M into E. In this result m is large and depends on M, not only on the cardinality of M. Observe that any isometric embedding of an equilateral space with n points (it is the simplest ultrametric) into a Euclidean space requires dimension $\geq n - 1$. So isometric embeddings of ultrametrics require large dimension. The situation changes if we allow some distortion. Bartal, Linial, Mendel and Naor [BLMN04] proved that there exist constants $C \geq 1$ and $c > 0$ such that any n-point ultrametric C-embeds into ℓ_p^k, for any $k \geq c \ln n$ and any $1 \leq p \leq \infty$. The goal of this section is to prove that there exists a universal constant K such that any n-point ultrametric embeds into any Banach space of dimension $\log_2 n$ with distortion $\leq K$.

It turns out that for embeddability of ultrametrics it is convenient to use the following, more restricted class of metrics.

Definition 2.14 ([Bar96]) For $k \geq 1$, a *k-hierarchically well-separated tree* (k-HST) is a metric space whose elements are the leaves of a rooted tree T. To each vertex $u \in T$ there is associated a label $\Delta(u) \geq 0$ such that $\Delta(u) = 0$ if and only if u is a leaf of T. It is required that if a vertex u is a child of a vertex v, then $\Delta(u) \leq \Delta(v)/k$. The distance between two leaves $x, y \in T$ is defined as $\Delta(\text{lca}(x, y))$, where $\text{lca}(x, y)$ is the

least common ancestor of x and y in T. A k-HST is said to be *exact* if $\Delta(u) = \Delta(v)/k$ for every two internal vertices (that is, neither u nor v is a leaf) where u is a child of v.

First, note that an ultrametric on a finite set and a (finite) 1-HST are identical concepts. Any k-HST is also a 1-HST, *i.e.*, an ultrametric. When we discuss k-HST's, we freely use the tree T as in Definition 2.14, which we refer to as *the tree defining the* HST. An internal vertex in T with out-degree 1 is said to be *degenerate*. If u is nondegenerate, then $\Delta(u)$ is the diameter of the sub-space induced on the subtree rooted by u. Degenerate nodes do not influence the metric on T's leaves; hence we may assume that all internal nodes are nondegenerate (note that this assumption need not hold for *exact* k-HST's).

By [BLMN05, Proposition 3.3], the class of k-HST coincides with $\text{comp}_k(EQ)$, where EQ denotes the class of finite equilateral spaces. Thus, by Corollary 2.7 and Corollary 2.13 there exists $k_0 \geq 2$ so that every k-HST with $k \geq k_0$ admits a bilipschitz embedding into any Banach space X with $\dim X \geq \log_2 D$, where D is the maximal out-degree of a vertex in the tree defining the k-HST, with a uniformly bounded distortion, which generalizes [BLMN04, Proposition 3].

Our next goal is to provide an alternative more direct proof of this result.

Proposition 2.15   *Any k-HST with $k > 17$ admits a bilipschitz embedding into any Banach space X with $\dim X \geq \log_2 D$, where D is the maximal out-degree of a vertex in the tree defining the k-HST, with distortion not exceeding $\frac{16k}{k-17}$.*

Proof   Let v be any of the non-leaf vertices of the tree defining the k-HST. The number of edges $E(v)$ contributing to the out-degree of v is at most D. Let $\delta = 1/8$. By Corollary 2.7 there is an injective map φ_v from $E(v)$ to the δ-net on the unit sphere of X. Combining φ_v for all non-leaf vertices v we get a (no longer injective) map φ from the edge set of the tree to the δ-net.

Now we define the map f on the set of leaves into X. For a leaf ℓ let

$$f(\ell) = \sum_{t \in [r, \overline{\ell}]} \Delta(t) \varphi(t\widetilde{t}),$$

where r is the root, $\overline{\ell}$ is the last non-leaf on the way from r to ℓ, $[r, \overline{\ell}]$ is the path joining r and $\overline{\ell}$ (path is regarded as a set of vertices), $\widetilde{t}$ is the next after t vertex on the path from r to ℓ, and $\Delta(t)$ is the label assigned according to Definition 2.14.

Let us estimate the distortion. Let ℓ_1 and ℓ_2 be two leaves in the tree, v be their least common ancestor, v_1 and v_2 be the first (after v) vertices on the paths $[v, \ell_1]$ and $[v, \ell_2]$, respectively. Then (we use the fact that $\Delta(v) = d(\ell_1, \ell_2)$)

$$\|f(\ell_1) - f(\ell_2)\| = \Big\| \sum_{t \in [v, \overline{\ell}_1]} \Delta(t) \varphi(t\widetilde{t}) - \sum_{t \in [v, \overline{\ell}_2]} \Delta(t) \varphi(t\widetilde{t}) \Big\|$$

$$\geq \Delta(v) \|\varphi(vv_1) - \varphi(vv_2)\| - \sum_{t \in [v_1, \overline{\ell}_1]} \Delta(t) \|\varphi(t\widetilde{t})\| - \sum_{t \in [v_2, \overline{\ell}_2]} \Delta(t) \|\varphi(t\widetilde{t})\|$$

$$\geq \frac{1}{8} \Delta(v) - 2\Delta(v)\Big(\frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots\Big) = d(\ell_1, \ell_2)\Big(\frac{1}{8} - \frac{2}{k-1}\Big).$$

On the other hand,

$$
\begin{aligned}
\|f(\ell_1) - f(\ell_2)\| &= \Big\| \sum_{t \in [v, \overline{\ell}_1]} \Delta(t) \varphi(t\widetilde{t}) - \sum_{t \in [v, \overline{\ell}_2]} \Delta(t) \varphi(t\widetilde{t}) \Big\| \\
&\leq \sum_{t \in [v, \overline{\ell}_1]} \Delta(t) \|\varphi(t\widetilde{t})\| + \sum_{t \in [v, \overline{\ell}_2]} \Delta(t) \|\varphi(t\widetilde{t})\| \\
&\leq 2\Delta(v)\Big(1 + \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots\Big) = d(\ell_1, \ell_2)\frac{2k}{k-1}. \qquad \blacksquare
\end{aligned}
$$

Since for any $k > 1$, any ultrametric is k-bilipschitz equivalent to a k-HST ([Bar99], see also [BLMN05, Lemma 3.5]), we obtain the following corollary of Proposition 2.15 (which is an arbitrary-space version of results of [BLMN04] and [BM04]).

Corollary 2.16   *Any n-point ultrametric embeds with uniformly bounded distortion into any Banach space X with $\dim(X) \geq \log_2 n$.*

Remark 2.17   It is natural to try to achieve distortions arbitrarily close to 1 in Proposition 2.15, provided that k is sufficiently large and the dimension is a sufficiently large multiple of $\log_2 D$. This is what was done for embeddings into ℓ_p in [BLMN04], as a consequence of the fact that n-point equilateral sets can be $(1 + \varepsilon)$-embedded into ℓ_p^k with $k \leq C(\varepsilon) \ln n$. By Corollary 2.13 (or a careful reading of the proof of Proposition 2.15) we obtain the same conclusion in every Banach space that satisfies the condition of Problem 2.5. See Section 2.2 for a list of classes of spaces known to satisfy this condition.

3   A New Example: Weighted Diamond Graphs

Our basic example is the family of weighted diamonds $\{W_n\}_{n=0}^{\infty}$ introduced in [Ost14]. Let us recall the definitions.

Definition 3.1 ([GNRS04])   Diamond graphs $\{D_n\}_{n=0}^{\infty}$ are defined as follows: the *diamond graph* of level 0 is denoted D_0; it contains two vertices joined by an edge. The *diamond graph* D_n is obtained from D_{n-1} as follows. Given an edge $uv \in E(D_{n-1})$, it is replaced by a quadrilateral u, a, v, b, with edges ua, av, vb, bu. (See Figure 3.1 for a sketch of D_2.)

Definition 3.2 ([Ost14])   We pick a number $\varepsilon \in (0, \frac{1}{2})$. The sequence $\{W_n\}_{n=0}^{\infty}$ of *weighted diamonds* is defined in terms of diamonds $\{D_n\}_{n=0}^{\infty}$ as follows (see Figure 3.2 for a sketch of W_2).

- W_0 is the same as D_0. The only edge of D_0 is given the weight 1.
- $W_1 = D_1 \cup W_0$ with edges of D_1 given weights $\left(\frac{1}{2} + \varepsilon\right)$; the weight of the edge of W_0 stays as 1 (as it was in the first step of the construction).
- $W_2 = D_2 \cup W_1$ with edges of D_2 given weights $\left(\frac{1}{2} + \varepsilon\right)^2$; weights of the edges of W_1 stay as they were in the previous step of the construction.

$\vdots$

- $W_n = D_n \cup W_{n-1}$ with edges of D_n given weights $\left(\frac{1}{2} + \varepsilon\right)^n$; weights of the edges of W_{n-1} stay as they were in the previous step of the construction.
- Graphs $\{W_n\}$ are endowed with their shortest path distances which we denote d_W.

Remark 3.3 Observe that the metric of W_n depends on ε although we do not reflect this fact in our notation.
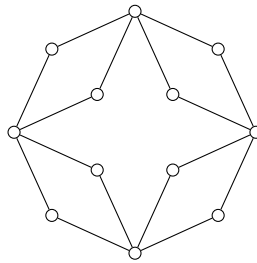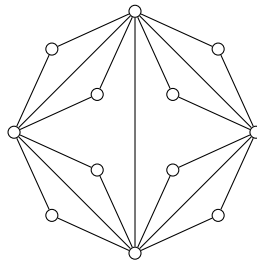

Figure 3.1: Diamond D_2.

