Compositio Math. 141 (2005) 359–384
DOI: 10.1112/S0010437X0400123X

# Rational cohomology of the moduli space of genus 4 curves

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Abstract

We compute the mixed Hodge structure on the rational cohomology of the moduli space of smooth genus 4 curves. Specifically, we prove that its Poincaré–Serre polynomial is $1 + t^2u^2 + t^4u^4 + t^5u^6$. We show this by producing a stratification of the space, such that all strata are geometric quotients of complements of discriminants.

## 1. Introduction and results

In this paper the rational cohomology of the moduli space $\mathcal{M}_4$ of non-singular complex genus 4 curves is computed. This is achieved by considering a natural stratification of $\mathcal{M}_4$, and determining the cohomology of each stratum. Non-singular genus 4 curves can be divided into three classes (see [Har77, IV.5.2.2 and IV.5.5.2]):

(i) curves whose canonical model is the complete intersection of a cubic surface and a non-singular quadric surface in $\mathbb{P}^3$;

(ii) curves whose canonical model is the complete intersection of a cubic surface and a quadric cone in $\mathbb{P}^3$;

(iii) hyperelliptic curves.

Denote by $C_0$ the locus in $M_4$ of curves of type (i), by $C_1$ the locus of curves of type (ii) and by $C_2$ the hyperelliptic locus. We have a three-step filtration

$$C_2 \subset \overline{C_1} \subset \overline{C_0} = \mathcal{M}_4. \tag{1}$$

The space $C_2$ can be studied from the theory of binary forms, and it is easy to show that it has the rational cohomology of a point. The spaces $C_0$ and $C_1$ are moduli spaces of smooth complete intersections, and their rational cohomology was not known. We choose to consider the elements of $C_0$ as representing isomorphism classes of non-singular curves of type $(3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Analogously, elements of $C_1$ can be regarded as isomorphism classes of non-singular curves of degree 6 on the weighted projective space $\mathbb{P}(1,1,2)$. The space $C_0$ is then the quotient of an open subset of the space of bihomogeneous polynomials of bidegree $(3,3)$ in the two sets of indeterminates $x_0, x_1$ and $y_0, y_1$, by the action of the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$. Analogously, $C_1$ is the quotient of an open subset of $\Gamma(\mathcal{O}_{\mathbb{P}(1,1,2)}(6))$ by the action of the automorphism group of $\mathbb{P}(1,1,2)$.

This means that in both cases we are in the situation of [PS03]. As a standard choice of notation, we denote the vector space by $V$, its dimension by $N$, the open subset which we are interested in quotienting by $X$, and the complement of $X$ in $V$ by $\Sigma$ ($V$ is a hypersurface called the *discriminant hypersurface*). We denote the group acting on $X$ by $G$, the geometric quotient by $\varphi : X \to X/G$,

Received 29 September 2003, accepted in final form 16 April 2004, published online 10 February 2005.
*2000 Mathematics Subject Classification* 14H10 (primary), 55R80, 14R30, 14F99 (secondary).
*Keywords:* rational cohomology, moduli of curves, discriminants.

and the orbit inclusion by $\rho$:

$$\begin{array}{ccc} G & \longrightarrow & X \\ g & \longmapsto & g(x) \end{array}$$

where $x$ is a fixed point of $X$. Recall (see [Che52]) that $H^\bullet(G)$ is an exterior algebra freely generated by classes $\eta_i \in H^{2r_i-1}(G)$. We show that in both cases in which we are interested, the following generalization of the Leray–Hirsch theorem applies.

Theorem 1.1 [PS03]. *Suppose there are subschemes $Y_i \subset \Sigma$ of pure codimension $r_i$ in $V$ whose fundamental classes map to a non-zero multiple of $\eta_i$ under the composition*

$$\bar{H}_{2(N-r_i)}(Y_i) \to \bar{H}_{2(N-r_i)}(\Sigma) \xrightarrow{\sim} H^{2r_i-1}(X) \xrightarrow{\rho^*} H^{2r_i-1}(G),$$

*where $\bar{H}_\bullet$ indicates the Borel–Moore homology of the space. Denote the image of $[Y_i]$ in $H^\bullet(X;\mathbb{Q})$ by $y_i$; then the map $a \otimes \eta_i \mapsto \varphi^* a \cup y_i$, $a \in H^\bullet(X/G;\mathbb{Q})$ extends to an isomorphism of graded $\mathbb{Q}$-vector spaces*

$$H^\bullet(X/G;\mathbb{Q}) \otimes H^\bullet(G;\mathbb{Q}) \xrightarrow{\sim} H^\bullet(X;\mathbb{Q}).$$

Note that Theorem 1.1 gives no information about the ring structure of the cohomology of $X, G$ and $X/G$.

