ON THE STEINER PROBLEM

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1. Introduction. Let M be a metric space with metric ρ which has the following properties.

1. M is finitely compact.
2. There exists a geodesic in M joining each two points of M .
3. For all a , $b \in M$, $\rho(a, b)$ is equal to the length of a geodesic joining a and b .

DEFINITION. Given N distinct points $b_1, \ldots, b_N$ in M , a tree U on the vertices $b_1, \ldots, b_N$ is a set of geodesics joining some of the $\binom{N}{2}$ pairs of points $b_i b_j$, with the property that any two vertices can be joined by a sequence of geodesics belonging to U in one and only one way. A geodesic $b_i b_j$ of U is called a branch of U , the length L(U) is the sum of the lengths of its branches, $\{b_i\}$ is the set of all vertices sending branches to the vertex b_i and $w(b_i)$ is the number of such vertices.

We consider the following problem.

S_n: Given a set $A = \{a_1, a_2, \ldots, a_n\}$ of $n \geq 3$ distinct points in M , to find the shortest tree(s) whose vertices contain these n points. In the Euclidean Plane S_n is called the Steiner Problem and we keep this title for our generalisations. Suppose a minimising tree U of S_n in the plane has additional vertices $s_1, \ldots, s_k$. Then

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P1. U is non self-intersecting.

P2. $w(s_i) = 3$, $i = 1, \ldots, k$.

P3. $w(a_j) \leq 3$, $j = 1, \ldots, n$.

P4. $0 \leq k \leq n-2$.

P5. Each s_i is the S-point of the triangle formed by $\{s_i\}$ (See [1] and [2]).

In [1] it is proved from these properties that the set of minimising trees of S_n in E_2 can be constructed using a finite set of Euclidean (ruler-compass) constructions. The proof contains errors, the algorithm is not clearly described and there is no attempt to make it more efficient.

In the first part of this paper we give another proof of this result which demonstrates the structure of minimising trees and continue with a discussion of how this structure can be used to reduce greatly the number of constructions. In the later sections we show that minimising trees of S_n in certain other spaces have the properties listed above and we conclude with a discussion of a generalisation of S_n posed in [1]. This discussion includes a proof (using the compactness of M) of the existence of all minimising trees mentioned in the paper.

2. An Effective Algorithm for the Steiner Problem in E_2 .

DEFINITIONS. A tree with vertices $\{a_1, \ldots, a_n\}$ and $\{s_1, \ldots, s_k\}$ (from this point on, these will be termed a-points and s-points respectively) has the property $\overline{\text{P}}4$ if $k = n-2$.

V is a subtree of a tree U if and only if (i) V is a tree and (ii) the set of geodesics of V is contained in the set of geodesics of U .

A tree U is an S-tree on A if it has properties P1, P2, P3, P4, P5 of section 1.

A tree U is an $\overline{\text{S}}$-tree on A if it has properties P1, P2, P3, $\overline{\text{P}}4$, P5.

A tree U is an S*-tree on A if it has properties P2, P3,

$\bar{P}4$, P5.

The finite set $\{A_1, A_2, \ldots, A_t, R\}$ $(t \geq 0)$ is a division of the set A if and only if

1. Each $A_i \subseteq A$, $R \subseteq A$.
2. $R \cap A_i = \phi$, $i = 1, \ldots, t$.
3. No a_j can be an element of more than 3 of the sets A_i.
4. Each A_i has 3 or more elements.
5. $A_1 \cup A_2 \cup \ldots \cup A_t \cup R = A$.

LEMMA 1. If U is any S-tree on A then for some division $\mathfrak{A} = \{A_1, A_2, \ldots, A_t, R\}$ of A there exist $\bar{S}$-subtrees of U on A_i for $i = 1, \ldots, t$.

Proof. If U contains no s-point, the required division is $\{A\}$. If S, the set of s-points of U, is non-empty, we define the relation "o" on S as follows: $s_i \text{ o } s_j$ if and only if the sequence of segments of U joining s_i to s_j contains no a-point of U. The relation is an equivalence relation and therefore partitions S into mutually exclusive and exhaustive sets $S_1, \ldots, S_t$ $(t > 0)$. For each $i = 1, \ldots, t$ define $A_i = \{a_j : a_j \in \{s_k\}$ for some $s_k \in S_i\}$, and $R = A - \bigcup_{i=1}^{t} A_i$. The set $\{A_1, A_2, \ldots, A_t, R\}$ is a division of A. It remains to show that there is an $\bar{S}$-subtree of U on each A_i. Let U_i be the subtree of U whose vertex set is $A_i \cup S_i$, $i = 1, \ldots, t$, (this is certainly a subtree of U by construction). U_i has the properties P1, P2, P3 and P5. We prove $\bar{P}4$. Let A_i contain p points, S_i q points and further suppose that S_i contains n_1, n_2 and n_3 s-points which directly join 1, 2 and 3 other s-points respectively. Then

$$(1) \qquad n_1 + n_2 + n_3 = q .$$

The number of branches of U_i connecting s-points is $(n_1 + 2n_2 + 3n_3)/2$. But by the defining property of S_i this number is $q-1$. Hence

$$(2) \qquad n_1 + 2n_2 + 3n_3 = 2(q-1) .$$

The number of branches of U_i connecting an a-point to an s-point is $2n_1 + n_2$ since the valency of each s-point is 3 . Hence

$$(3) \qquad 2n_1 + n_2 = p .$$

From equations (1) (2) and (3) we deduce $q = p-2$. Therefore U_i satisfies $\overline{P}4$ and is an $\overline{S}$-subtree of U . Hence the Lemma.