Figure 3.2: Graph W_2. The longest edge has weight 1, the shortest edges have weights $\left(\frac{1}{2} + \varepsilon\right)^2$, the edges of intermediate length have weights $\left(\frac{1}{2} + \varepsilon\right)$.

Note that D_k has 4^k edges and that in each step we introduce 2 new vertices of W_k per each edge of D_{k-1}. Hence W_k, $k \geq 1$, has $2(4^{k-1} + 4^{k-2} + \cdots + 1) + 2$ vertices. Thus $\frac{1}{2}4^n \leq |W_n| < 4^n$ for $n \geq 2$ and

(3.1)
$$2n - 1 \leq \log_2 |W_n| < 2n.$$

Using a mixture of δ-net arguments (Lemma 2.6) and some "linear" manipulations we prove that W_n's admit bounded-distortion embeddings into all Banach spaces with dimension bounds which are substantially smaller than the ones implied by the Dvoretzky-type Theorem 2.1. Namely, in Section 3.3 we will prove the following result (Corollary 3.22).

Corollary 3.4   *For every $\varepsilon \in (1/2, 1)$, there exist constants $c, C > 1$ so that for every $n \geq C$, W_n can be embedded in every Banach space X with $\dim X \geq \exp\big(c(\log \log |W_n|)^2 \big)$ with the distortion bounded from above by a constant which depends only on ε.*

Before proving Corollary 3.4 we study the structure of the weighted diamonds W_n. We show that they are snowflaked versions of standard diamonds D_n and that W_n's are not included in the set of examples presented in Section 2.

3.1   Weighted Diamonds Are Bilipschitz-equivalent to Snowflaked Diamonds

The following definition is standard (see [Hei01, p. 98]).

Definition 3.5   Let (X, d_X) be a metric space and $0 < \alpha < 1$. The space X endowed with a modified metric $(d_X(u, v))^\alpha$ is called a *snowflake* of (X, d_X). We also say that (X, d_X^α) is an *α-snowflaked version* of (X, d_X).

One of the standard metrics on diamonds $\{D_k\}_{k=1}^\infty$ is the shortest path distance obtained under the assumption that each edge in D_k has length $\left(\frac{1}{2}\right)^k$. Let us denote this metric d_{D_k}.

Proposition 3.6   *For any $\varepsilon \in (0, 1/2)$ there exists $\alpha \in (0, 1)$ so that the natural identity bijections of (vertex sets of) weighted diamonds $\{W_k\}$ onto (vertex sets of) α-snowflaked versions of diamonds $\{(D_k, d_{D_k})\}$ have uniformly bounded distortions for all $k \in \mathbb{N}$.*

For the proof we need the following fact about the structure of shortest paths in W_n, which was proved in [Ost14, Claim 4.1].

Lemma 3.7 ([Ost14, Claim 4.1])   *A shortest path between two vertices in W_n can contain edges of each possible length:*

$$1, \left(\frac{1}{2} + \varepsilon\right), \left(\frac{1}{2} + \varepsilon\right)^2, \left(\frac{1}{2} + \varepsilon\right)^3, \ldots$$

at most twice. Actually for 1 this can happen only once because there is only one such edge. If there are two longest edges, they are adjacent.

Proof (This proof is a slightly modified version of the proof in [Ost14]; we include it here for the convenience of the readers.) Let e be one of the longest edges in the path and $\left(\frac{1}{2} + \varepsilon\right)^k$ be its length. We assume that $k \geq 1$; the case $k = 0$ can be considered on the same lines, it is even easier.

As for diamonds, we define *weighted subdiamonds* to be subsets of W_n that evolved from an edge (as sets of vertices they coincide with the subdiamonds defined in [JS09, Ost14]). The edge from which a subdiamond evolved is called its *diagonal*.

Consider the subdiamond S containing e with diagonal of length $\left(\frac{1}{2} + \varepsilon\right)^{k-1}$. Let $e = uv$. Without loss of generality we may assume that u is one of the ends of the diagonal of S, denote the other end by t.

The rest of the path consists of two pieces, starting at u and v, respectively. We claim that the part which starts at v can never leave S. It obviously cannot leave through u.

It cannot leave through t, because otherwise the piece of the path between u and t could be replaced by the diagonal of S, which is strictly shorter.

This implies that the part of the path in S which starts at v can contain edges only shorter than $(\frac{1}{2} + \varepsilon)^k$. For the next edge in this part of the path we can repeat the argument and get, by induction, that lengths of edges in the remainder of the path in S are strictly decreasing.

The part of the path which starts at u can be considered similarly. The last statement of the Lemma is immediate from the proof. ∎

Proof of Proposition 3.6 Let $\varepsilon \in (0, 1/2)$ and let $(\frac{1}{2} + \varepsilon)$ be the weight of edges of W_1 which are not in W_0. Pick $\alpha \in (0, 1)$ so that $(\frac{1}{2})^\alpha = (\frac{1}{2} + \varepsilon)$. Since every edge of D_k has length $(\frac{1}{2})^k$, if we raise its length to the power α, we get the length of the same edge in W_k. Therefore (since $d_{D_k}^\alpha$ satisfies the triangle inequality) for any two vertices x, y of D_k (or W_k) we have $(d_{D_k}(x, y))^\alpha \le d_{W_k}(x, y)$.

To get the inequality in the other direction, let uv be the one of the (at most two) longest edges in the shortest xy-path in W_k. We claim that

$$(3.2) \qquad d_{D_k}(x, y) \ge \frac{1}{2} d_{D_k}(u, v).$$

If (3.2) is satisfied, then by Lemma 3.7 and since uv is an edge we have

$$d_{W_k}(x, y) \le 2 d_{W_k}(u, v) \Big(1 + \Big(\frac{1}{2} + \varepsilon \Big) + \Big(\frac{1}{2} + \varepsilon \Big)^2 + \cdots \Big)$$

$$= \frac{2 d_{W_k}(u, v)}{\frac{1}{2} - \varepsilon} = \frac{2}{\frac{1}{2} - \varepsilon} (d_{D_k}(u, v))^\alpha \le \frac{2^{\alpha+1}}{\frac{1}{2} - \varepsilon} (d_{D_k}(x, y))^\alpha$$

$$= \frac{8}{1 - 4\varepsilon^2} (d_{D_k}(x, y))^\alpha.$$

We assume without loss of generality that the shortest xy-path visits u before v. To prove (3.2) we consider three possible cases.

Case 1: Both x and y are contained in the subdiamond S with diagonal uv.
Case 2: Exactly one of x, y is contained in the subdiamond S with the diagonal uv.
Case 3: None of x, y is contained in the subdiamond S with the diagonal uv.

Let $m_0 \in \mathbb{N}$ be the smallest number such that

$$\Big(\frac{1}{2} + \varepsilon \Big) + \Big(\frac{1}{2} + \varepsilon \Big)^2 + \cdots + \Big(\frac{1}{2} + \varepsilon \Big)^{m_0} \ge 1 + \Big(\frac{1}{2} + \varepsilon \Big)^{m_0}.$$

It is clear that $m_0 \ge 3$ if $\varepsilon \in (0, \frac{1}{2})$.

We show that in Case 1 y is contained in one of the subdiamonds of S whose diagonal has length $\le (\frac{1}{2} + \varepsilon)^{m_0-1} d_{W_k}(u, v)$, and v is one of its ends. In fact, otherwise the vy-path in W_k, which is a part of the shortest xy-path which we consider, contains a vertex z of the subdiamond S for which zv is an edge of W_k satisfying

$$d_{W_k}(z, v) = \Big(\frac{1}{2} + \varepsilon \Big)^t d_{W_k}(u, v)$$

for some positive integer $t \le m_0 - 1$. But then there is a uz-path P_s in W_k which is strictly shorter than any uz-path P_l through v, contrary to our assumption that one
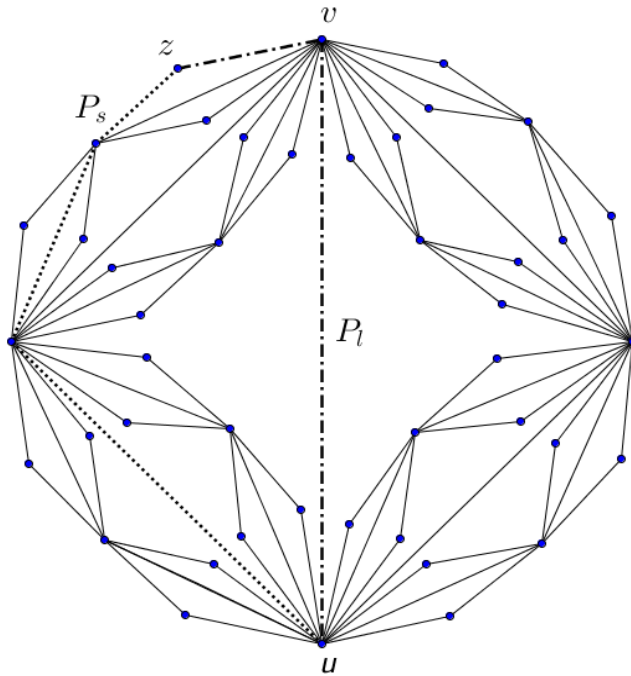
Figure 3.3: Paths P_s and P_l in the subdiamond S

of the shortest xy-paths passes through u and v. To see this we observe that we can pick P_s (see Figure 3.3) of length

$$\left(\left(\tfrac{1}{2} + \varepsilon \right) + \left(\tfrac{1}{2} + \varepsilon \right)^2 + \cdots + \left(\tfrac{1}{2} + \varepsilon \right)^t \right) d_{W_k}(u, v),$$

and that P_l is of length at least

$$\left(1 + \left(\tfrac{1}{2} + \varepsilon \right)^t \right) d_{W_k}(u, v).$$

The conclusion length(P_s) < length(P_l) follows from $t \le m_0 - 1$ and the definition of m_0.