We use Theorem 1.1 to interpret $H^\bullet(X;\mathbb{Q})$ as a tensor product of the rational cohomology of $X/G$ and $G$. This allows us to recover $H^\bullet(C_j;\mathbb{Q})$ from the other data. The problem is then essentially reduced to the computation of the rational cohomology of $X$. Vassiliev invented a general topological method for calculating the cohomology of complements of discriminants, or, more generally, of spaces of non-singular functions. First, the computation of the cohomology of $X$ is reduced to that of the (Alexander dual) Borel–Moore homology of its complement $\Sigma$. Secondly, Vassiliev constructed a simplicial resolution of $\Sigma$, and a stratification of this resolution, based on the study of possible configurations of the singularities of the elements of $\Sigma$. The spectral sequence associated with this filtration is proved to converge to the Borel–Moore homology of $\Sigma$. Vassiliev used this method in [Vas99] to determine, for instance, the real cohomology of the space of non-singular quadric curves in $\mathbb{P}^2$, and of non-singular cubic surfaces in $\mathbb{P}^3$. Gorinov [Gor01] modified Vassiliev's method so that it applies to a wider range of situations. In this way, he could calculate the real cohomology of the space of quintic curves in $\mathbb{P}^2$. We compute the cohomology of the complements of the discriminants we are interested in by a modification of Gorinov–Vassiliev's method. In this paper we include an exposition of the method we use. The functoriality of the whole construction is emphasized by describing it in the language of cubical schemes. Note that Gorinov–Vassiliev's method respects the mixed Hodge structure of the space $X$. Hence, we can use it to find the mixed Hodge structure of the cohomology of the complement of the discriminant.

By applying the techniques above, we show the following results. They are stated by means of Poincaré–Serre polynomials. The Poincaré–Serre polynomial of a variety $Z$ is the polynomial in two indeterminates $t, u$ such that the coefficient of $t^i u^j$ is the dimension of the weight $j$ subquotient of the cohomology of $Z$ in degree $i$.

Theorem 1.2. *The space $C_0$ of non-singular genus 4 curves whose canonical model lies on a non-singular quadric in $\mathbb{P}^3$ has Poincaré–Serre polynomial $1 + t^5u^6$.*

Theorem 1.3. *The space $C_1$ of smooth genus 4 curves whose canonical model lies on a quadric cone in $\mathbb{P}^3$ has the rational cohomology of a point.*

Theorems 1.2 and 1.3 allow us to compute the spectral sequence associated with the filtration (1). This establishes the main result.

Theorem 1.4. *The Poincaré–Serre polynomial of $\mathcal{M}_4$ is $1 + t^2u^2 + t^4u^4 + t^5u^6$.*

In particular, Theorem 1.4 yields the Serre characteristic of the cohomology with compact support of $\mathcal{M}_4$ as $L^9 + L^8 + L^7 - L^6$.

Theorem 1.4 agrees with what was previously known about the cohomology of $\mathcal{M}_4$. In particular, its Euler characteristic had been computed by Harer and Zagier in [HZ86], and found to be 2. It also agrees with the known algebraic classes of $\mathcal{M}_4$, as computed in [Fab90].

The plan of the paper is as follows. In § 2 we explain our version of Gorinov–Vassiliev's method for computing cohomology of complements of discriminants. We formulate it by using the language of $\mathcal{A}$-cubical schemes, which is also introduced. Moreover, we show or recall some useful homological results. In §§ 3 and 4 we calculate explicitly the rational cohomology of $C_0$ and $C_1$ respectively, establishing Theorems 1.2 and 1.3. In § 5 we give a short proof, in the style of the paper, of the fact that the moduli space of hyperelliptic curves of genus $g \geqslant 2$ has the rational cohomology of a point.

### 1.1 Notation and conventions

The symbol $\mathbb{C}^n$ denotes the Euclidean space $\mathbb{C}^n \cong \mathbb{R}^{2n}$. The complex projective space of dimension $n$ over $\mathbb{C}$ is denoted by $\mathbb{P}^n$. The Grassmannian of linear subspaces of dimension $m$ in the vector space $V$ is denoted by $G(m, V)$. The symbol $S_n$ denotes the symmetric group in $n$ elements, $\mathrm{GL}(n)$ and $\mathrm{PGL}(n)$ indicate, respectively, the general linear group and the projective linear group of dimension $n$ over $\mathbb{C}$.

The symbol $\bar{H}_\bullet(Z, S)$ denotes Borel–Moore homology of the space $Z$ with the local system of coefficients $S$. In this paper we make an extensive use of Borel–Moore homology, i.e. homology with locally finite support. For its definition and for the properties we use, please refer, for instance, to [Ful84, Chapter 19]. When appropriate, we consider the Poincaré–Serre polynomial of the Borel–Moore homology, defined as the polynomial in $\mathbb{Z}[t, u^{-1}]$ such that the coefficient of $t^i u^{-j}$ is the weight $(-j)$ subquotient of the Borel–Moore homology in degree $i$.

