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Cyclic Cubic Fields of Given Conductor and Given Index

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Abstract. The number of cyclic cubic fields with a given conductor and a given index is determined.

1 Introduction

Let K be a cyclic cubic extension of $\mathbb{Q}$ so that $[K:\mathbb{Q}] = 3$ and $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}$. By the Kronecker–Weber theorem [10, p. 289] there exists a positive integer m such that the cyclotomic field $\mathbb{Q}\left(e^{2\pi i/m}\right) \supseteq K$. The smallest such m is called the conductor of K and is denoted by $f(K)$. The discriminant of K is given by $d(K) = f(K)^2$ [8, p. 831]. The conductor $f(K)$ of a cyclic cubic field is of the form

$$f = p_1 p_2 \cdots p_r, \tag{1.1}$$

where $r \in \mathbb{N}$ and $p_1, \dots, p_r$, are distinct integers from the set

$$P = \{9\} \cup \{p \text{ (prime)} \equiv 1 \pmod 3\} = \{7, 9, 13, 19, 31, 37, \dots\}, \tag{1.2}$$

see [8, p. 831]. Moreover each positive integer f of the form (1.1) is the conductor of some cyclic cubic field; indeed it is the conductor of 2^{r-1} cyclic cubic fields [8, p. 831]. For any cubic field K it is known that its field index $i(K) = 1$ or 2 [5, p. 234]. For f of the form (1.1) and $i \in \{1, 2\}$, we define

$$N(f, i) = \text{number of cyclic cubic fields } K \text{ with } f(K) = f \text{ and } i(K) = i, \tag{1.3}$$

so that

$$N(f, 1) + N(f, 2) = 2^{r-1}. \tag{1.4}$$

In this paper we determine $N(f, 1)$ and $N(f, 2)$.

It is well known that each prime $p \equiv 1 \pmod 3$ has a unique representation in the form

$$4p = a^2 + 27b^2, \quad a, b \in \mathbb{N}, \tag{1.5}$$

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see [1, Theorem 3.1.3, p. 105; Lemma 3.0.1, p. 101]. Clearly for such a representation we have $a \equiv b \pmod 2$ and

$$\gcd(a,b) = 1 \text{ or } 2. \tag{1.6}$$

It is a classical result of Gauss that 2 is a cubic residue (mod p) if and only if $\gcd(a,b) = 2$, see [1, Theorem 7.1.1, p. 213]. We set

$$P_1 = \{9\} \cup \{p\,(\text{prime}) \equiv 1 \pmod 3, 4p = a^2 + 27b^2, \gcd(a,b) = 1\} \tag{1.7}$$

and

$$P_2 = \{p(\text{prime}) \equiv 1 \pmod 3, 4p = a^2 + 27b^2, \gcd(a,b) = 2\}, \tag{1.8}$$

so that

$$P_1 \cup P_2 = P, \quad P_1 \cap P_2 = \phi. \tag{1.9}$$

Clearly

$$P_1 = \{7, 9, 13, 19, 37, \dots\}, \quad P_2 = \{31, 43, 109, 127, \dots\}.$$

If p is a prime in P_1, then $a \equiv b \equiv 1 \pmod 2$. Replacing b by $-b$, if necessary, we may suppose that $a \equiv b \pmod 4$. Set $x = (a-b)/4 \in \mathbb{Z}$, $y = b \in \mathbb{Z}$. Then $4x^2 + 2xy + 7y^2 = p$. Conversely if $p = 4x^2 + 2xy + 7y^2$ for some $x, y \in \mathbb{Z}$ then y is odd, $\gcd(x,y) = 1$ and $4p = a^2 + 27b^2$ with $a = |4x + y|$, $b = |y|$ and $\gcd(a,b) = \gcd(4x + y, y) = \gcd(4x, y) = \gcd(x, y) = 1$. Thus the primes in P_1 are precisely those which can be expressed in the form $4x^2 + 2xy + 7y^2$ for some $x, y \in \mathbb{Z}$. The primes in P_2 are precisely those which can be expressed in the form $x^2 + 27y^2$ for some $x, y \in \mathbb{Z}$.