The statement that y is contained in one of the subdiamonds of S whose diagonal has length $\le \left(\tfrac{1}{2} + \varepsilon \right)^{m_0-1} d_{W_k}(u, v)$, and v is one of its ends implies that $d_{D_k}(y, v) \le \left(\tfrac{1}{2} \right)^{m_0-1} d_{D_k}(u, v) \le \tfrac{1}{4} d_{D_k}(u, v)$ since $m_0 \ge 3$. Similarly we prove that $d_{D_k}(x, u) \le \tfrac{1}{4} d_{D_k}(u, v)$. We conclude that

$$d_{D_k}(x, y) \ge d_{D_k}(u, v) - d_{D_k}(y, v) - d_{D_k}(x, u) \ge \left(1 - \tfrac{2}{4} \right) d_{D_k}(u, v) = \tfrac{1}{2} d_{D_k}(u, v),$$

so the inequality (3.2) is satisfied when Case 1 holds.

In Cases 2 and 3 we consider the subdiamond of W_k with the diagonal of the length $(\frac{1}{2} + \varepsilon)^{m-1}$ (where $(\frac{1}{2} + \varepsilon)^m$ is the length of uv in W_k). We may assume without loss of generality that the diagonal of this subdiamond is of the form uw. We denote by $\widetilde{v}$ the other vertex for which both $\widetilde{v}u$ and $\widetilde{v}w$ have length $(\frac{1}{2} + \varepsilon)^m$.

Because of lack of symmetry we consider separately Case 2a, where x is and y is not in the subdiamond S with the diagonal uv, and Case 2b, where y is and x is not in the subdiamond S with the diagonal uv.

In Case 2a the vertex y is contained in the subdiamond with the diagonal vw (because the edge uw is shorter than any other uw-path). Also $d_{D_k}(x, u) \leq \frac{1}{4} d_{D_k}(u, v)$ for the same reason as in Case 1. Hence $d_{D_k}(x, y) > \frac{3}{4} d_{D_k}(u, v)$.

In Case 2b there are two subcases: where x is in the subdiamond with the diagonal uw, and where x is not there. In both cases it is easy to see that the shortest in D_k xy-path cannot be shorter than $d_{D_k}(u, y) \geq \frac{3}{4} d_{D_k}(u, v)$.

In Case 3 the only situation in which the desired inequality is not immediate is the situation where x is in the subdiamond with the diagonal $\widetilde{v}w$ and y is in the subdiamond with the diagonal vw. In this case the shortest path contains both the edge $\widetilde{v}u$ and uv. Replacing these edges by the edges $\widetilde{v}w$ and wv, we get one of the previously considered cases. ∎

In this context it is natural to recall the following well-known result of Assouad [Ass83] (see also [Hei01, Chapter 12]).

Definition 3.8 A metric space is called *doubling* if there exists $N < \infty$ such that each ball in this space can be covered by at most N balls of twice smaller radius.

Theorem 3.9 ([Ass83]) *Each snowflaked version of a doubling metric space admits a bilipschitz embedding into a Euclidean space.*

Remark 3.10 In the original proof of Theorem 3.9 the dimension N of the receiving Euclidean space and the distortion of the embedding depend both on the doubling constant of the metric space and on the amount α of snowflaking, with N going to ∞, as α approaches 1. Recently, Naor and Neiman [NN12] (*cf.* also [DS13]) obtained estimates of N depending only on the doubling constant and independent of α, for $\alpha \in (1/2, 1)$.

Remark 3.11 The spaces $\{W_k\}$ do not satisfy the assumptions of Theorem 3.9, *i.e.*, the spaces $\{D_k\}_{k=1}^{\infty}$ are not uniformly doubling. Indeed, balls of radius $(\frac{1}{2})^k$ centered at the bottom of D_k contain 2^k vertices of mutual distance $(\frac{1}{2})^{k-1}$, in addition to the bottom vertex, and thus no pair of such vertices is contained in any ball of radius $(\frac{1}{2})^{k+1}$.

Note that the doubling condition is important for ℓ_2-embeddability in Theorem 3.9. Consider, for example, the space $L_1(0, 1)$ and $\alpha \in (\frac{1}{2}, 1)$, and apply [AB14, Lemma 2.1]. Thus the results about embeddings of $\{W_k\}$ in [Ost14] and in the present paper are not covered by the Assouad theorem (Theorem 3.9).

3.2 Weighted Diamonds Are Not Included in the Set of Examples Presented in Section 2

The goal of this section is to show that the bilipschitz embeddability of W_n's into an arbitrary Banach space of dimension $\exp(\Omega((\log\log|W_n|)^2))$ with uniformly bounded distortion does not follow from results of Section 2.

We start from two simple results about nonembeddability of W_n's with uniformly bounded distortion into low-dimensional Euclidean spaces and equilateral spaces.

Proposition 3.12   *The distortions of embeddings of W_n into $\ell_2^{k(n)}$ can be uniformly bounded only if $k(n) \geq cn \approx c\log(|W_n|)$ for some $c > 0$.*

Proof   This is an immediate consequence of Lemma 2.2 and the following lemma.   ∎

Lemma 3.13   *The spaces W_n contain equilateral subsets of sizes 2^m for all $m \leq n$.*

Proof   The bottom vertex b of W_n is adjacent in $W_m \subseteq W_n$ to 2^m vertices $\{a_j\}_{j=1}^{2^m}$ with edges of length $(\frac{1}{2} + \varepsilon)^m$ joining them and b, and thus for all $1 \leq i, j \leq 2^m$, with $i \neq j$, $d_{W_n}(a_i, a_j) = 2(\frac{1}{2} + \varepsilon)^m$.   ∎

Remark 3.14   The same argument shows non-embeddability of W_n's with uniformly bounded distortion into any low-dimensional Banach spaces.

Proposition 3.15   *The spaces W_n cannot be embedded with uniformly bounded distortion into any equilateral space.*

Proof   The maximal distance between two elements in W_n is ≥ 1, and the minimal distance is $(\frac{1}{2} + \varepsilon)^n$. Hence any embedding of W_n into an equilateral space has distortion greater than or equal to $(\frac{1}{2} + \varepsilon)^{-n}$. Thus, since $\varepsilon \in (0, \frac{1}{2})$, the distortions are not uniformly bounded.   ∎

In the next two propositions we show that W_n's do not admit bilipschitz embeddings with uniformly bounded distortions into spaces of the form $M_\beta(\mathcal{N})$ where $\beta \geq 1/2$, the collection $\mathcal{N}$ of metric spaces is such that W_n does not admit bilipschitz embeddings with uniformly bounded distortions into any $N \in \mathcal{N}$ and M is either an equilateral space or a metric space that admits a bounded-distortion embedding into a $O((\log\log|W_n|)^2)$-dimensional Euclidean space.

Proposition 3.16   *For $n \in \mathbb{N}$, let $A_n, B_n > 0$ be constants so that there exists a finite equilateral metric space M, $\beta \geq 1/2$, and $\mathcal{N} = \{N_x\}_{x \in M}$ a collection of finite metric spaces so that for any $x \in M$, W_n cannot be embedded into N_x with distortion $\leq A_n/B_n$, and so that there exists an embedding $\phi: W_n \to M_\beta(\mathcal{N})$ such that for all $u_1, u_2 \in W_n$,*

(3.3) $B_n d_{W_n}(u_1, u_2) \leq d_\beta(\phi(u_1), \phi(u_2)) \leq A_n d_{W_n}(u_1, u_2).$

Then

$$\lim_{n \to \infty} \frac{A_n}{B_n} = \infty.$$

Proof Since M is equilateral, we have

$$d_\beta(y_1, y_2) = \begin{cases} d_{N_x}(y_1, y_2) & \text{if } y_1, y_2 \in N_x, \\ \beta\gamma & \text{if } y_1 \in N_{x_1}, y_2 \in N_{x_2}, x_1 \neq x_2, \end{cases}$$

where $\gamma = \max_{x \in M} \text{diam}(N_x)$.