Incidentally one can also prove $n_1 - n_3 = 2$ from (1), (2), and (3), which implies that $n_1 \geq 2$. We note that the non self-intersection property is not involved in the establishment of these equations and state that an S*-tree on a set A has at least two s-points which directly join exactly one other s-point and two a-points. This fact will be used in the next Lemma.

We call the subtrees U_i $(i = 1, \ldots, t)$ the components of U and suggest that the components of a minimising tree U may be considered as stability sets for U in the following sense. If one a-point belonging to a component U_j is slightly perturbed, there is a minimising tree U' for the new set of n a-points which is identical to U except for a small perturbation of U_j .

If P, Q are points in the plane we shall denote by (PQ) and (QP) the third vertices of the equilateral triangle on PQ as base, (PQ) being the point to the left of P looking from P along PQ .

The construction we now explain is crucial to the proof. Let U be an $\overline{S}$-tree on $A = \{a_1, \ldots, a_n\}$ with s-points $s_1, \ldots, s_k$; then there exists (see note following Lemma 1) an s-point say s_1 which is connected directly to two a-points say a_1 and a_2 . In fact a portion of the tree appears as in Fig. 1.

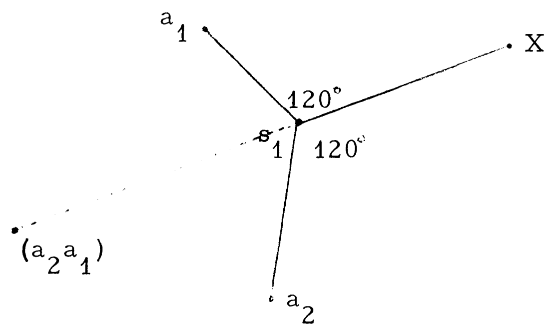

Figure 1

X is the third point of $\{s_1\}$ in U . Since s_1 is the S-point of $\{s_1\}$, the line Xs_1 produced passes through (a_2a_1) and the tree U' on $A' = \{(a_2a_1), a_3, \ldots, a_n\}$ with s-points $s_2, \ldots, s_k$ formed from U by replacing the branches a_1s_1 , a_2s_1 , s_1X by the single branch $(a_2a_1)X$ is an S*-tree on A' . (We cannot say that U' is an $\bar{S}$-tree since the non-self intersection property may have been contradicted.) It is easy to show that $a_1s_1 + a_2s_1 + s_1X = (a_2a_1)X$ and so the trees U and U' have equal lengths. Henceforth we shall refer to the above as the "Equilateral Construction" .

We next define the term "Association" of an $\bar{S}$-tree U on a set A . From U , we form a tree U' and set A' as above. The construction is repeated forming a new tree U'' and set A'' , U''' and A''' etc. until the set $A^{(r)}$ contains only two points. (Actually r is equal to the number of s-points in the original tree U). The two points of $A^{(r)}$ can be expressed in terms of the original a-points of U and the equilateral triangle bracketing notation defined above. This representation of $A^{(r)}$ we call an "Association" of the tree U . We give a simple example below. We note the following:

(i) The process is always possible since at every stage the tree $U^{(k)}$ is an S*-tree on $A^{(k)}$ and hence has an s-point which directly joins two a-points (in fact at least 2 such s-points by the note following Lemma 1).

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(ii) It follows from (i) that every $\bar{S}$-tree on A has an association (certainly not an unique association).

(iii) At each stage length is preserved. Thus if P,Q are the points of $A^{(r)}$ PQ is equal to the length of the original tree U .

(iv) No two $\bar{S}$-trees on A have a common association.

EXAMPLE. In Fig. 2, U is an $\bar{S}$-tree on $A = \{1, 2, 3, 4, 5\}$ with s-points s_1, s_2, s_3 .