Now suppose that f is of the form (1.1) with

$$p_1, p_2, \dots, p_u \in P_1 \quad \text{and} \quad p_{u+1}, p_{u+2}, \dots, p_r \in P_2,$$

where $u \in \{0, 1, \dots, r\}$. In Section 5 we prove the following result.

Theorem *With the above notation, we have*

$$N(f,1) = \frac{1}{3}(2^r - (-1)^u 2^{r-u}), \quad N(f,2) = \frac{1}{3}(2^{r-1} + (-1)^u 2^{r-u}).$$

In Sections 2, 3, 4 we give some results on representations of integers by binary quadratic forms which will be needed in the proof of this theorem.

2 The Form Class Group $H(d)$

Let $H(d)$ denote the set of classes of primitive, positive-definite, integral binary quadratic forms $(a, b, c) = ax^2 + bxy + cy^2$ of discriminant $d = b^2 - 4ac \equiv 0$ or $1 \pmod 4$ under the action of the modular group. As $ax^2 + bxy + cy^2$ is positive-definite, we have $a > 0$ and $d < 0$. The class of the form (a, b, c) is denoted by $[a, b, c]$. Multiplication of classes of $H(d)$ is due to Gauss and is described, for example, in [2]. With respect to multiplication, $H(d)$ is a finite abelian group called the form class group of discriminant d. The order of $H(d)$ is called the form class number of discriminant d and is denoted by $h(d)$. The identity I of the group $H(d)$ is the principal class

$$I = \begin{cases} [1, 0, -d/4] & \text{if } d \equiv 0 \pmod 4, \\ [1, 1, (1-d)/4] & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

The inverse of the class $K = [a, b, c] \in H(d)$ is the class $K^{-1} = [a, -b, c] \in H(d)$. Each class of $H(d)$ contains one and only one form (a, b, c) with

$$(2.1) \qquad -a < b \le a \le c, \quad b \ge 0 \ \text{ if } a = c, \quad b^2 - 4ac = d, \quad \gcd(a, b, c) = 1,$$

see [4, pp. 68-71]. Let $n \in \mathbb{N}$. If x and y are integers such that $n = ax^2 + bxy + cy^2$, then (x, y) is called a representation of the positive integer n by the form (a, b, c). As (a, b, c) is a positive-definite form, the number $R_{(a,b,c)}(n)$ of representations of n by the form (a, b, c) is finite. If in addition the representation (x, y) satisfies $\gcd(x, y) = 1$, then the representation is called primitive. The number of primitive representations of n by the form (a, b, c) is denoted by $P_{(a,b,c)}(n)$. Clearly,

$$(2.2) \qquad R_{(a,b,c)}(n) = \sum_{e^2 \mid n} P_{(a,b,c)}(n/e^2).$$

If (A, B, C) is a form equivalent to (a, b, c) it is well known that $R_{(A,B,C)}(n) = R_{(a,b,c)}(n)$ and $P_{(A,B,C)}(n) = P_{(a,b,c)}(n)$. Hence we can define the number of representations of $n \in \mathbb{N}$ by the class $K \in H(d)$ by

$$(2.3) \qquad R_K(n) = R_{(a,b,c)}(n) \text{ for any } (a, b, c) \in K$$

and the number of primitive representations of $n \in \mathbb{N}$ by the class $K \in H(d)$ by

$$(2.4) \qquad P_K(n) = P_{(a,b,c)}(n) \text{ for any } (a, b, c) \in K.$$

From (2.2)–(2.4) we deduce that for $n \in \mathbb{N}$ and $K \in H(d)$

$$(2.5) \qquad R_K(n) = \sum_{e^2 \mid n} P_K(n/e^2).$$

In particular, if $n \in \mathbb{N}$ is squarefree, we have

$$(2.6) \qquad R_K(n) = P_K(n).$$

As each representation (x, y) of n by (a, b, c) gives a representation $(x, -y)$ of n by $(a, -b, c)$ and conversely, we have for $n \in \mathbb{N}$ and $K \in H(d)$

$$R_K(n) = R_{K^{-1}}(n), \quad P_K(n) = P_{K^{-1}}(n). \tag{2.7}$$

For $n_1, n_2 \in \mathbb{N}$ with n_1 squarefree, n_2 squarefree and $\gcd(n_1, n_2) = 1$, it is known that