Since for any $x \in M$, W_n cannot be embedded into N_x with distortion $\leq A_n/B_n$, there exist $v_1, v_2 \in W_n$ so that

(3.4) $$\phi(v_1) \in N_{x_1}, \quad \phi(v_2) \in N_{x_2}, \quad \text{and} \quad x_1 \neq x_2.$$

In W_n we can travel between any pair of vertices using edges of the smallest available length, *i.e.*, there exists a sequence of vertices $\{u_j\}_{j=0}^{j_0}$ so that $u_0 = v_1$, $u_{j_0} = v_2$, and for all $0 < j \leq j_0$, $d_{W_n}(u_j, u_{j-1}) = (\frac{1}{2} + \varepsilon)^n$. For $0 < j \leq j_0$, let $z_j = \phi(u_j)$. By (3.4), there exists $0 < i \leq j_0$ so that $z_{i-1} \in N_x$, $z_i \in N_{x'}$, and $x \neq x'$. Thus $d_\beta(z_{i-1}, z_i) = \beta\gamma$. Hence by (3.3), we have

(3.5) $$B_n\left(\frac{1}{2} + \varepsilon\right)^n \leq \beta\gamma \leq A_n\left(\frac{1}{2} + \varepsilon\right)^n.$$

Now let $w_1, w_2 \in W_k$ be such that $d_{W_k}(w_1, w_2) = 1$. Then

(3.6) $$B_n \leq d_\beta(\phi(w_1), \phi(w_2)) \leq A_n.$$

If there exists $x \in M$ so that $\phi(w_1), \phi(w_2) \in N_x$, then $d_\beta(\phi(w_1), \phi(w_2)) = d_{N_x}(\phi(w_1), \phi(w_2)) \leq \gamma$. Therefore, combining (3.5) and (3.6), since $\beta \geq 1/2$, we obtain

$$\frac{A_n}{B_n} \geq \beta\left(\frac{1}{\frac{1}{2} + \varepsilon}\right)^n \geq \frac{1}{2}\left(\frac{1}{\frac{1}{2} + \varepsilon}\right)^n.$$

If $\phi(w_1) \in N_{x_1}$, $\phi(w_2) \in N_{x_2}$, where $x_1 \neq x_2$, then $d_\beta(\phi(w_1), \phi(w_2)) = \beta\gamma$. Therefore, combining (3.5) and (3.6), we obtain

$$\frac{A_n}{B_n} \geq \left(\frac{1}{\frac{1}{2} + \varepsilon}\right)^n.$$

Since $(\frac{1}{2} + \varepsilon) < 1$, in either case the proposition is proved. ∎

Proposition 3.17 *Let $C \geq 1$. For $n \in \mathbb{N}$, let $A_n, B_n > 0$ be constants, possibly depending on C, so that there exists a finite metric space M which admits a C-embedding into a $O((\log\log|W_n|)^2)$-dimensional Euclidean space, $\beta \geq 1/2$, and $\mathcal{N} = \{N_x\}_{x \in M}$ a collection of finite metric spaces so that for any $x \in M$, W_n cannot be embedded into N_x with distortion $\leq A_n/B_n$, and so that there exists an embedding $\phi\colon W_n \to M_\beta(\mathcal{N})$ such that for all $u_1, u_2 \in W_n$,*

$$B_n d_{W_n}(u_1, u_2) \leq d_\beta(\phi(u_1), \phi(u_2)) \leq A_n d_{W_n}(u_1, u_2).$$

Then

$$\lim_{n \to \infty} \frac{A_n}{B_n} = \infty.$$

Proof By Lemma 3.13, since $\lfloor\sqrt{n}\rfloor \le n$, the spaces W_n contain equilateral subsets S_n of sizes $2^{\lfloor\sqrt{n}\rfloor}$ with distances between their elements equal to $2(\frac{1}{2}+\varepsilon)^{\lfloor\sqrt{n}\rfloor}$. By (3.1) we have $\lim_{n\to\infty}\frac{\lfloor\sqrt{n}\rfloor}{(\log_2\log_2|W_n|)^2} = \infty$. Thus, by Lemma 2.2, S_n do not admit bilipschitz embeddings with uniformly bounded distortions into a $O((\log\log|S_n|)^2) = O((\log\log|W_n|)^2)$-dimensional Euclidean space, and thus into M. Therefore ϕ maps some elements of $u_1, u_2 \in S_n$ into the same set N_{x_0} for some $x_0 \in M$ (see Definition 2.8). We have

$$B_n 2\Big(\frac{1}{2}+\varepsilon\Big)^{\lfloor\sqrt{n}\rfloor} \le d_{N_{x_0}}(\phi(u_1),\phi(u_2)) \le A_n 2\Big(\frac{1}{2}+\varepsilon\Big)^{\lfloor\sqrt{n}\rfloor}$$

and

$$(3.7) \qquad \max_{x\in M}\operatorname{diam}(N_x) \ge \operatorname{diam}(N_{x_0}) \ge d_{N_{x_0}}(\phi(u_1),\phi(u_2)) \ge B_n 2\Big(\frac{1}{2}+\varepsilon\Big)^{\lfloor\sqrt{n}\rfloor}.$$

On the other hand, our assumptions imply that all elements of W_n are not mapped into the same set N_x, *i.e.*, there exist vertices v_1 and v_2 of W_n which are mapped into sets N_{x_1} and N_{x_2} with $x_1 \ne x_2$. Using the same argument as in the proof of (3.5) in Proposition 3.16, we obtain that there exist $y_1 \ne y_2 \in M$ so that

$$(3.8) \qquad B_n\Big(\frac{1}{2}+\varepsilon\Big)^n \le \beta\gamma d_M(y_1,y_2) \le A_n\Big(\frac{1}{2}+\varepsilon\Big)^n,$$

where $\gamma = \frac{\max_{x\in M}\operatorname{diam}(N_x)}{\min_{x\ne y\in M} d_M(x,y)}$. By (3.7) and (3.8) we get

$$A_n\Big(\frac{1}{2}+\varepsilon\Big)^n \ge \beta\gamma d_M(y_1,y_2) = \beta\cdot\frac{\max_{x\in M}\operatorname{diam}(N_x)}{\min_{x\ne y\in M} d_M(x,y)}\, d_M(y_1,y_2)$$

$$\ge \beta\max_{x\in M}\operatorname{diam}(N_x) \ge \beta B_n 2\Big(\frac{1}{2}+\varepsilon\Big)^{\lfloor\sqrt{n}\rfloor}.$$

Therefore

$$\frac{A_n}{B_n} \ge \Big(\frac{1}{2}+\varepsilon\Big)^{\lfloor\sqrt{n}\rfloor-n}.$$

Since $(\frac{1}{2}+\varepsilon) < 1$, the proposition is proved. ∎

We are now ready to prove the main result of this subsection. We denote by $\mathcal{E}$ the class of all finite equilateral spaces, by $\mathcal{L}_{n,C}$, for $n \in \mathbb{N}$ and $C \ge 1$, the class of all finite metric spaces that admit embeddings into $O((\log n)^2) = O((\log\log|W_n|)^2)$-dimensional Euclidean spaces with distortion $\le C$, and let $\mathcal{M}_{n,C} = \mathcal{E} \cup \mathcal{L}_{n,C}$.

Theorem 3.18 *For any $C, \beta \ge 1$, the spaces W_n do not admit embeddings with a uniformly bounded distortion into metric spaces $V \in \operatorname{comp}_\beta(\mathcal{M}_{n,C})$.*

Proof The proof is by induction on the level m of complexity of spaces $V \in \operatorname{comp}_\beta(\mathcal{M}_{n,C})$ as defined in (2.4). The base case $m = 0$ follows from Propositions 3.12 and 3.15. By Proposition 2.11, the inductive step follows from Propositions 3.16 and 3.17. ∎

3.3 Embeddings of Weighted Diamonds Into Low Dimensional Banach Spaces With Uniformly Bounded Distortion

We prove first that weighted diamonds can be embedded into low dimensional spaces with a basis with distortion which is bounded by a constant that depends only on ε and the basis constant of the target space.

Theorem 3.19   *For every $C \geq 1$ and $\varepsilon \in (0, 1/2)$ there exists a constant $D \geq 1$ so that for every $n \geq 2$, W_n D-embeds into every Banach space X with a Schauder basis with basis constant smaller than or equal to C and of dimension $\geq \frac{1}{2}(\log_2 |W_n|)^2$.*

To apply Theorem 3.19 to embeddings into arbitrary finite dimensional spaces we need to know what is the best estimate for the dimension of a subspace with a basis constant C in any n-dimensional Banach spaces. More precisely, we are interested in lower bounds for the following problem.

Problem 3.20   Let $C \in (1, \infty)$. Define the function $f_C(n)$ to be the largest $k \in \mathbb{N}$ so that each n-dimensional Banach space contains a k-dimensional subspace with basis constant at most C. What are the estimates for $f_C(n)$?

Known upper estimates can be found in [MT93]. Many experts believe that the techniques of [MT93] (which go back to Gluskin [Glu81] and Szarek [Sza83]) can be used to achieve the upper bound of order $n^{1/2}$ (perhaps multiplied by some power of a logarithm), but it does not seem that anyone has worked this out.

The best lower bound for $f_C(n)$ that we have found in the literature is a result Szarek and Tomczak-Jaegermann [ST09], where they studied the nontrivial projection problem. Thus they were interested in 'large' subspaces with 'large' codimension which have small projection constants in comparison with their dimension, but since the subspaces found in [ST09] were close to ℓ_p^k with $p \in \{1, 2, \infty\}$, their result can be used for our purposes. It appears that techniques of [AM83, Rud95, ST09, Tal95] could be useful for further work on lower estimates for Problem 3.20.

We state here the result of [ST09] in the form closest to the answer to Problem 3.20.

Theorem 3.21 ([ST09])   *There exist absolute constants $A, B, C > 0$ so that for every $n \geq A$ and for every n-dimensional normed space X, there exists a subspace $Y \subseteq X$ so that $\dim Y \geq B \exp(\frac{1}{2}\sqrt{\ln n})$ and Y is C-isomorphic to an ℓ_p-space for some $p \in \{1, 2, \infty\}$.*

As an immediate consequence of Theorems 3.19 and 3.21 we obtain that W_n's can be embedded with uniformly bounded distortion in an arbitrary Banach space of dimension $\exp(c(\log\log |W_n|)^2)$ for some fixed $c > 1$.