The $N$-dimensional standard simplex in $\mathbb{R}^{N+1}$ is denoted by $\Delta_N$ and $\mathring{\Delta}_N$ denotes its interior.

All complex varieties are considered with the analytic topology. We denote the Tate Hodge structure on $\mathbb{Q}$ of weight $-2k$ by $\mathbb{Q}(k)$ (for the definition, see [Del71]).

If not otherwise specified, all cohomology and Borel–Moore homology groups are considered with rational coefficients.

## 2. Topological tools

### 2.1 Gorinov–Vassiliev's method

We explain here the method we use for computing the cohomology of complements of discriminants. It has been formulated by Vassiliev (see, for instance, [Vas99]), and successively modified by Gorinov [Gor01] in a way that allows it to be applied to a wider range of situations. We slightly modified Gorinov–Vassiliev's method though, in order to adapt it to configurations on a projective algebraic variety, not necessarily smooth.

Definition 2.1. Let $Z$ be a projective variety. A subset $S \subset Z$ is called a *configuration* in $Z$ if it is compact and non-empty. The space of all configurations in $Z$ is denoted by $\mathrm{Conf}(Z)$.

We can choose a metric on $Z \subset \mathbb{P}^N$, by considering the restriction $\mu$ of the Fubini–Study metric on $\mathbb{P}^N$. We have that $(Z, \mu)$ is a compact complete metric space. The metric $\mu$ induces a function $\tilde{\mu} : \mathrm{Conf}(Z) \times \mathrm{Conf}(Z) \longrightarrow \mathbb{R}$, by posing

$$\tilde{\mu}(S_1, S_2) = \max\{\mu(x, S_2) : x \in S_1\} + \max\{\mu(S_1, y) : y \in S_2\}.$$

Proposition 2.2 [Gor01]. *The pair* $(\mathrm{Conf}(Z), \tilde{\mu})$ *is a compact complete metric space.*

Let us fix $Z$, and consider a vector space $V$ such that there is a map

$$\begin{array}{ccc} V & \longrightarrow & \mathrm{Conf}(Z) \cup \{\emptyset\} \\ v & \longmapsto & K_v, \end{array}$$

such that $K_0 = Z$, and $L(K) := \{v \in V : K \subset K_v\}$ is a linear space for all $K \in \mathrm{Conf}(Z)$.

The most natural case is that in which the elements of $V$ can be seen as sections in a line bundle on $Z$ and $K_v$ is the set of singular points of $v$.

We define the discriminant as

$$\Sigma := \{v \in V : K_v \neq \emptyset\}.$$

The method is based on the fact that there is a direct relation between the cohomology of the complement of $\Sigma$ in $V$ and the Borel–Moore homology of $\Sigma$. The cap product with the fundamental class $[\Sigma]$ of the discriminant induces for all indices $i$ an isomorphism

$$\tilde{H}^i(V \setminus \Sigma; \mathbb{Q}) \cong H^{i+1}(V, V \setminus \Sigma; Q) \overset{\cap[\Sigma]}{\cong} \bar{H}_{2M-1-i}(\Sigma; \mathbb{Q})(-M),$$

where we have denoted the complex dimension of $V$ by $M$, and the reduced cohomology of the space by $\tilde{H}$.

Our aim is to compute the Borel–Moore homology of $\Sigma$. Gorinov constructed a simplicial resolution for $\Sigma$, starting from a collection $X_1, \dots, X_N$ of families of configurations in $Z$. For his construction to work, the $X_i$ have to satisfy some axioms. We list below the properties that we require for $X_1, \dots, X_N$.

List 2.3.

(i) For every element $v \in \Sigma$, $K_v$ must belong to some $X_i$.

(ii) If $x \in X_i$, $y \in X_j$, $x \subsetneq y$, then $i < j$.

(iii) For every index $j = 1, \dots, N$ the space $L(x) \subset \Sigma$ has the same dimension $d_j$ for every configuration $x \in X_j$. Moreover, for all indices $j$ the map:

$$\begin{array}{cccc} \varphi_j : & X_j & \longrightarrow & G(d_j, V) \\ & v & \longmapsto & L(K_v) \end{array}$$

from the configuration space $X_j$ to the Grassmannian of linear subspaces of dimension $d_j$ of $V$, is continuous.

(iv) $X_i \cap X_j = \emptyset$ for $i \neq j$.

(v) Any $x \in \bar{X}_i \setminus X_i$ belongs to some $X_j$ with $j < i$.

(vi) For every $i$ the space

$$\mathcal{T}_i = \{(p, x) \in Z \times X_i : p \in x\},$$

with the evident projection, is the total space of a locally trivial bundle over $X_i$.

(vii) Suppose $X_i$ consists of finite configurations. Then for all $y, x$ such that $x \in X_i$, $y \subsetneq x$, the configuration $y$ belongs to $X_j$ for some index $j < i$.