$$R_K(n_1 n_2) = \frac{1}{w(d)} \sum_{K_1 K_2 = K} R_{K_1}(n_1) R_{K_2}(n_2), \tag{2.8}$$

where K_1, K_2 run through all the classes of $H(d)$ whose product is K, and

$$w(d) = 6, 4 \text{ or } 2 \text{ according as } d = -3,\ d = -4 \text{ or } d < -4, \tag{2.9}$$

see [9, (29) and Lemma 5.5]. The largest positive integer f such that $f^2 \mid d$ with $\Delta = d/f^2 \equiv 0$ or $1 \pmod 4$ is called the conductor of d. By a theorem of Dirichlet, see [6], we have for $\gcd(n, f) = 1$

$$\sum_{K \in H(d)} R_K(n) = w(d) \sum_{e \mid n} \left(\frac{d}{e}\right) = w(d) \sum_{e \mid n} \left(\frac{\Delta}{e}\right), \tag{2.10}$$

where $\left(\frac{d}{*}\right)$ is the Legendre–Jacobi–Kronecker symbol of discriminant d. If p is a prime such that $\left(\frac{d}{p}\right) = 1$, then there is at least one class $C \in H(d)$ which represents p. If $C = C^{-1}$, then C is the only class of $H(d)$ representing p and $R_C(p) = 2w(d)$. If $C \neq C^{-1}$, then C and C^{-1} are the only classes of $H(d)$ representing p and $R_C(p) = R_{C^{-1}}(p) = w(d)$. See [9, Lemma 5.3].

3 Representations of Integers by [1,0,3]

From (2.1) with $d = -12$ we find

$$H(-12) = \{I\}, \quad h(-12) = 1,$$

where

$$I = [1, 0, 3].$$

Here $f = 2$ and $\Delta = -3$.

Lemma 3.1 *Let $p_1, \dots, p_t$ $(t \geq 0)$ be distinct primes $\equiv 1 \pmod 3$. Then*

$$R_I(p_1 \cdots p_t) = 2^{t+1}, \quad P_I(p_1 \cdots p_t) = 2^{t+1},$$
$$R_I(9 p_1 \cdots p_t) = 2^{t+1}, \quad P_I(9 p_1 \cdots p_t) = 0.$$

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Proof If $n \in \mathbb{N}$ is such that $\gcd(n,2) = 1$, by (2.9) and (2.10) with $d = -12$, we have

$$R_I(n) = 2\sum_{e|n}\left(\frac{-3}{e}\right). \tag{3.1}$$

Taking $n = p_1 \cdots p_t$, as $(\frac{-3}{p_i}) = 1$ $(i = 1, \ldots, t)$, we obtain

$$R_I(p_1 \cdots p_t) = 2 \sum_{e|p_1\cdots p_t} 1 = 2 \cdot 2^t = 2^{t+1}.$$

Then, appealing to (2.6), we obtain

$$P_I(p_1 \cdots p_t) = 2^{t+1}.$$

Taking $n = 9p_1 \cdots p_t$ in (3.1), since $\left(\frac{-3}{3}\right) = 0$ we obtain

$$R_I(9p_1 \cdots p_t) = 2 \sum_{e|9p_1\cdots p_t} \left(\frac{-3}{e}\right) = 2 \sum_{e|p_1\cdots p_t} \left(\frac{-3}{e}\right) = 2^{t+1}.$$

Finally, by (2.5), we have

$$R_I(9p_1 \cdots p_t) = P_I(9p_1 \cdots p_t) + P_I(p_1 \cdots p_t),$$

so that

$$P_I(9p_1 \cdots p_t) = 2^{t+1} - 2^{t+1} = 0.$$

This completes the proof of the lemma. ∎

4 Representations of Integers by [1,0,27] and [4,2,7]

From (2.1) with $d = -108$ we find

$$H(-108) = \{I, A, A^2\} \simeq \mathbb{Z}/3\mathbb{Z}, \quad h(-108) = 3,$$

where

$$I = [1, 0, 27], \quad A = [4, 2, 7], \quad A^2 = [4, -2, 7], \quad A^3 = I.$$

Here $f = 6$ and $\Delta = -3$.