Corollary 3.22   *For every $\varepsilon \in (1/2, 1)$, there exist constants $c, C > 1$ so that for every $n \geq C$, W_n can be embedded in every Banach space X such that $\dim X \geq \exp(c(\log\log |W_n|)^2)$ with the distortion bounded from above by a constant that depends only on ε.*

The remainder of this section is devoted to the proof Theorem 3.19.

Proof of Theorem 3.19   Fix $C \geq 1$, $\varepsilon > 0$, and $n \geq 2$. Let X be any Banach space with $\dim X = d \geq \frac{1}{2}(\log_2 |W_n|)^2$, and with a Schauder basis $\{x_i\}_{i=0}^{d-1}$ with basis constant at most C so that $\|x_i\| = 1$ for all i. Let

$$Y_0 = \text{span}\{x_0\}, \quad Y_1 = \text{span}\{x_1\}, \quad Y_m = \text{span}\Big\{ x_j : \Big(\sum_{k=1}^{m-1} k\Big) + 1 \leq j \leq \sum_{k=1}^{m} k \Big\},$$

for $m = 2, \ldots, n$. Note that, for $m = 1, \ldots, n$, $\dim Y_m = m$, and thus, by (3.1) and since $n \geq 2$,

$$1 + \sum_{k=1}^{n} k = 1 + \frac{n(n+1)}{2} \leq \frac{(2n-1)^2}{2} \leq \frac{1}{2}(\log_2 |W_n|)^2 \leq d.$$

Thus there are enough basis vectors in X to define all these subspaces. For $m = 0, \ldots, n$, let $\{y_{m,k}\}_{k=1}^{2 \cdot 4^{m-1}}$ be elements of the unit sphere of Y_m satisfying the conditions of Lemma 2.6 with $\delta = 1/16$. It is easy to see that for any $m = 0, \ldots, n$, any $1 \leq k \neq k' \leq 2 \cdot 4^{m-1}$, and any a with $0 \leq a \leq 1$, we have

(3.9) $\|y_{m,k} - ay_{m,k'}\| \geq \delta/2$.

Note that for $m > m'$, $y_{m,k}$ and $y_{m',k'}$ are supported on disjoint intervals with respect to the basis $\{x_i\}_{i=0}^{d-1}$, and therefore

$$\|y_{m,k} - y_{m',k'}\| \geq \frac{1}{C}\|y_{m',k'}\| = \frac{1}{C}.$$

We construct an embedding $S_n : W_n \to X$ in the following way.

- The map S_n maps the vertices of D_0 to 0 and x_0, respectively. It is clear that $S_n|_{W_0}$ is an isometric embedding.
- The map $S_n|_{W_m}$, $1 \leq m \leq n$, is an extension of the map $S_n|_{W_{m-1}}$. Note that for each $m \geq 1$, $|W_m \setminus W_{m-1}| = 2 \cdot 4^{m-1}$. Let $\sigma_m : W_m \setminus W_{m-1} \to \{1, \ldots, 2 \cdot 4^{m-1}\}$ be any bijective map. Each vertex $w \in W_m \setminus W_{m-1}$ corresponds to a pair of vertices of W_{m-1}: w is the vertex of a 2-edge path joining u and v. We map the vertex w to

$$\frac{1}{2}(S_n u + S_n v) + \varepsilon\Big(\frac{1}{2} + \varepsilon\Big)^{m-1} y_{m,\sigma_m(w)}.$$

Now we estimate the distortion of S_n. First we observe that the map S_n is 1-Lipschitz. This can be proved for $S_n|_{W_m}$ by induction on $m = 0, 1, \ldots, n$. It suffices to observe that for each edge uv in W_{m-1} and each vertex w satisfying the condition of the previous paragraph we have $d_{W_n}(u, w) = (\frac{1}{2} + \varepsilon)^m$ and

$$\|S_n u - S_n w\| \leq \frac{1}{2}\|S_n u - S_n v\| + \varepsilon\Big(\frac{1}{2} + \varepsilon\Big)^{m-1} \leq \Big(\frac{1}{2} + \varepsilon\Big)^m.$$

To estimate from above the Lipschitz constant of S_n^{-1} we consider any shortest path P between two vertices w, z in W_n. Let $(\frac{1}{2} + \varepsilon)^t$ be the length of the longest edge in it. By Lemma 3.7,

(3.10) $d_{W_n}(w, z) \leq \dfrac{2\big(\frac{1}{2} + \varepsilon\big)^t}{\big(\frac{1}{2} - \varepsilon\big)}.$

On the other hand, since the subspaces Y_m are supported on disjoint intervals with respect to the basis $\{x_i\}_{i=0}^{d-1}$, for every $m \in \{1, \dots, n\}$, we have

$$(3.11) \qquad \|S_n w - S_n z\| \geq \frac{1}{2C} \left\| (S_n w - S_n z)\big|_{Y_m} \right\|$$

where $x|_{Y_m}$ denotes the natural projection onto Y_m.

Let $m_0 \in \mathbb{N}$ be the smallest number such that

$$(3.12) \qquad \left(\frac{1}{2} + \varepsilon\right) + \left(\frac{1}{2} + \varepsilon\right)^2 + \cdots + \left(\frac{1}{2} + \varepsilon\right)^{m_0} > 1 + \left(\frac{1}{2} + \varepsilon\right)^{m_0}.$$

Such a number m_0 obviously exists if $\varepsilon > 0$. It is clear that $m_0 \geq 3$ if $\varepsilon < \frac{1}{2}$ and that m_0 depends only on ε.

Now we turn to estimates of $\|S_n w - S_n z\|$ from below. Let xy be one of the edges of the largest length $(\frac{1}{2} + \varepsilon)^t$ in the path P from w to z (by Lemma 3.7 we know that path P contains at most two such edges and that if there are two of them, they share a vertex). We restrict our attention to the case where at least one of the vertices x, y is not in W_0; the excluded case can be considered along the same lines. Without loss of generality we assume that $y \in W_t \setminus W_{t-1}$ and $x \in W_{t-1}$. Let $\overline{x}$ be the vertex in W_{t-1} so that $x\overline{x}$ is an edge of the length $(\frac{1}{2} + \varepsilon)^{t-1}$ and y belongs to the subdiamond with diagonal $x\overline{x}$. We assume that our notation is chosen in such a way that z is closer to y than to x. Then the part of the path P from y to z does not contain an edge of length $(\frac{1}{2} + \varepsilon)^t$, and z is either in the subdiamond with diagonal yx, or in the subdiamond with diagonal $y\overline{x}$.

To simplify the notation, let us denote the vector $\varepsilon(\frac{1}{2} + \varepsilon)^{t-1} y_{t,\sigma_t(y)}$ by $\pi_{t,y}$.

Lemma 3.23   (i) *If z is in the subdiamond with the diagonal $y\overline{x}$, then*

$$S_n z\big|_{Y_t} = \rho_1 \pi_{t,y}$$

for some $\rho_1 \geq \left(\frac{1}{2}\right)^{m_0-1}$.

(ii) *If z is in the subdiamond with diagonal yx, then $(S_n y - S_n z)\big|_{Y_t} = \rho_2 \pi_{t,y}$ for some $0 \leq \rho_2 \leq \left(\frac{1}{2}\right)^{m_0-1}$.*

(iii) *If w is in the subdiamond with diagonal yx, then $S_n w\big|_{Y_t} = \rho_3 \pi_{t,y}$ for some $0 \leq \rho_3 \leq \left(\frac{1}{2}\right)^{m_0-1}$.*

(iv) *If w is not in the subdiamond with the diagonal yx, then $S_n w\big|_{Y_t} = \rho_4 y_{t,k}$, for some $k \neq \sigma_t(y)$ and $\rho_4 \in [0,1]$.*

Proof   (i) Let z be in the subdiamond with diagonal $y\overline{x}$. Observe that ends of edges of length $\leq (\frac{1}{2} + \varepsilon)^{m_0+t-1}$ with one end at $\overline{x}$ and the other end in the subdiamond with the diagonal $\overline{x}y$ cannot be in P because then, by (3.12), the path through $\overline{x}$ would be shorter. Therefore,

$$S_n z = (1 - b) S_n \overline{x} + b S_n y + \overline{z}_t,$$

where

$$S_n y \in B_t \overset{\text{def}}{=} \operatorname{span}\left\{ x_j : j \leq \sum_{k=1}^{t} k \right\} = \operatorname{span}\left(\bigcup_{m=1}^{t} Y_t \right), \quad S_n \overline{x} \in B_{t-1},$$

$$\overline{z}_t \in T_t \overset{\text{def}}{=} \operatorname{span}\left\{ x_j : j > \sum_{k=1}^{t} k \right\}, \quad \text{and} \quad b \geq \left(\frac{1}{2}\right)^{m_0-1}.$$

Note that

$$(3.13) \qquad S_n y = \frac{1}{2}(S_n x + S_n \overline{x}) + \pi_{t,y},$$

where $S_n x, S_n \overline{x} \in B_{t-1}$ and $\pi_{t,y} \in Y_t$. Hence, $S_n z|_{Y_t} = b\pi_{t,y}$ and the conclusion follows.