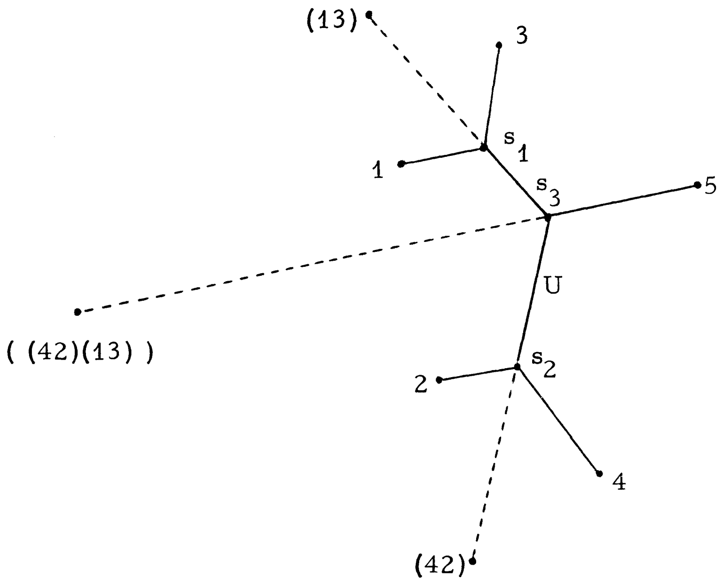

Figure 2

We "pair" the points 1 and 3 giving

$A' = \{(13), 2, 4, 5\}$ and U' with branches $(13)s_3$, $s_3 5$, $s_2 s_3$, $2s_2$, $4s_2$.

Next we pair 4 and 2

$A'' = \{(13), (42), 5\}$, U'' has branches $(42)s_3$, $(13)s_3$, $s_3 5$.

Finally we pair (13) and (42)

$A''' = \{\underline{((42)(13)), 5}\}$, U''' has branch ((42)(13))5 .

The underlined portion i.e., A''' without the set parentheses, is an association of U . The length of U is the length of the

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branch of U''' .

LEMMA 2. If it is known that U , an $\bar{S}$-tree on A , has a certain association α , we can construct U by a finite number of Euclidean constructions.

Proof. The Lemma is true for n = 3 . Assume it is true for n = N and let $A_{N+1} = \{a_1, \ldots, a_{N+1}\}$ be any plane set of N + 1 points on which U is an $\bar{S}$-tree with association α . Suppose the labelling of points in A_{N+1} is such that a_1, a_2 in this order in α have no brackets or comma separating them. We now consider the set $A_N = \{(a_1a_2), a_3, \ldots, a_{N+1}\}$. From the equilateral construction there exists U' , an $\bar{S}$-tree on A_N , which has association α except that (a_1a_2) is now regarded as a single point. By the inductive hypothesis we can construct U' by a finite number of Euclidean constructions. Let $(a_1a_2)X$, the branch of U' connecting (a_1a_2) , be replaced by the branches a_1s, a_2s, sX, where s is the point of intersection of the circle through $(a_1a_2), a_1, a_2$ with the line $(a_1a_2)X$. The resulting tree U , by the equilateral construction, is the (unique by note iv) $\bar{S}$-tree on A_{N+1} with association α . Hence the Lemma by induction.

LEMMA 3. The set of all $\bar{S}$-trees on $A = \{a_1, \ldots, a_n\}$ is finite and may be constructed by a finite number of Euclidean constructions.

Proof. Any combination of $A = \{a_1, \ldots, a_n\}$ by the above equilateral bracketing notation to form just two points, will be called an association of A . Then the set of all associations of all $\bar{S}$-trees on A is a subset of the finite set $\mathcal{B}$ of all associations of A (in fact a proper subset for $n > 3$) . If for each $b \in \mathcal{B}$ we perform the finite number of Euclidean constructions (Lemma 2) that construct the $\bar{S}$-tree on A with association b (if such a tree exists), we shall construct all the $\bar{S}$-trees on A . Hence the Lemma.

THEOREM 1. For every n , there exists a finite number of Euclidean constructions yielding all the minimising trees of the problem S_n.

<u>Proof.</u> The minimising trees of S_n are S-trees on A. It is therefore sufficient to show that all S-trees on A can be constructed by a finite number of Euclidean constructions.

Form the finite set $\mathfrak{A}$ of all divisions of A. Let $\mathfrak{A}_i \in \mathfrak{A}$ and suppose $\mathfrak{A}_i = \{A_{i1}, A_{i2}, \ldots, A_{it_i}, R_i\}$. For each $A_{ij} \in \mathfrak{A}_i$ we can construct, by a finite sequence of Euclidean constructions, the finite set C_{ij} of all $\bar{S}$-trees on A_{ij} (Lemma 3). If during this procedure we find an A_{ij} for which $C_{ij} = \phi$, we reject the division and move on to the next element of $\mathfrak{A}$. Suppose now $\mathfrak{A}_i \in \mathfrak{A}$ is such that $C_{ij} \neq \phi$ for $j = 1, \ldots, t_i$; we call such a division "acceptable".