Note that the maps $\varphi_j$ of condition (iii) are always continuous, if $L(\{z\})$ has the same dimension $d$ for all $z \in Z$ and the map to the Grassmannian of linear subspaces of dimension $d$ of $V$,

$$\begin{array}{cccc} L : & Z & \longrightarrow & G(d, V) \\ & z & \longmapsto & L(\{z\}) \end{array}$$

is continuous.

In the construction of the resolution, we use the language of *cubical spaces*.



https://doi.org/10.1112/S0010437X0400123X Published online by Cambridge University Press

DEFINITION 2.4. A *cubical space* over an index set $\mathcal{A}$ (briefly, an $\mathcal{A}$-cubical space) is a collection of topological spaces $\{Y(I)\}_{I\subset\mathcal{A}}$ such that for each inclusion $I \subset J$ we have a natural continuous map $f_{IJ}: Y(J) \to Y(I)$ such that $f_{IK} = f_{IJ} \circ f_{JK}$ whenever $I \subset J \subset K$.

We are ready to define the cubical space and the index set we work with:

$$\begin{gathered}\mathcal{A} = \{1, 2, \ldots, N\}\\ \Lambda(I) := \Big\{A \in \prod_{i\in I} X_i : i < j \Rightarrow A_i \subset A_j\Big\}.\\ \mathcal{X}(I) := \{(F, A) \in \Sigma \times \Lambda(I) : K_F \supset A_{\max(I)}\} \text{ if } I \neq \emptyset,\\ \mathcal{X}(\emptyset) := \Sigma.\end{gathered}$$

Analogously, we define the following (auxiliary) cubical spaces:

$$\begin{gathered}\tilde{\Lambda}(I) := \Big\{A \in \prod_{i\in I} \overline{X}_i : i < j \Rightarrow A_i \subset A_j\Big\},\\ \tilde{\mathcal{X}}(I) := \{(F, A) \in \Sigma \times \tilde{\Lambda}(I) : K_F \supset A_{\max(I)}\} \text{ if } I \neq \emptyset,\\ \tilde{\mathcal{X}}(\emptyset) := \Sigma.\end{gathered}$$

The subsets $I \subset \mathcal{A}$ represent the vertices of the cube $\square_{\mathcal{A}}$, which are in one-to-one correspondence with the faces $\Delta_I$ of the simplex

$$\Delta_{\mathcal{A}} := \Big\{(f : \mathcal{A} \to [0, 1]) : \sum_{a\in\mathcal{A}} f(a) = 1\Big\}.$$

By definition, $\Delta_I = \{f \in \Delta_{\mathcal{A}} : f|_{\mathcal{A}\setminus I} = 0\}$. There is a natural map associated with every inclusion $I \subset J$, which we denote by $e_{IJ} : \Delta_I \to \Delta_J$.

DEFINITION 2.5. Let $Y(\bullet)$ be a cubical space over an index set $\mathcal{A}$.

Note that $Y(\bullet)$ has a natural augmentation towards $Y(\emptyset)$. Then the *geometric realization* of $Y(\bullet)$ is defined as the map

$$|\epsilon| : \big|Y(\bullet)\big| \longrightarrow Y(\emptyset)$$

induced from the natural augmentation on the space

$$\big|Y(\bullet)\big| = \coprod_{I\subset\mathcal{A}} (\Delta_I \times Y(I))/R,$$

where $R$ is the equivalence relation given by

$$(f, y)\ R\ (f', y') \Leftrightarrow f' = e_{IJ}(f), y = f_{IJ}(y').$$

Note that $\Delta_\emptyset = \emptyset$ so that $Y(\emptyset)$ does not appear in the construction of the geometric realization.

We construct the geometric resolution of all the cubical spaces defined above. We can define a surjective map $\varphi$ between $\big|\tilde{\Lambda}(\bullet)\big|$ and $\big|\Lambda(\bullet)\big|$ as follows. Let $(t, A) \in \Delta_I \times \tilde{\Lambda}(I)$, and let $[t, A]$ be the corresponding class in $\big|\tilde{\Lambda}(\bullet)\big|$. Note that by conditions (iv) and (v) in List 2.3, for each $A_i$ $(i \in I)$ there exists a unique index $k(i) \in \mathcal{A}$ such that $A_i \in X_{k(i)}$.

We define $\varphi([t,A])$ as the class in $\left|\Lambda(\bullet)\right|$ of the element $(s,B)\in\Delta_J\times\Lambda(J)$, where

$$J := \{k\in\mathcal{A} : A_i\in X_k \text{ for some } i\in I\} = \{k(i) : i\in I\};$$
$$B := \prod_{k\in J} B_k,\quad B_k := A_i \quad \text{for any index } i : k(i)=k;$$
$$s : J \longrightarrow [0,1],\quad s(k) := \sum_{i\in I : k(i)=k} t(i).$$

Observe that $\varphi$ acts by contracting all closed simplices in $\left|\tilde\Lambda(\bullet)\right|$ corresponding to inclusions of configurations of the form $x=x=x=\cdots=x$.