(ii) If z is in the subdiamond with diagonal yx, since yx is a part of a shortest path, we conclude that the longest edge in the part of the path P from y to z has length $\leq (\frac{1}{2} + \varepsilon)^{t+m_0}$, where m_0 satisfies (3.12). Thus

$$(3.14) \qquad S_n z = aS_n y + (1-a)S_n x + z_t,$$

where $S_n y \in B_t$, $S_n x \in B_{t-1}$, $z_t \in T_t$, and

$$1 \geq a \geq 1 - \sum_{k=m_0}^{\infty} \left(\frac{1}{2}\right)^k = 1 - \left(\frac{1}{2}\right)^{m_0-1}.$$

By (3.13) we get $(S_n y - S_n z)|_{Y_t} = \pi_{t,y} - a\pi_{t,y} = (1-a)\pi_{t,y}$ and the conclusion follows.

(iii) If w is in the subdiamond for which xy is the diagonal, similarly as in (3.14), we obtain $S_n w = cS_n x + (1-c)S_n y + w_t$, where $S_n x \in B_{t-1}$, $S_n y \in B_t$, $w_t \in T_t$, and $1 \geq c \geq 1 - (\frac{1}{2})^{m_0-1}$. By (3.13), $S_n w|_{Y_t} = (1-c)\pi_{t,y}$ and we are done in this case.

(iv) If w is not in the subdiamond for which xy is the diagonal, let $q \in W_t \setminus W_{t-1}$, $q \neq y$, be the vertex which is an endpoint of an edge of length $(\frac{1}{2} + \varepsilon)^t$ which is a diagonal of the subdiamond that contains w. By construction, the projection of $S_n w$ onto the subspace Y_t is a multiple of $y_{t,\sigma_t(q)} \neq y_{t,\sigma_t(y)}$, with some coefficient $\rho_4 \in [0,1]$. ∎

Observe that Lemma 3.23 implies the estimate for the Lipschitz constant of S_n^{-1}, and thus Theorem 3.19, in all of the cases except the case where both (i) and (iii) hold. Consider, for example the case where (i) and (iv) hold. Then (we use (3.10), (3.11), the conclusions of (i) and (iv), the definition of $\pi_{t,y}$ and (3.9))

$$\frac{d_{W_n}(w,z)}{\|S_n w - S_n z\|} \leq \frac{2(\frac{1}{2} + \varepsilon)^t}{(\frac{1}{2} - \varepsilon)\frac{1}{2C}\|\rho_1 \pi_{t,y} - \rho_4 y_{t,k}\|}$$

$$\leq \frac{4C(\frac{1}{2} + \varepsilon)^t}{(\frac{1}{2} - \varepsilon)\frac{\delta}{2}\rho_1 \varepsilon(\frac{1}{2} + \varepsilon)^{t-1}} \leq \frac{8C(\frac{1}{2} + \varepsilon)}{(\frac{1}{2} - \varepsilon)\delta\varepsilon(\frac{1}{2})^{m_0-1}},$$

and this number depends only on C and ε.

It remains to consider the case when both (i) and (iii) hold. In this case we estimate from below the norm of $(S_n w - S_n z)|_{Y_{t-1}}$. We use $S_n z = (1-b)S_n \overline{x} + bS_n y + \overline{z}_t$ and $S_n w = cS_n x + (1-c)S_n y + w_t$ with $1 \geq b \geq (\frac{1}{2})^{m_0-1}$ and $1 \geq c \geq 1 - (\frac{1}{2})^{m_0-1}$. The value of b actually does not matter for our argument, it is only important that $0 \leq b \leq 1$. Recall also that $S_n y = \frac{1}{2}(S_n x + S_n \overline{x}) + \pi_{t,y}$.

Therefore

$$(S_n z - S_n w)|_{Y_{t-1}} = \left(\left(1 - \frac{1}{2}b - \frac{1-c}{2}\right)S_n \overline{x} - \left(c + \frac{1-c}{2} - \frac{1}{2}b\right)S_n x\right)\Big|_{Y_{t-1}}$$

Observe that each of the coefficients of $S_n \overline{x}$ and $S_n x$ in this sum is at least

$$\left(\frac{1}{2} - \left(\frac{1}{2} \right)^{m_0} \right).$$

There are two possible cases:

Case (A): $x \in W_{t-1} \setminus W_{t-2}, \overline{x} \in W_{t-2}$;

Case (B): $\overline{x} \in W_{t-1} \setminus W_{t-2}, x \in W_{t-2}$.

The cases are similar, so we consider case (A) only. Let $o \in W_{t-2}$ be such that x is in the subdiamond with diagonal $o\overline{x}$, so $S_n \overline{x}, S_n o \in B_{t-2}$,

$$S_n x = \frac{1}{2}(S_n \overline{x} + S_n o) + \varepsilon \left(\frac{1}{2} + \varepsilon \right)^{t-2} y_{t-1, \sigma_{t-1}(x)},$$

and

$$(S_n z - S_n w)|_{Y_{t-1}} = -\left(c + \frac{1-c}{2} - \frac{1}{2}b \right) \varepsilon \left(\frac{1}{2} + \varepsilon \right)^{t-2} y_{t-1, \sigma_{t-1}(x)}$$

We get

$$\frac{d_{W_n}(w, z)}{\|S_n w - S_n z\|} \leq \frac{2(\frac{1}{2} + \varepsilon)^t}{(\frac{1}{2} - \varepsilon)\frac{1}{2C}(c + \frac{1-c}{2} - \frac{1}{2}b)\varepsilon(\frac{1}{2} + \varepsilon)^{t-2}}$$

$$\leq \frac{4C(\frac{1}{2} + \varepsilon)^2}{(\frac{1}{2} - \varepsilon)\varepsilon(\frac{1}{2} - (\frac{1}{2})^{m_0})}.$$

The obtained number depends only on C and ε. This concludes the proof. ∎

4 More General Examples

The goal of this section is to generalize the results of Section 3 to more general "hierarchically built weighted graphs", which we denote $\{G_i\}_{i=0}^{\infty}$ and call *corals* because they are more chaotic than diamonds.

Definition 4.1 We pick $\lambda \in (\frac{1}{2}, 1)$ and a sequence $\{N_i\}_{i=0}^{\infty}$ of natural numbers so that $N_0 = 2$ and $N_i \geq 1$ for all $i \geq 1$. The sequence $\{G_n\}_{n=0}^{\infty}$ of *corals* is defined inductively. Vertices and edges of a coral come in *generations* denoted $\{V_i\}_{i=0}^{\infty}$ and $\{E_i\}_{i=0}^{\infty}$, respectively. We proceed as follows (see Figure 4.1 for a sample graph G_1).

- G_0 is the same as D_0, *i.e.*, V_0 consists of two vertices v_0, v_1 which are joined by one edge of weight 1. Thus $G_0 = (V_0, E_0)$, where $|V_0| = 2$, $|E_0| = 1$.
- Suppose that $\bigcup_{i=0}^{k} V_i$, $\bigcup_{i=0}^{k} E_i$, and G_k have been already defined. Let V_{k+1} be a set of cardinality N_{k+1}, disjoint with $\bigcup_{i=0}^{k} V_i$. The vertex set of the graph G_{k+1} is $\bigcup_{i=0}^{k+1} V_i$. The set E_{k+1} of new edges is a subset of edges joining the vertices of V_{k+1} with $\bigcup_{i=0}^{k} V_i$. Every edge in E_{k+1} is is given weight λ^{k+1}. Edges in E_{k+1} are chosen so that each vertex in V_{k+1} has degree 1 or 2 and if a vertex $v \in V_{k+1}$ has degree 2, then it is adjacent to vertices $u, w \in \bigcup_{i=0}^{k} V_i$ which are joined by an edge uw in E_k, *i.e.*, uw is of length λ^k in G_k.

Remark 4.2 The graph G_n depends on λ, the numbers $N_1, N_2, \ldots, N_n$, and on the choices that we make when we attach new vertices to already existing ones. For brevity, we do not reflect these dependencies in the notation for G_n.
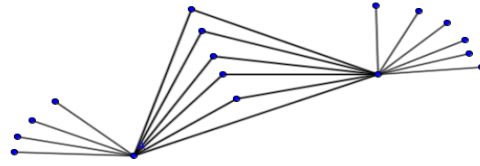
Figure 4.1: Sketch of G_1

Note that a *coral* can be regarded as a chaotically branching snowflaked diamond in which we allow attaching smaller diamonds to vertices of larger diamonds. In particular the weighted graph W_n is an example of a very regular coral. Also one can think of the coral as constructed in a fractal-like fashion, where we start from two vertices joined by one edge. On the next step we replace the one edge by a copy of G_1 (see Figure 4.1) so the set of edges is now $E_0 \cup E_1$, where every edge in E_1 has length λ. We now replace every edge in E_1 by a scaled (by λ) version of G_1, obtaining set E_2 of additional edges of length λ^2. We continue for arbitrary number of generations. The difference between this procedure and a true fractal is that every scaled copy of G_1 can have different number of vertices and edges, so the final graph can be very chaotic (see Figure 4.2).
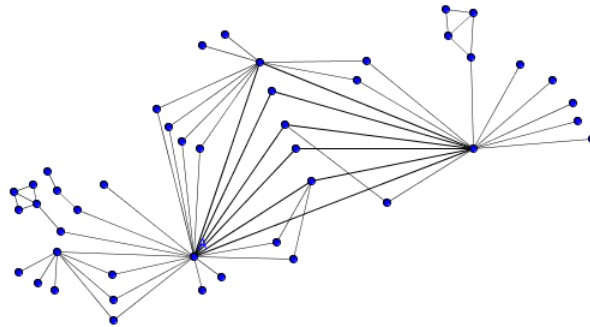

Figure 4.2: An example of a coral with a few generations

Our goal is to prove that Corollary 3.22 can be generalized for corals.
We introduce the following function $L: \mathbb{N} \to \mathbb{N}$,

$$L(i) = \begin{cases} 1 & \text{if } i = 1, 2, \\ 2 & \text{if } i = 3, 4, \\ \lceil \log_4 i \rceil & \text{if } i \geq 5. \end{cases}$$

The function $L(i)$ shows the dimension, which is sufficient to accommodate i δ-separated points, for $\delta = 1/16$, in the unit sphere, *cf.* Lemma 2.6.