For each $A_{ij} \in \mathfrak{A}_i$ we choose an element of C_{ij} and join up these trees to each other and the residual points R_i so as to form an S-tree on A. (The $\bar{S}$-trees on non-disjoint sets A_{ip}, A_{iq} are automatically joined. The joining process involves linking certain pairs of a-points to connect the graph but, of course, must not contradict the S-tree properties P1-P5.) By Lemma 1, if we take all the finite number of selections from the C_{ij}, use all the finite number of ways of joining them and the R_i which form S-trees on A, and do this for all $\mathfrak{A}_i$ in the finite set of acceptable divisions of A, we shall construct all S-trees on A by a finite number of Euclidean constructions. Hence the Theorem.

<u>Towards an Efficient Algorithm.</u> Our first result shows that a certain class of divisions of A are not "acceptable" and hence need not be considered by the algorithm. Let $\mathfrak{A} = \{A_1, \ldots, A_t, R\}$ be a division of A and conv A_i be the convex hull of A_i. If for any i, j, conv A_i - conv A_j is disconnected, then $\mathfrak{A}$ is not acceptable, for in such a case there are points a_{i1}, a_{i2} of A_i which are separated by conv A_j and a_{j1}, a_{j2} of A_j separated by A_i. The sequences of branches joining these pairs in any tree constructed from this division of

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A will intersect, contradicting one of the properties P1, P2, P3.

For each $i, j, i \neq j$ let L_{ij} be the set of points strictly inside both the circles circumscribing the equilateral triangles with $a_i a_j$ as base. In order that $(a_i a_j)$ may appear in the association of an $\bar{S}$-tree on A_i i.e. in order that a_i and a_j may be paired directly in a component, it is necessary that $A_i \cap L_{ij} = \phi$. Assume to the contrary that a_i, a_j are linked directly to S in a minimum tree U and there exists $a_k \in A_i \cap L_{ij}$. Then the longer of the two branches $a_i S, a_j S$ may be replaced by the shorter of the lines $a_i a_k$, $a_j a_k$, reducing the length of the assumed minimum tree. Thus a large class of associations cannot construct minimum trees and may be omitted from the algorithm.

If a set A_i on which we are to construct the set of all $\bar{S}$-trees forms a convex polygon, only adjacent vertices of the polygon can be paired in an association for otherwise the constructed tree would be self intersecting.

Since we are only interested in minimum length $\bar{S}$-trees on subsets A_i of A and the association method will evaluate the length of an $\bar{S}$-tree from an association before the s-points are actually constructed, we do not need to construct the majority of $\bar{S}$-trees on each A_i at all.

Finally we note that having constructed minimum length $\bar{S}$-trees on the sets $A_1, \ldots, A_t$ of a division of A, we wish to join these and the residual set R_i so that the resulting S-tree on A is of minimum length. An appropriate algorithm for this joining process can be obtained from [3].

3. S_n in Euclidean m-space E_m $(m \geq 3)$.

THEOREM 2. The minimising trees of S_n in $E_m (m \geq 3)$ have the properties P1-P5 listed in Section 1.

Proof. We first show each vertex of a minimising tree U which has a-points $a_1, \ldots, a_n$ and s-points $s_1, \ldots, s_n$ has valency ≤ 3. For suppose U has a vertex x and branches xx_i along the directions of the unit vectors u_i, $i = 1, \ldots, 4$. Then one of the angles at x (say $\measuredangle\ x_1xx_2$) is less than $120°$ since $u_i \cdot u_j \leq -\frac{1}{2}$ for all $i \neq j$ implies that

$$w^2 = [\sum_{i=1}^{4} u_i]^2 = \sum_{i=1}^{4} u_i^2 + 2 \sum_{i \neq j} u_i \cdot u_j < 4 + 2.6.(-\frac{1}{2}) = -2$$

which is impossible. It follows (see [1]), that xx_1, xx_2 is not the minimum length network connecting xx_1x_2 contrary to assumption. Similarly branches of U may not intersect except at a vertex.

Therefore U has properties P1, P3 and $w(s_i) \leq 3$ for $i = 1, \ldots, k$. But $w(s_i) \geq 3$ or there would be no gain in introducing the additional vertex s_i. Hence P2. Property P5 is immediate from [1]. It remains to prove P4. The number of branches leading to s-points (using P2) is $\geq 3k/2$, the connectivity of U assures us that the number of branches from a-points is $\geq n/2$, and a tree with $n + k$ vertices has $n + k - 1$ branches. Therefore $(n + 3k)/2 \leq n + k - 1$ from which we deduce $k \leq n - 2$. Hence P4.

4. Steiner Problem on a Surface D in E_3.

We shall assume that D has no singularities of any kind. The main purpose of this section is to prove that minimising trees of S_n in D have properties which are identical to those in E_m. We first prove two results which show that the $120°$ property of additional vertices (see [1]) holds in D.