Let p be a prime with $p \equiv 1 \pmod 3$. Then

$$\left(\frac{d}{p}\right) = \left(\frac{-108}{p}\right) = \left(\frac{-2^2 \cdot 3^3}{p}\right) = \left(\frac{-3}{p}\right) = 1,$$

so that p is represented by some class in $H(-108)$. If p is represented by I, then (as $I = I^{-1}$) I is the only class representing p, and

$$R_I(p) = 4, \quad R_A(p) = R_{A^2}(p) = 0. \tag{4.1}$$

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If p is represented by A or A^2, then (as $A \neq A^{-1}$) the only classes of $H(-108)$ representing p are A and A^2, and

$$R_I(p) = 0, \quad R_A(p) = R_{A^2}(p) = 2. \tag{4.2}$$

Now let m be a product of distinct primes $\equiv 1 \pmod 3$. By (2.9) and (2.10) we have

$$R_I(m) + R_A(m) + R_{A^2}(m) = 2 \sum_{e \mid m} \left(\frac{-3}{e} \right) = 2^{\tau(m)+1},$$

where $\tau(m)$ denotes the number of primes dividing m. As $R_A(m) = R_{A^{-1}}(m) = R_{A^2}(m)$ by (2.7), we deduce that

$$R_A(m) = R_{A^2}(m) = 2^{\tau(m)} - \frac{1}{2} R_I(m). \tag{4.3}$$

By (2.8) we have for $p \nmid m$

$$R_I(pm) = \frac{1}{2} \left(R_I(p) R_I(m) + R_A(p) R_{A^2}(m) + R_{A^2}(p) R_A(m) \right). \tag{4.4}$$

Appealing to (4.1)–(4.4), we obtain

$$R_I(pm) = \begin{cases} 2R_I(m) & \text{if } R_I(p) > 0, \\ 2^{\tau(m)+1} - R_I(m) & \text{if } R_A(p) > 0. \end{cases} \tag{4.5}$$

We now use (4.5) to prove the following result.

Lemma 4.1 *Let $p_1, \ldots, p_l$ be $l (\geq 0)$ distinct primes $\equiv 1 \pmod 3$, which are represented by $I = [1, 0, 27]$, and let $q_1, \ldots, q_m$ be $m (\geq 0)$ distinct primes $\equiv 1 \pmod 3$, which are represented by $A = [4, 2, 7]$. Then*

$$R_I(p_1 \cdots p_l q_1 \cdots q_m) = \frac{1}{3} \left(2^{l+m+1} + (-1)^m 2^{l+2} \right),$$

$$R_A(p_1 \cdots p_l q_1 \cdots q_m) = R_{A^2}(p_1 \cdots p_l q_1 \cdots q_m) = \frac{1}{3} \left(2^{l+m+1} - (-1)^m 2^{l+1} \right).$$

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Proof By (4.5) we obtain

$$
\begin{aligned}
R_I(p_1 \cdots p_l q_1 \cdots q_m)& \\
&= 2R_I(p_1 \cdots p_{l-1} q_1 \cdots q_m) \\
&= 2^2 R_I(p_1 \cdots p_{l-2} q_1 \cdots q_m) \\
&= \cdots \\
&= 2^l R_I(q_1 \cdots q_m) \\
&= 2^l \big(2^m - R_I(q_1 \cdots q_{m-1})\big) \\
&= 2^l \big(2^m - 2^{m-1} + R_I(q_1 \cdots q_{m-2})\big) \\
&= \cdots \\
&= 2^l \big(2^m - 2^{m-1} + 2^{m-2} - \cdots + (-1)^{m-2} 2^2 + (-1)^{m-1} R_I(q_1)\big) \\
&= 2^l \big(2^m - 2^{m-1} + 2^{m-2} - \cdots + (-1)^{m-2} 2^2\big) \\
&= \frac{1}{3}\big(2^{l+m+1} + (-1)^m 2^{l+2}\big),
\end{aligned}
$$