Theorem 4.3 *Let $C \geq 1$ and $\lambda \in (1/2, 1)$. Then there exists a constant $D = D(C, \lambda)$, so that every coral G_n with parameters λ and $\{N_i\}_{i=0}^n \subset \mathbb{N}$, D-embeds into any Banach space X which contains a basic sequence of length $\sum_{i=0}^n L(N_i)$ with basis constant $\leq C$.*

Remark 4.4 Note that in the case when $G_n = W_n$, Theorem 4.3 reduces to Theorem 3.19.

Remark 4.5 Analogs of Theorem 3.18 and Proposition 3.12 do not hold for some families of corals. In fact, certain families of corals can embed in low dimensional Euclidean spaces, for example a family consisting of a triangle with progressively longer tails embeds into $\mathbb{R}$ with uniformly bounded distortions.

For the proof of Theorem 4.3 we will need an analogue of Lemma 3.7.

Lemma 4.6 (This is a version of [Ost14, Claim 4.1]) *A shortest path between two vertices in G_n can contain edges of each possible length: $1, \lambda, \lambda^2, \lambda^3, \ldots$ at most twice. Actually for 1 this can happen only once because there is only one such edge. If there are two longest edges, they are adjacent.*

The proof of this lemma is a slightly modified version of the proof of Lemma 3.7. We start with a definition of a notion analogous to the notion of a subdiamond.

Definition 4.7 We define a *(degree 2) subcoral* of a coral G_n grown out of an edge uv to be the subgraph of G_n induced by the set of vertices containing u, v and viewed as constructed in steps such that the following conditions are satisfied.

- All vertices except u and v can be included into the subcoral only if they have degree 2 when they appear for the first time;
- All vertices that have degree 2 when they appear, with both ends in the subcoral, get into the subcoral.

The edge from which a subcoral evolved is called its *diagonal*.

Proof of Lemma 4.6 Let $e = uv$ be one of the longest edges in the path and λ^k be its length. For each edge of the graph G_n except the initial edge, one of the ends was introduced later. Assuming that e is not the initial edge of the graph, we may assume that the vertex v was introduced later than u.

There are two cases.

Case 1: The vertex v was attached to two vertices of an edge uw. Let S be the subcoral which evolved from uw, so S contains e and has a diagonal of length λ^{k-1}

Case 2: The vertex v was attached to the vertex u only.

The rest of the path consists of two pieces: (1) The one which starts at v and (2) the one which starts at u.

We claim that in Case 1 the part which starts at v can never leave S. It obviously cannot leave through u, nor can it leave through the w, because otherwise the piece

of the path between u and w could be replaced by the diagonal of S, which is strictly shorter. This implies that the part of the path in S which starts at v can contain edges only strictly shorter than λ^k.

The same is true in Case 2 because only vertices of further generations will be attached to v in this case and they are attached using edges of length $\le \lambda^{k+1}$.

For the next edge in the part of the path which starts at v we can repeat the argument and get (by induction) that lengths of edges in the remainder of the path are strictly decreasing.

The part of the path which starts at u can be considered similarly.

The last statement of the Lemma is immediate from the proof.   ∎

Proof of Theorem 4.3 The proof is very similar to the proof of Theorem 3.19. Let $L = \sum_{i=0}^{n} L(N_i)$, X be a Banach space and $\{x_i\}_{i=0}^{L-1}$ be a basic sequence in X with $\|x_i\| = 1$ for all i, and a basis constant $\le C$. Let $Y_0 = \mathrm{span}\{x_0\}$, and $Y_m = \mathrm{span}\{x_j : \left(\sum_{k=0}^{m-1} L(k)\right) + 1 \le j \le \sum_{k=0}^{m} L(k)\}$, for $m = 1, \ldots, n$. Thus $\dim Y_m = L(N_m)$ for $m = 0, \ldots, n$. Let $\{y_{m,k}\}_{k=1}^{N_m}$ be elements of the unit sphere of Y_m satisfying the conditions of Lemma 2.6 with $\delta = 1/16$ (observe that the definition of $L(N_m)$ is such that this is always possible). Note that for $m > m'$, $y_{m,k}$ and $y_{m',k'}$ are supported on disjoint intervals with respect to the basis $\{x_i\}$, hence

$$\|y_{m,k} - y_{m',k'}\| \ge \frac{1}{C} \|y_{m',k'}\| = \frac{1}{C}.$$

We construct an embedding $T \colon G_n \to X$ in the following way. We define it in steps for vertices of $V_0, V_1, \ldots, V_n$

- The map T maps the two vertices of V_0 to 0 and x_0, respectively. It is clear that it is an isometric embedding.
- Suppose that we have already constructed the restriction of T to $\bigcup_{i=0}^{m-1} V_i$. Our next step is to extend T to V_m. Observe that our notation is such that there exists a bijection between V_m and $\{y_{m,k}\}_{k=1}^{N_m}$. Let $w \in V_m$. We denote the vector corresponding to a vertex w by $y_{m,\sigma_m(w)}$. Then we define Tw as follows.
  - If the vertex w is attached to two vertices $u, v \in \bigcup_{i=0}^{m-1} V_i$, we let

$$Tw = \frac{1}{2}(Tu + Tv) + \left(\lambda - \frac{1}{2}\right)\lambda^{m-1} y_{m,\sigma_m(w)}.$$

  - If the vertex w is attached to one vertex $u \in \bigcup_{i=0}^{m-1} V_i$, we let

(4.1) $$Tw = Tu + \lambda^m y_{m,\sigma_m(w)}.$$

Now we estimate the distortion of T. First we show that the map T is 1-Lipschitz. This can be proved for $T|_{G_m}$ by induction on $m = 0, 1, \ldots, n$ (observe that the metric induced on G_m from G_{m+1} coincides with the metric of G_m). It suffices to prove that for each $w \in V_m$ and each edge uw in G_m we have $d_W(u, w) = \lambda^m$ and $\|Tu - Tw\| \le \lambda^m$.

The equality $d_W(u, w) = \lambda^m$ follows immediately from our definitions. To prove that $\|Tu - Tw\| \le \lambda^m$, we need to consider two cases: (a) w has degree 1 in G_m; (b) w has degree 2 in G_m.

Since $\|y_{m,\sigma_m(w)}\| = 1$, the desired inequality in case (a) follows immediately from (4.1). In case (b) we get

$$\|Tu - Tw\| \le \frac{1}{2}\|Tu - Tv\| + \left(\lambda - \frac{1}{2}\right)\lambda^{m-1} \le \lambda^m,$$

where for the last inequality we use the assumption that uv is an edge in G_{m-1} and therefore $\|Tu - Tv\| \le \lambda^{m-1}$.

To estimate the Lipschitz constant of T^{-1} from above we consider any shortest path P between two vertices w, z in G_n. Let λ^t be the length of the longest edge in it. By Lemma 4.6,

$$(4.2) \qquad\qquad d_W(w,z) \le 2\lambda^t/(1-\lambda) .$$

On the other hand, since the subspaces Y_m are supported on disjoint intervals with respect to the basis $\{x_i\}$, we have for every $m \in \{0, 1, \dots, n\}$,

$$(4.3) \qquad\qquad \|Tw - Tz\| \ge \frac{1}{2C}\|(Tw - Tz)\big|_{Y_m}\|,$$

where by $x\big|_{Y_m}$ we denote the natural projection of x onto Y_m.

Let $m_0 \in \mathbb{N}$ be the smallest number such that

$$(4.4) \qquad\qquad \lambda + \lambda^2 + \dots + \lambda^{m_0} > 1 + \lambda^{m_0}.$$

Such number m_0 obviously exists since $\lambda > \frac{1}{2}$. It is clear that $m_0 \ge 3$ since $\lambda < 1$, and that m_0 depends only on λ.

Now we turn to estimates of $\|Tw - Tz\|$ from below. Let xy be one of the edges of the largest length λ^t in the path P from w to z (by Lemma 4.6 we know that the path P contains at most two such edges; and that if there are two of them, they share a vertex). Without loss of generality we assume that $y \in V_t$ and $x \in \bigcup_{i=0}^{t-1} V_i$.

(1) In the case where y is of degree 2 in G_t, let $\overline{x}$ be the vertex in V_{t-1} so that $x\overline{x}$ is an edge of the length λ^{t-1} in G_{t-1} and $\overline{x}y$ is an edge of length λ^t in G_t. Then the part of the path P from y to z does not contain an edge of length λ^t. Furthermore, some part of this path (from y to z), starting at y (possibly all of the path from y to z) is either in the subcoral with diagonal yx, or in the subcoral with diagonal $y\overline{x}$, and then leaves for parts of the coral which are attached to older parts of the coral through one vertex. Let $\overline{z}$ be the vertex at which this happens. We let $\overline{z} = z$ if this never happens.

(2) In the case where y is of degree 1 in G_t, we do the same, but in this case the only option which is available is the option of subcoral with the diagonal xy.

Similarly we define $\overline{w}$. For simplicity we denote the vector $(\lambda - \frac{1}{2})\lambda^{t-1}y_{t,\sigma_t(y)}$ by $\pi_{t,y}$.