Suppose A, B, C are distinct points in D and $P \notin \{A, B, C\}$ minimises the sum $\rho(P, A) + \rho(P, B) + \rho(P, C)$. We prove that the angles at P between the geodesics PA, PB, PC are each $120°$. Let $\rho(P, A) = a$, $\rho(P, B) = b$ and $\rho(P, C) = c$. Consider the geodesic ellipse E (= the locus of points Z such that $\rho(Z, A) + \rho(Z, B) = a + b$) and the geodesic circle $\rho(Z, C) = c$. These

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closed curves touch at P , for otherwise there would be a point Y interior to both curves such that $\rho(Y, A) + \rho(Y, B) < a + b$ and $\rho(Y, C) < c$ contradicting the minimum property of P . Since geodesic PC meets $\rho(Z, C) = c$ orthogonally, geodesic PC meets E orthogonally. By a result of classical differential geometry [4, page 120] E bisects the angle between the geodesic parallels $\rho(Z, A) = a$ and $\rho(Z, B) = b$ and therefore, since the geodesics AP, BP meet these circles orthogonally, the angles α and β of Fig. 3 are equal. Therefore

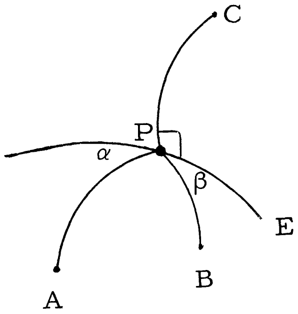

Figure 3

$$\measuredangle\ APC = \measuredangle\ BPC . \tag{4}$$

Similarly by considering the geodesic ellipse $\rho(Z, A) + \rho(Z, C) = a + c$, we prove $\measuredangle$ APB = $\measuredangle$ BPC and this together with (4) proves the result.

Secondly we show that if ABC is a geodesic triangle on D with the angle at A less than 120° , then A does not minimise the sum $\rho(Z, A) + \rho(Z,B) + \rho(Z, C)$. Let V be an ϵ-neighborhood of A sufficiently small so that for all $r, s \in V$ there is only one geodesic joining them. Let B(V) intersect the geodesics AB , AC in X and Y . Consider the following 1-1 mapping of the geodesic triangle AXY onto the tangent plane at A . For Q in the geodesic triangle AXY with $\rho(A, Q) = q$, the corresponding point Q' is the point on the tangent line to the geodesic AQ at A such that AQ' = q . Since the angle A of the plane triangle AX'Y' is less than 120° , there exists P' in the tangent plane such that AP' + X'P' + Y'P' < AX' + AY' (see [1]) and furthermore the difference is proportional to ϵ i.e.,

$$AX' + AY' - (AP' + X'P' + Y'P') = \epsilon k_1 \text{ for some } k_1 .$$

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If ϵ is sufficiently small, for all $Q_1, Q_2 \epsilon V$, $|Q_1'Q_2' - \rho(Q_1, Q_2)| < L\epsilon^2$ where L is constant (the proof of this is elementary but long) and hence if $P \epsilon V$ corresponds to P' in the tangent plane,

$$|(AP' + X'P' + Y'P') - \{\rho(A,P) + \rho(X,P) + \rho(Y,P)\}| < k_2\epsilon^2 \text{ for some } k_2 .$$

Therefore

$$\{\rho(A,X) + \rho(A,Y)\} - \{\rho(A,P) + \rho(X,P) + \rho(Y,P)\}$$

$$= (AX' + AY') - (AP' + X'P' + Y'P')$$

$$+ (AP' + X'P' + Y'P') - \{\rho(A,P) + \rho(X,P) + \rho(Y,P)\}$$

$$> k_1\epsilon - k_2\epsilon^2 > 0 \text{ if } \epsilon \text{ is sufficiently small.}$$

Therefore

$$\rho(A,X) + \rho(A,Y) > \rho(A,P) + \rho(X,P) + \rho(Y,P) .$$

If we now add $\rho(X,B) + \rho(X,C)$ to each side and apply the triangle inequality on the right we obtain

$$\rho(A,B) + \rho(A,C) > \rho(A,P) + \rho(B,P) + \rho(C,P)$$

showing that A does not minimise $\rho(Z,A) + \rho(Z,B) + \rho(Z,C)$ as required.

Using these 120° properties and a proof identical to that of Theorem 2 we deduce that U, a minimising tree of S_n in D with extra vertices $s_1, \ldots, s_k$, has the properties P1, P2, P3, P4 listed in Section 1 and the following analog of P5 :

P'5 : For each $i = 1, \ldots, k$ if $\{s_i\}$ contains points p_i, q_i, r_i then each of the angles at s_i between the geodesics p_is_i, q_is_i, r_is_i is 120° .

5. The Steiner Problem in Plane Minkowski Metric Spaces.

Let Σ be a centrally symmetric convex surface in E_m with centre 0 . The m-dimensional Minkowski Metric Space

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M_m associated with Σ is obtained by defining the distance $\rho(x, y)$ for $x, y \in E_m$ as follows. If $x = y$, $\rho(x, y) = 0$. If $x \neq y$ let the ray with initial point 0 which is parallel to xy meet Σ at P. Then $\rho(x, y) = xy/OP$ where xy and OP are usual Euclidean distances. M_m is a metric space satisfying conditions 1, 2, 3 of the introduction. In this section, for brevity, we omit most of the proofs which depend only on the following easily verified properties of M_m (see [5], page 21). x and y are to be considered as m-dimensional vectors. For all $x, y \in M_m$

(i) $\rho(x, y) = \rho(0, x-y)$ and more generally, any translation is an isometry.