as required. Then, by (4.3), we obtain

$$
\begin{aligned}
R_A(p_1 \cdots p_l q_1 \cdots q_m) &= 2^{l+m} - \frac{1}{2}\Big(\frac{1}{3}\big(2^{l+m+1} + (-1)^m 2^{l+2}\big)\Big) \\
&= \frac{1}{3}\big(2^{l+m+1} - (-1)^m 2^{l+1}\big),
\end{aligned}
$$

as asserted. ■

5 Proof of Theorem

There is a one-to-one correspondence between cyclic cubic fields K and triples $(a, b, f) \in \mathbb{N}^3$ with

$$
\begin{gathered}
a^2 + 27b^2 = 4f, \quad \gcd(a, b) = 1 \text{ or } 2, \quad f = p_1 \cdots p_r, \\
r \in \mathbb{N}, \quad p_1, \dots, p_r \in P, \quad p_i \neq p_j \quad (1 \le i < j \le r),
\end{gathered}
$$

see [3, Section 6.4.6, pp. 336–343]. The cyclic cubic field corresponding to the triple (a, b, f) is $K = \mathbb{Q}(\theta)$, where $\theta^3 - 3f\theta + fa = 0$. The conductor of K is f. The index of K is

$$
i(K) = \begin{cases} 2 & \text{if } a \text{ is even}, \\ 1 & \text{if } a \text{ is odd}, \end{cases}
$$

see [7, Theorem 4, p. 585]. If a is even, then b is even and $\left(\frac{a}{2}\right)^2 + 27\left(\frac{b}{2}\right)^2 = f$ with $\gcd\left(\frac{a}{2}, \frac{b}{2}\right) = 1$. Thus

$$
N(f, 2) = \frac{1}{4} P_{[1,0,27]}(f).
$$

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First suppose that $9 \nmid f$. We may suppose that $p_1, \dots, p_u \in P_1$ (so they are represented by $[4,2,7]$) and $p_{u+1}, \dots, p_r \in P_2$ (so they are represented by $[1,0,27]$) with $u \in \{0,1,2,\dots,r\}$. Then, by (2.6) and Lemma 4.1 (with $l = r-u$ and $m = u$), we have

$$P_{[1,0,27]}(f) = R_{[1,0,27]}(f) = \frac{1}{3}(2^{r+1} + (-1)^u 2^{r-u+2}),$$

so that

$$N(f,2) = \frac{1}{3}(2^{r-1} + (-1)^u 2^{r-u}), \quad 9 \nmid f.$$

Now suppose that $9 \mid f$. We may suppose that $p_1 = 9$, $p_2, \dots, p_u \in P_1$ (so they are represented by $[4,2,7]$) and $p_{u+1}, \dots, p_r \in P_2$ (so they are represented by $[1,0,27]$). As $9 \mid f$ we have

$$f = x^2 + 27y^2 \Longleftrightarrow f/9 = (x/3)^2 + 3y^2$$

so that

$$R_{[1,0,27]}(f) = R_{[1,0,3]}(f/9).$$

As $f/9$ is squarefree, we have

$$R_{[1,0,27]}(f/9) = P_{[1,0,27]}(f/9).$$

From (2.5) we deduce

$$R_{[1,0,27]}(f) = P_{[1,0,27]}(f) + P_{[1,0,27]}(f/9).$$

Thus

$$P_{[1,0,27]}(f) = R_{[1,0,3]}(f/9) - R_{[1,0,27]}(f/9).$$

Appealing to Lemma 3.1 (with $t = r-1$) and Lemma 4.1 (with $l = r-u$ and $m = u-1$), we obtain

$$P_{[1,0,27]}(f) = 2^r - \frac{1}{3}\left(2^r + (-1)^{u-1} 2^{r-u+2}\right) = \frac{1}{3}\left(2^{r+1} + (-1)^u 2^{r-u+2}\right)$$

so that

$$N(f,2) = \frac{1}{3}(2^{r-1} + (-1)^u 2^{r-u}), \quad 9 \mid f.$$

Finally, from (1.4), we obtain in both cases

$$N(f,1) = 2^{r-1} - N(f,2) = \frac{1}{3}(2^r - (-1)^u 2^{r-u}).$$

This completes the proof of the theorem. ■

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