We analogously define the map

$$\begin{array}{rccc}\psi: & \left|\tilde{\mathcal{X}}(\bullet)\right| & \longrightarrow & \left|\mathcal{X}(\bullet)\right| \\ & [(F,A),t] & \longmapsto & (F,\varphi(A,t)).\end{array}$$

In the rest of the paper, we consider the spaces $\left|\tilde\Lambda(\bullet)\right|$ and $\left|\tilde{\mathcal{X}}(\bullet)\right|$ with the quotient topology under the equivalence relation $R$ of the direct product topology of the spaces $\tilde\Lambda(I)$ (and the spaces $\mathcal{X}(I)$, respectively). The topology on $\left|\Lambda(\bullet)\right|$ and $\left|\mathcal{X}(\bullet)\right|$ is the topology induced by $\varphi$ (respectively, $\psi$).

Proposition 2.6 [Gor01]. *The geometric realization of $\mathcal{X}(\bullet)$,*

$$|\epsilon| : \left|\mathcal{X}(\bullet)\right| \longrightarrow \mathcal{X}(\emptyset)=\Sigma,$$

*is a homotopy equivalence and induces an isomorphism on Borel–Moore homology groups.*

*Proof.* It is enough to prove that $|\epsilon|$ is a proper map, and that its fibers are contractible.

Fix a compact subset $W\subset\Sigma$. We define a cubical space as follows:

$$\tilde{\mathcal{X}}_W(I) := \{(F,A)\in W\times\tilde\Lambda(I) : K_F\supset A_{\max(I)}\} \text{ if } I\neq\emptyset,$$
$$\tilde{\mathcal{X}}_W(\emptyset) := W.$$

Note that $\tilde{\mathcal{X}}_W(I)$ is compact for all $I\in\mathcal{A}$, and so is the space $\left|\tilde{\mathcal{X}}_W(\bullet)\right|$.

Then $|\epsilon|^{-1}(W)$ is compact, because it coincides with the image of the continuous map

$$\begin{array}{rccc}\psi_W: & \left|\tilde{\mathcal{X}}_W(\bullet)\right| & \longrightarrow & \left|\mathcal{X}(\bullet)\right| \\ & [(F,A),t] & \longrightarrow & (F,\varphi(A,t)).\end{array}$$

Hence, the map $\left|\epsilon\right|$ is proper.

Next, we show that the fibers of $\left|\epsilon\right|$ are contractible. Consider the fiber over $v\in\Sigma$. By conditions (i) and (iv) of List 2.3, there is a unique index $j$ such that $K_v\in X_j$. By definition, $|\epsilon|^{-1}(v)$ is a cone with vertex $[(K_v,v)]\in\left|\mathcal{X}|_{\{j\}}(\bullet)\right|\hookrightarrow\left|\mathcal{X}(\bullet)\right|$, so it is clearly contractible. ☐

For every index set $I\subset\mathcal{A}$, we can restrict $\Lambda(\bullet)$ and $\mathcal{X}(\bullet)$ to the index set $I$, obtaining the two $I$-cubical spaces $\Lambda|_I(\bullet)$ and $\mathcal{X}|_I(\bullet)$. Then for every $I\subset J\subset\mathcal{A}$, there are natural embeddings $\left|\Lambda|_I(\bullet)\right|\hookrightarrow\left|\Lambda|_J(\bullet)\right|$ and $\left|\mathcal{X}|_I(\bullet)\right|\hookrightarrow\left|\mathcal{X}|_J(\bullet)\right|$. In this way, we can define an increasing filtration on $\left|\Lambda(\bullet)\right|$ by posing

$$\mathrm{Fil}_j\left|\Lambda(\bullet)\right| := \mathrm{Im}\left(\left|\Lambda|_{\{1,2,\dots,j\}}\right|\hookrightarrow\left|\Lambda(\bullet)\right|\right)$$

for $j=1,\dots,N$. We analogously define the filtration $\mathrm{Fil}_j\left|\mathcal{X}(\bullet)\right|$ on $\left|\mathcal{X}(\bullet)\right|$. We use the notation $F_j := \mathrm{Fil}_j\left|\mathcal{X}(\bullet)\right|\setminus\mathrm{Fil}_{j-1}\left|\mathcal{X}(\bullet)\right|$, $\Phi_j := \mathrm{Fil}_j\left|\Lambda(\bullet)\right|\setminus\mathrm{Fil}_{j-1}\left|\Lambda(\bullet)\right|$.

The filtration $\mathrm{Fil}_j\left|\mathcal{X}(\bullet)\right|$ defines a spectral sequence that converges to the Borel–Moore homology of $\Sigma$. Its term $E^1_{p,q}$ is isomorphic to $\bar H_{p+q}(F_p;\mathbb{Q})$.