Lemma 4.8   (i) *If $\overline{z}$ is in the subcoral with the diagonal $y\overline{x}$, then*

$$Tz\big|_{Y_t} = T\overline{z}\big|_{Y_t} = \alpha\pi_{t,y}$$

for some $\alpha \ge \left(\frac{1}{2}\right)^{m_0-1}$.

     (ii) *If $\overline{z}$ is in the subcoral with diagonal yx, then*

$$(Ty - Tz)\big|_{Y_t} = (Ty - T\overline{z})\big|_{Y_t} = \beta\pi_{t,y}$$

for some $0 \le \beta \le \left(\frac{1}{2}\right)^{m_0-1}$.

(iii) *If $\overline{w}$ is in the subcoral with diagonal yx, then*

$$Tw|_{Y_t} = T\overline{w}|_{Y_t} = \gamma \pi_{t,y}$$

for some $0 \le \gamma \le (\frac{1}{2})^{m_0-1}$.

(iv) *If $\overline{w}$ is not in the subcoral with the diagonal yx, then*

$$Tw|_{Y_t} = T\overline{w}|_{Y_t} = \omega y_{t,k},$$

for some $k \ne \sigma_t(y)$ and $\omega \in [0,1]$.

Proof Observe that the leftmost equalities in each of the statements follow immediately from (4.1), so we shall focus only on the rightmost equalities.

(i) Let $\overline{z}$ be in the subcoral with diagonal $y\overline{x}$. Observe that ends of edges of length $\le \lambda^{m_0+t-1}$ with one end at $\overline{x}$ and the other end in the subcoral with the diagonal $\overline{x}y$ cannot be in P because then, by (4.4), the path through $\overline{x}$ would be shorter. Therefore, $T\overline{z} = (1-b)T\overline{x} + bTy + \overline{z}_t$, where

$$Ty \in B_t \overset{\text{def}}{=} \text{span}\{x_j : j \le \sum_{k=0}^{t} L(N_k)\} = \text{span}(\bigcup_{k=0}^{t} Y_k), \quad T\overline{x} \in B_{t-1},$$

$$\overline{z}_t \in T_t \overset{\text{def}}{=} \text{span}\{x_j : j > \sum_{k=0}^{t} L(N_k)\}, \quad b \ge \left(\frac{1}{2}\right)^{m_0-1}.$$

Note that

(4.5) $$Ty = \frac{1}{2}(Tx + T\overline{x}) + \pi_{t,y},$$

where $Tx, T\overline{x} \in B_{t-1}$ and $\pi_{t,y} \in Y_t$. Hence $T\overline{z}|_{Y_t} = b\pi_{t,y}$ and the conclusion follows.

(ii) If $\overline{z}$ is in the subcoral with diagonal yx, since yx is a part of a shortest path, we conclude that the longest edge in the part of the path P from y to $\overline{z}$ has length $\le \lambda^{t+m_0}$, where m_0 satisfies (4.4). Thus

(4.6) $$T\overline{z} = aTy + (1-a)Tx + z_t,$$

where $Ty \in B_t$, $Tx \in B_{t-1}$, $z_t \in T_t$, and

$$1 \ge a \ge 1 - \sum_{k=m_0}^{\infty} \left(\frac{1}{2}\right)^k = 1 - \left(\frac{1}{2}\right)^{m_0-1}.$$

By (4.5) we get $(Ty - T\overline{z})|_{Y_t} = \pi_{t,y} - a\pi_{t,y} = (1-a)\pi_{t,y}$ and the conclusion follows.

(iii) If $\overline{w}$ is in the subcoral for which xy is the diagonal, similarly as in (4.6), we obtain $T\overline{w} = cTx + (1-c)Ty + w_t$, where $Tx \in B_{t-1}$, $Ty \in B_t$, $w_t \in T_t$, and $1 \ge c \ge 1 - (\frac{1}{2})^{m_0-1}$. By (4.5), $T\overline{w}|_{Y_t} = (1-c)\pi_{t,y}$ and we are done in this case.

(iv) If $\overline{w}$ is not in the subcoral for which xy is the diagonal, let $q \in V_t$, $q \ne y$, be the vertex which is an endpoint of an edge of length λ^t which is a diagonal of the subcoral that contains $\overline{w}$. By construction, the projection of $T\overline{w}$ onto the subspace Y_t is a multiple of $y_{t,\sigma_t(q)} \ne y_{t,\sigma_t(y)}$, with some coefficient $\omega \in [0,1]$. ∎

Observe that Lemma 4.8 implies the estimate for the Lipschitz constant of T^{-1}, and thus Theorem 4.3, in all of the cases except the case where both (i) and (iii) hold. Consider, for example the case where (i) and (iv) hold. Then (we use (4.2), (4.3), the conclusions of (i) and (iv), the definition of $\pi_{t,y}$ and the observation that for any two

elements $y_1 \neq y_2$ of the set constructed in Lemma 2.6, the inequality $\|y_1 - ay_2\| \geq \delta/2$ holds for all a with $0 \leq a \leq 1$)

$$
\begin{aligned}
\frac{d_W(w, z)}{\|S_n w - S_n z\|} &\leq \frac{2\lambda^t}{(1-\lambda)\frac{1}{2C}\|\alpha\pi_{t,y} - \omega y_{t,k}\|} \\
&\leq \frac{4C\lambda^t}{(1-\lambda)\frac{\delta}{2}\alpha(\lambda - \frac{1}{2})\lambda^{t-1}} = \frac{8C\lambda}{(1-\lambda)\delta(\lambda - \frac{1}{2})(\frac{1}{2})^{m_0-1}},
\end{aligned}
$$

and this number depends only on C and λ.

It remains to consider the case where both (i) aand (iii) hold. In this case we estimate from below the norm of $(T\overline{w} - T\overline{z})|_{Y_{t-1}}$. We use $T\overline{z} = (1-b)T\overline{x} + bTy + \overline{z}_t$ and $T\overline{w} = cTx + (1-c)Ty + w_t$ with $1 \geq b \geq (\frac{1}{2})^{m_0-1}$ and $1 \geq c \geq 1 - (\frac{1}{2})^{m_0-1}$. The value of b actually does not matter for our argument, it is only important that $0 \leq b \leq 1$. Recall also that $Ty = \frac{1}{2}(Tx + T\overline{x}) + \pi_{t,y}$.

Therefore

$$
(T\overline{z} - T\overline{w})|_{Y_{t-1}} = \left(\left(1 - \frac{1}{2}b - \frac{1-c}{2}\right)T\overline{x} - \left(c + \frac{1-c}{2} - \frac{1}{2}b\right)Tx \right)\Big|_{Y_{t-1}}
$$

Observe that each of the coefficients of $T\overline{x}$ and Tx in this sum is at least

$$
\left(\frac{1}{2} - \left(\frac{1}{2}\right)^{m_0}\right).
$$

There are two possible cases.

Case A: $x \in V_{t-1}$, $\overline{x} \in \bigcup_{i=0}^{t-2} V_i$;

Case B: $\overline{x} \in V_{t-1}$, $x \in \bigcup_{i=0}^{t-2} V_i$.

The cases are similar, so we consider Case (A) only.

Subcase 1: There is $o \in \bigcup_{i=0}^{t-2} V_i$ such that x is in the subcoral with diagonal $o\overline{x}$, so $T\overline{x}, To \in B_{t-2}$, and

$$
Tx = \frac{1}{2}(T\overline{x} + To) + \left(\lambda - \frac{1}{2}\right)\lambda^{t-2}y_{t-1, \sigma_{t-1}(x)},
$$

and

$$
(T\overline{z} - T\overline{w})|_{Y_{t-1}} = -\left(c + \frac{1-c}{2} - \frac{1}{2}b\right)\left(\lambda - \frac{1}{2}\right)\lambda^{t-2}y_{t-1, \sigma_{t-1}(x)}.
$$

We get

$$
\begin{aligned}
\frac{d_W(w, z)}{\|T\overline{z} - T\overline{w}|} &\leq \frac{2\lambda^t}{(1-\lambda)\frac{1}{2C}(c + \frac{1-c}{2} - \frac{1}{2}b)(\lambda - \frac{1}{2})\lambda^{t-2}} \\
&\leq \frac{4C\lambda^2}{(1-\lambda)(\lambda - \frac{1}{2})(\frac{1}{2} - (\frac{1}{2})^{m_0})}.
\end{aligned}
$$

The obtained number depends only on C and λ.

Subcase 2: The vertex x has degree 1 in G_{t-1} so $T\overline{x} \in B_{t-2}$,

$$
Tx = T\overline{x} + \lambda^{t-1}y_{t-1, \sigma_{t-1}(x)},
$$

and

$$\left(T\overline{z} - T\overline{w}\right)\big|_{Y_{t-1}} = -\left(c + \frac{1-c}{2} - \frac{1}{2}b\right)\lambda^{t-1}y_{t-1,\sigma_{t-1}(x)}.$$

We get

$$\frac{d_W(w,z)}{\|T\overline{z} - T\overline{w}\|} \leq \frac{2\lambda^t}{(1-\lambda)\frac{1}{2C}\left(c + \frac{1-c}{2} - \frac{1}{2}b\right)\lambda^{t-1}}$$

$$\leq \frac{4C\lambda}{(1-\lambda)\left(\frac{1}{2} - \left(\frac{1}{2}\right)^{m_0}\right)}.$$

The obtained number depends only on C and λ. This concludes the proof. ∎

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