(ii) The triangle inequality is strict provided that Σ is strictly convex and the three points involved are non-collinear.

(iii) For Σ strictly convex

$$\rho(0, x+y) \leq \rho(0, x) + \rho(0, y)$$

and this inequality is strict unless $0, x, y$ are collinear with x, y lying on the same side of 0.

For the remainder of this section by M_2 we shall mean a plane Minkowski metric space the defining curve of which is strictly convex.

PROPOSITION 1. Given any n distinct non-collinear points $a_1, \ldots, a_n$ in M_2, there exists a unique point z minimising the function $f(z) = \sum_{i=1}^{n} \rho(z, a_i)$.

<u>Proof.</u> A simple compactness argument shows that a minimum exists and it is easily verified that for all $z_1, z_2 \in M_2$,

$$f\left(\frac{z_1 + z_2}{2}\right) < \frac{1}{2} f(z_1) + \frac{1}{2} f(z_2) .$$

Hence $f(z)$ is strictly convex and has an unique minimum.

PROPOSITION 2. Let A, P be any two points in M_2. Then there exist two points B, one on either side of the line AP, each having the following properties:

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1. P minimises the function $f(z) = \rho(z, A) + \rho(z, B) + \rho(z, P)$.

2. For any point C strictly within the angle APB , P does not minimise the function $\rho(z, A) + \rho(z, C) + \rho(z, P)$.

The angles APB of Proposition 2 are called Critical Angles. For the Euclidean Metric each critical angle is 120° . In Minkowski spaces critical angles will vary in their Euclidean magnitude.

PROPOSITION 3. Let A, B, C be any three distinct points in M_2 and suppose P minimises $\rho(z, A) + \rho(z, B) + \rho(z, C)$. Then either

(i) $P \in \{A, B, C\}$,

or

(ii) there exists an unique point X at which the sides of the triangle subtend critical angles and P = X .

PROPOSITION 4. Let angles APB , BPC be supplementary angles. Then P cannot minimise both of the functions

$$f_1(z) = \rho(z, A) + \rho(z, B) + \rho(z, P)$$

and

$$f_2(z) = \rho(z, C) + \rho(z, B) + \rho(z, P) .$$

At this point we digress and state the following fact which will be used in the next section. Suppose A, B, C are distinct points in an m-dimensional Minkowski Metric Space with strictly convex defining surface S and let the plane π defined by A, B, C meet S in the curve Σ . Then the above theory holds in the plane Minkowski Space defined on π by Σ i.e., we can apply Propositions 1-4 to three points ABC in an m-dimensional Minkowski Metric Space.

THEOREM 3. Let U be a minimising tree of S_n in M_2 with additional vertices $s_1, \ldots, s_k$. Then U has the properties P1-P4 of Section 1 and the following analog of P5 :

P''5 : For each $i = 1, \ldots, k$, s_i minimises $\rho(z, a) + \rho(z, b) + \rho(z, c)$ where a, b, c are the points of $\{s_i\}$ and each angle as_ib, bs_ic, cs_ia is a critical angle.

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Proof. Suppose the branches AB, CD intersected at P (not a vertex of U). Then for some pair of points from A, B, C, D say A, B the point which minimises $\rho(z, A) + \rho(z, B) + \rho(z, P)$ is $X \neq P$ (Proposition 4) and a replacement of AP, PB by the three lines XP, XA, XB shortens the assumed minimising tree, which proves P1. A similar argument shows that no vertex x of U can have $w(x) > 3$. Thus U satisfies P3 and has the property $w(s_i) \leq 3$ for all $i = 1, \ldots, k$. Also $w(s_i) \nless 3$ (hence P2) and P4 holds, the proofs being identical to those given in Theorem 2; finally P''5 is immediate from Proposition 3.

In [6] an example is given where the defining convex curve of the Minkowski Metric is not strictly convex. The property $w(x) \leq 3$ for each vertex x of a minimising tree of S_n does not hold.

6. The Problem $S_{n\alpha\beta\gamma}$.

Consider the following generalisation of the Steiner Problem:
$S_{n\alpha\beta\gamma}$: Given three non-negative real numbers α, β, γ and n distinct points $a_1, \ldots, a_n \in M$ to find an integer k and k points $s_1, \ldots, s_k \in M$ and to construct the tree(s) U on the vertices $a_1, \ldots, a_n$, $s_1, \ldots, s_k$ so as to minimise the sum

$$T = L(U) + \alpha \sum_{j=1}^{n} w(a_j) + \beta \sum_{j=1}^{k} w(s_j) + \gamma k .$$

DEFINITION. U is a μ-tree on $A = \{a_1, \ldots, a_n\}$ if U is a tree with vertices $a_1, \ldots, a_n$, $s_1, \ldots, s_k$ and

(i) $w(s_i) \geq 3$, $i = 1, \ldots, k$.