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PROPOSITION 2.7 [Gor01]. (1) *For every $j = 1, \dots, N$, the stratum $F_j$ is a complex vector bundle of rank $d_j$ over $\Phi_j$. The space $\Phi_j$ is, in turn, a fiber bundle over the configuration space $X_j$.*

(2) *If $X_j$ consists of configurations of $m$ points, the fiber of $\Phi_j$ over any $x \in X_j$ is an $(m-1)$-dimensional open simplex, which changes its orientation under the homotopy class of a loop in $X_j$ interchanging a pair of points in $x_j$.*

(3) *If $X_N = \{Z\}$, $F_N$ is the open cone with vertex a point (corresponding to the configuration $Z$), over $\mathrm{Fil}_{N-1}\left|\Lambda(\bullet)\right|$.*

We recall here the topological definition of an open cone.

DEFINITION 2.8. Let $B$ be a topological space. Then a space is said to be an *open cone* over $B$ with vertex a point if it is homeomorphic to the space

$$B \times [0,1)/R,$$

where the equivalence relation is $R = (B \times \{0\})^2$.

*Proof of Proposition 2.7.* The second and the third point are clear by construction. The first point is trivial for $F_j$. For the map $\Phi_j \longrightarrow X_j$, the fiber over a configuration $x$ is given by a union of simplices with vertices determined by the points of $x \subset Z$. Thus, the fibration is locally trivial as a consequence of condition (vi) in List 2.3. □

### 2.2 Homological lemmas

The fiber bundle $\Phi_j \to X_j$ of Proposition 2.7 is, in general, non-orientable. As a consequence, we have to consider the homology of $X_j$ with coefficients not in $\mathbb{Q}$, but in some local system of rank one. Therefore, we recall here some results and constructions about the Borel–Moore homology of configuration spaces, with twisted coefficients.

DEFINITION 2.9. Let $Z$ be a topological space. Then for every $k \geqslant 1$ we have the space of ordered configurations of $k$ points in $Z$,

$$F(Z,k) = Z^k \Big\backslash \bigcup_{1 \leqslant i < j \leqslant k} \{(z_1, \dots, z_k) \in Z^k : z_i = z_j\}.$$

There is a natural action of the symmetric group $S_k$ on $F(k,Z)$. The quotient is called the space of unordered configurations of $k$ points in $Z$,

$$B(Z,k) = F(Z,k)/S_k.$$

The *sign representation* $\pi_1(B(Z,k)) \to \mathrm{Aut}(\mathbb{Z})$ maps the paths in $B(Z,k)$ defining odd (respectively, even) permutations of $k$ points to multiplication by $-1$ (respectively, 1). The local system $\pm\mathbb{Q}$ over $B(Z,k)$ is the one locally isomorphic to $\mathbb{Q}$, but with monodromy representation equal to the sign representation of $\pi_1(B(Z,k))$. We often call $\bar{H}_\bullet(B(Z,k), \pm\mathbb{Q})$ the *Borel–Moore homology of $B(Z,k)$ with twisted coefficients* or, simply, the *twisted Borel–Moore homology of $B(Z,k)$*.

LEMMA 2.10 [Vas92]. *We have $\bar{H}_\bullet(B(\mathbb{C}^N, k); \pm\mathbb{Q}) = 0$ if $k \geqslant 2$.*

*Proof.* In [Vas92, Theorem 4.3, Corollary 2], it is proved that the Borel–Moore homology of $B(\mathbb{R}^{2N}, k)$ with coefficients in the system $\pm\mathbb{Z}$ is a finite group. Since $\pm\mathbb{Q} = \pm\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$, the claim follows. □

LEMMA 2.11 [Vas99]. *We have $\bar{H}_\bullet(B(\mathbb{P}^N, k); \pm\mathbb{Q}) = H_{\bullet - k(k-1)}(G(k, \mathbb{C}^{N+1}); \mathbb{Q})$. In particular, $\bar{H}_\bullet(B(\mathbb{P}^N, k); \pm\mathbb{Q})$ is trivial if $k > N+1$.*

Lemma 2.12. *The Poincaré–Serre polynomial of* $\bar{H}_\bullet(B(\mathbb{C}^*,2);\pm\mathbb{Q})$ *is* $t^2u^{-2}+t^3u^{-4}$. *For constant rational coefficients, it is* $t^2(1+tu^{-2})^2$.

*Proof.* We recover the Borel–Moore homology of $B(\mathbb{C}^*,2)$ from the known situation for $B(\mathbb{C},2)$. Since $\bar{H}_\bullet(B(\mathbb{C},2);\pm\mathbb{Q})$ is trivial, we have that $\bar{H}_j(B(\mathbb{C},2);\mathbb{Q})$ and $\bar{H}_j(\mathbb{C}^2\setminus\{(x,y)\in\mathbb{C}^2 : x=y\};\mathbb{Q})$ are isomorphic and, in particular, both have Poincaré–Serre polynomial $t^3u^{-2}+t^4u^{-4}$.

The configuration space $B(\mathbb{C}^*,2)$ can be considered as an open subset of $B(\mathbb{C},2)$, with complement isomorphic to $\mathbb{C}^*$. Then the claim follows from the long exact sequence

$$\cdots\to\bar{H}_k(\mathbb{C}^*;\mathbb{Q})\to\bar{H}_k(B(\mathbb{C},2);S)\to\bar{H}_k(B(\mathbb{C}^*,2);S)\to\bar{H}_k(\mathbb{C}^*;\mathbb{Q})\to\cdots$$

where $S=\mathbb{Q}$ or $\pm\mathbb{Q}$. ☐

Lemma 2.13. *The Poincaré–Serre polynomial of* $\bar{H}_\bullet(B(\mathbb{P}^1\times\mathbb{P}^1,1);\pm\mathbb{Q})$ *is* $(t^2u^{-2}+1)^2$. *The Poincaré–Serre polynomial of* $\bar{H}_\bullet(B(\mathbb{P}^1\times\mathbb{P}^1,2);\pm\mathbb{Q})$ *is* $2t^2u^{-2}(t^4u^{-4}+t^2u^{-2}+1)$. *The Poincaré–Serre polynomial of* $\bar{H}_\bullet(B(\mathbb{P}^1\times\mathbb{P}^1,3);\pm\mathbb{Q})$ *is* $t^4u^{-4}(t^2u^{-2}+1)^2$. *The Poincaré–Serre polynomial of* $\bar{H}_\bullet(B(\mathbb{P}^1\times\mathbb{P}^1,4);\pm\mathbb{Q})$ *is* $t^8u^{-8}$. *The twisted Borel–Moore homology of* $B(\mathbb{P}^1\times\mathbb{P}^1,k)$ *is trivial for* $k\geqslant 5$.

*Proof.* Here, we modify Vassiliev's arguments in the proof of Lemma 2.11 in [Vas99]. The technique we use is that of decomposing $B(\mathbb{P}^1\times\mathbb{P}^1,k)$ into spaces of which the twisted Borel–Moore homology is known. In particular, it is possible to decompose $\mathbb{P}^1\times\mathbb{P}^1$ by fixing two lines $l,m$ in different rulings and considering the filtration $S_1\subset S_2\subset S_3\subset S_4$, where

$$S_1:=l\cap m,\quad S_2:=l,\quad S_3:=l\cup m,\quad S_4:=\mathbb{P}^1\times\mathbb{P}^1.$$

This means that $\mathbb{P}^1\times\mathbb{P}^1$ is the disjoint union of spaces isomorphic to $\{*\},\mathbb{C},\mathbb{C},\mathbb{C}^2$, respectively.

Let us fix $k\geqslant 1$. To any configuration of points in $B(\mathbb{P}^1\times\mathbb{P}^1,k)$ we can associate an ordered partition $(a_1,a_2,a_3,a_4)$, where $a_i$ is the number of points contained in $S_i\setminus S_{i-1}$. We can consider each possible partition of $k$ as defining a stratum in $B(\mathbb{P}^1\times\mathbb{P}^1,k)$, and order such strata by lexicographic order of the index of the partition. Note that all strata with $a_i\geqslant 2$ for some $i$ have no twisted Borel–Moore homology by Lemma 2.10, so we need not consider them. This is the case for all strata, when $k\geqslant 5$.

As an example, we consider the case $k=2$ explicitly. The situation is analogous for the other values. There are six admissible partitions for $k=2$.

- *Partitions* $(1,1,0,0)$ *and* $(1,0,1,0)$. Both these strata are isomorphic to $\mathbb{C}$, hence they have homology $\mathbb{Q}(1)$ in degree 2, and trivial homology in all other degrees.
- *Partitions* $(1,0,0,1)$ *and* $(0,1,1,0)$. Both these strata are isomorphic to $\mathbb{C}^2$, hence they have homology $\mathbb{Q}(2)$ in degree 4, and trivial homology in all other degrees.
- *Partitions* $(0,1,0,1)$ *and* $(0,0,1,1)$. These strata are both isomorphic to $\mathbb{C}^3$, hence they have homology $\mathbb{Q}(3)$ in degree 6, and trivial homology in all other degrees.

This gives precisely that $\bar{H}_j(B(\mathbb{P}^1\times\mathbb{P}^1,2);\pm\mathbb{Q})$ is $\mathbb{Q}(j/2)^2$ for $j=2,4,6$, and is trivial otherwise. ☐

Lemma 2.14. *Let us consider a quadric cone* $U\subset\mathbb{P}^3$. *Define* $U_0$ *as* $U$ *minus its vertex. Then* $\bar{H}_\bullet(B(U_0,2);\pm\mathbb{Q})$ *has dimension 1 in degree 6 and is trivial otherwise. The twisted Borel–Moore homology* $\bar{H}_\bullet(B(U_0,k);\pm\mathbb{Q})$ *is always trivial for* $k\geqslant 3$. *The Hodge weight of the homology in degree* $d$ *is* $-d$.