(ii) $0 \leq k \leq n-2$.

PROPOSITION 5. If a solution of $S_{n\alpha\beta\gamma}$ exists, it is a μ-tree on A.

Proof. $w(s_i) \geq 2$ for each $i = 1, \ldots, k$. Suppose $w(s_i) = 2$ for some i and $\{s_i\} = \{x, y\}$. Then the tree formed by replacing branches $s_i x$, $s_i y$ by the branch xy has a smaller value of T. Hence (i), and (ii) is proved exactly as in Theorem 2.

DEFINITION. Let U be a μ-tree on $A = \{a_1, \ldots, a_n\}$ with additional vertices $s_1, \ldots, s_k$.

By the association of U we mean the integer k and the sets $\{a_i\}$ and $\{s_j\}$ $(i = 1, \ldots, n, \quad j = 1, \ldots, k)$.

THEOREM 4. The problem $S_{n\alpha\beta\gamma}$ is reducible to a finite number of minimum length problems.

Proof. The relation "has the same association as" on the set of all μ-trees on A is an equivalence relation. We show that the number of equivalence classes is finite. Suppose there are k additional vertices. Then we have $n + k$ points on which to construct a tree i.e., $n + k - 1$ branches must be selected from the possible $\binom{n+k}{2}$ geodesics. Thus the number of associations with k extra vertices is not greater than

$$g(n, k) = \binom{\binom{n+k}{2}}{n + k - 1},$$

and the total number of associations is not greater than $\sum_{k=0}^{n-2} g(n, k)$ and hence is finite.

Let the equivalence classes be $C_1, \ldots, C_N$ and let C_i be any one of these classes. The tree(s) which minimise T in C_i are precisely the tree(s) of minimum length of C_i since the association common to all the tree(s) of C_i fixes the other three terms of T i.e., for all $U \in C_i$

$$\alpha \sum_{i=1}^{n} w(a_i) + \beta \sum_{j=1}^{k} w(s_j) + \gamma k \text{ is constant .}$$

The minimising trees of $S_{n\alpha\beta\gamma}$ are a subset of the trees belonging to $C_1, \ldots, C_N$ (Proposition 5). Therefore each minimising tree of $S_{n\alpha\beta\gamma}$ is a solution to one of the following N minimum length problems: For $i = 1, \ldots, N$ to construct the tree(s) $U \in C_i$ which minimise $L(U)$.

The next Theorem and its Corollary prove the existence of all minimising trees mentioned in this paper.

THEOREM 5. Let $C \in \{C_1, \ldots, C_N\}$; there exists a tree of minimum length in C.

Proof. The association of trees in C stipulates which of the pairs $s_i a_j$, $s_i s_j$, $a_i a_j$ will be joined by geodesics as branches of trees in C. We exclude the case $k = 0$ for which the Theorem is obvious. Let $A_i = \{a_{i1}, a_{i2}, \ldots, a_{i\lambda_i}\} = \{a_t : a_t \in A$ and $s_i a_t$ is a branch of trees in $C\}$. Let R_1, R_2 be sets of unordered pairs of integers, defined as follows:

$$R_1 = \{(i, j) : s_i s_j \text{ is a branch of trees in } C\},$$

$$R_2 = \{(i, j) : a_i a_j \text{ is a branch of trees in } C\}.$$

Then the length of a tree in C is

$$f(s_1, \ldots, s_k) = \sum_{i=1}^{k} \sum_{j=1}^{\lambda_i} \rho(s_i, a_{ij})$$

$$+ \sum_{(i,j)\in R_1} \rho(s_i, s_j) + \sum_{(i,j)\in R_2} \rho(a_i, a_j).$$

Suppose L is the length of the shortest tree with vertices $a_1, \ldots, a_n$ only. Let $Z = \{z : z \in M$ and $\min_i \rho(z, a_i) \leq L\}$. Then every s-point of a tree of shortest length in C is in Z for otherwise the length of the tree would necessarily be greater than L.

Thus if $\{s_1, \ldots, s_k\}$ is a set of s-points of a minimum length tree of C then $\{s_1, \ldots, s_k\}$ is an element of the cartesian product Z^k. Now since Z is closed and bounded and M is finitely compact, Z is compact implying by the Tychonoff Theorem that Z^k is compact. But $f(s_1, \ldots, s_k)$ is continuous on Z^k and so has a minimum value on Z^k. Hence the Theorem.

COROLLARY. There exist minimising trees of $S_{n\alpha\beta\gamma}$.

Proof. Immediate from this Theorem and Theorem 4.

THEOREM 6. If M is a Minkowski Metric space M_m for which the defining surface Σ is strictly convex, then there is an unique tree of minimum length in $C \epsilon \{C_1, \ldots, C_N\}$.

Proof. Suppose to the contrary that f has minima, value ℓ, at $\{s_1, \ldots, s_k\}$ and $\{t_1, \ldots, t_k\}$ where $t_i \neq s_i$ for some i. Consider the set

$$\left\{ \frac{s_1+t_1}{2}, \ldots, \frac{s_k+t_k}{2} \right\} .$$

Using Properties (i) - (iii) of Minkowski Spaces (Section 5)

$$\rho\left(\frac{s_i+t_i}{2}, a_{ij}\right) = \rho\left(0, \frac{s_i-a_{ij}}{2} + \frac{t_i-a_{ij}}{2}\right)$$

$$\leq \rho\left(0, \frac{s_i-a_{ij}}{2}\right) + \rho\left(0, \frac{t_i-a_{ij}}{2}\right)$$

$$= \frac{1}{2}\rho(0, s_i-a_{ij}) + \frac{1}{2}\rho(0, t_i-a_{ij})$$

$$= \frac{1}{2}\rho(s_i, a_{ij}) + \frac{1}{2}\rho(t_i, a_{ij}) .$$

This inequality is strict unless a_{ij}, s_i, t_i are collinear with s_i, t_i on the same side of a_{ij}. Therefore

$$\sum_{i=1}^{k} \sum_{j=1}^{\lambda_i} \rho\left(\frac{s_i+t_i}{2}, a_{ij}\right) \leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{\lambda_i} \{\rho(s_i, a_{ij}) + \rho(t_i, a_{ij})\}$$

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and the inequality is strict unless for each $i = 1, \ldots, k$, a_{i1}, $a_{i2}, \ldots, a_{i\lambda_i}$, s_i, t_i are collinear with s_i, t_i occupying suitable positions on the line. Such a situation cannot occur in a minimum length tree of C. Assuming $n > 2$ (otherwise the problem is trivial), there exists i for which $\lambda_i \geq 2$ and s_i joins only one other s-point. If $\lambda_i > 2$ a simple application of the triangle inequality proves that the assumed tree could not be minimum length in C; the case $\lambda_i = 2$ is disposed of using Proposition 4. Thus we can conclude

$$(5) \quad \sum_{i=1}^{k} \sum_{j=1}^{\lambda_i} \rho\left(\frac{s_i+t_i}{2}, a_{ij}\right) < \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{\lambda_i} \rho(s_i, a_{ij}) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{\lambda_i} (t_i, a_{ij}) .$$

By a similar use of the properties of M_m we can show

$$\sum_{(i,j)\in R_1} \rho\left(\frac{s_i+t_i}{2}, \frac{s_j+t_j}{2}\right) \leq \frac{1}{2} \sum_{(i,j)\in R_1} \rho(s_i, s_j) + \frac{1}{2} \sum_{(i,j)\in R_1} \rho(t_i, t_j).$$

Adding this to (5) and $\sum_{(i,j)\in R_2} \rho(a_i, a_j)$ to both sides we obtain

$$f\left(\frac{s_1+t_1}{2}, \ldots, \frac{s_k+t_k}{2}\right) < \frac{1}{2} f(s_1, \ldots, s_k) + \frac{1}{2} f(t_1, \ldots, t_k) = \ell$$

which contradicts the minimum property of ℓ.

COROLLARY. In M_m, $S_{n\alpha\beta\gamma}$ and S_n have a finite number of minimising trees.

Proof. Immediate from Theorems 4, 5 and 6.

For suitable values of α, β, γ (see [1]), $S_{n\alpha\beta\gamma}$ reduces to the problem P_n: Given any n distinct points $a_1, \ldots, a_n$ in M, to find the point z which minimises $\sum_{i=1}^{n} \rho(z, a_i)$. Work is currently in progress on a proof, using Galois Theory, that in the plane P_n (and hence $S_{n\alpha\beta\gamma}$) is in general not solvable by Euclidean Constructions.

Similar proofs to those given in this section may be used to establish identical results when the function to be minimised is

$$F\left(L(U)\ ,\ \sum_{i=1}^{n} w(a_i)\ ,\ \sum_{j=1}^{k} w(s_j)\ ,\ k\right)$$

where F is any positive function which is strictly increasing in each of its four variables.

REFERENCES

1. Z.A. Melzak, On the Problem of Steiner. Canadian Mathematical Bulletin, Vol. 4, (1961), pp. 143-148.

2. R. Courant and H. Robbins, What is Mathematics? (New York 1941).

3. R.C. Prim, Shortest Connecting Networks. Bell System Tech. Journal 31, pp. 1398-1401.

4. C.E. Wetherburn, Differential Geometry of Three Dimensions. (Cambridge 1947).

5. L.M. Blumenthal, Theory and Applications of Distance Geometry. (Oxford 1953).

6. M. Hanan, On the Steiner Problem with Rectilinear Distance. S.I.A.M. Journal of Applied Mathematics, Vol. 14 (1966), pp. 255-265.

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