*Proof.* The proof is analogous to that of the previous lemma, by using the fact that $U_0$ can be decomposed as the disjoint union of $\mathbb{C}$ (a line of the ruling) and a space homeomorphic to $\mathbb{C}^2$. ☐

Lemma 2.15. *Let us consider the local system $T$ on $\mathbb{C}^*$ locally isomorphic to $\mathbb{Q}$ and changing its sign when the point moves along a loop in $\mathbb{C}^*$ whose homotopy class is an odd multiple of the generator of $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$. Then $\bar{H}_\bullet(\mathbb{C}^*; T) = 0$.*

*Proof.* Let us consider the map $q : \mathbb{C}^* \longrightarrow \mathbb{C}^*$, $q(t) = t^2$. Clearly $q_*\mathbb{Q} = \mathbb{Q} \oplus T$. Since $\bar{H}_\bullet(\mathbb{C}^*; q_*\mathbb{Q}) \cong \bar{H}_\bullet(\mathbb{C}^*; \mathbb{Q})$, we have $\bar{H}_\bullet(\mathbb{C}^*; T) = 0$. □

### 2.3 Some cases where the Borel–Moore homology is trivial

In §§ 3 and 4 we use the fact that most strata of the filtration we consider over the domain of the geometric realization of discriminants give no contribution to its Borel–Moore homology. We give some ideas here about the reason why this happens. In general they are either situations where the space is a locally trivial fiber bundle whose fibers have trivial Borel–Moore homology in the induced system of coefficients, or situations were non-discrete configurations are involved.

Lemma 2.16. *Let $C$ be an open cone with vertex a point over a compact connected space $B$. Then $C$ admits a compactification $\bar{C}$ with border homeomorphic to $B$. There are the following isomorphisms:*

$$\bar{H}_\bullet(C; \mathbb{Q}) \cong H_\bullet(\bar{C}, B; \mathbb{Q}) \cong H_{\bullet-1}(B, \mathit{point}; \mathbb{Q}).$$

*Proof.* The first isomorphism comes from the characterization of Borel–Moore homology as relative homology of the one-point compactification of the space modulo the added point. The second is the border isomorphism of the exact sequence associated with the triple $(\text{point}, B, \bar{C})$. □

Lemma 2.17. *Suppose we have a variety $Z$ and the following families of configurations in $Z$:*

$$\begin{aligned}
&X_1 = B(Z, 1);\\
&X_2 = \{\{p, q\} \in B(Z, 2) :\ p \text{ and } q \text{ lie on a line } l \subset Z\};\\
&X_3 = \{\{p, q, r\} \in B(Z, 3) :\ p, q \text{ and } r \text{ lie on a line } l \subset Z\};\\
&X_4 = \{\text{lines on } Z\}.
\end{aligned}$$

*Construct the cubical space $\Lambda(\bullet)$, its geometric realization and the filtration as in § 2.1. Then the space $\Phi_4$ has trivial Borel–Moore homology.*

*Proof.* The space $\Phi_4$ is a fiber bundle over $X_4$. Its fiber $\Psi$ over a line $l \subset Z$ is the union of all simplices with four vertices $p, q, r, l$ for all $\{p, q, r\} \in B(Z, 3)$. Note that we have to take *closed* simplices minus their face with vertices $p, q, r$, so they are indeed open cones with vertex $l$ over the closed simplices with vertices corresponding to the points $p, q, r$. The system of coefficients induced on the fiber changes its sign if we interchange two of the points $p, q, r$. The union of all *open* simplices with vertices $p, q, r, l$ is a non-oriented simplices bundle over $B(l, 3)$. Hence the Borel–Moore homology of the union of open simplices with three vertices on $l$ is trivial, because $\bar{H}_\bullet(B(l, 3); \pm\mathbb{Q})$ is trivial (see Lemma 2.11). This means that we only need to consider the Borel–Moore homology of the union of the external faces of the simplices considered before. This space admits a filtration of the form $A_0 = \{l\}$, $A_1$ is the union of the open segments joining $l$ and a point of $l$, and $A_2$ is the union of the open simplices with vertices $l$ and two distinct points on $l$. This gives a spectral sequence converging to the Borel–Moore homology of $\Psi$, with the following $E^1$ term:

$$\begin{array}{r|ccc}
1 & & \mathbb{Q} & \mathbb{Q}\\
0 & & & \\
-1 & \mathbb{Q} & \mathbb{Q} & \\
\hline
 & 1 & 2 & 3
\end{array}$$

Our space is an open cone, and it is a consequence of Lemma 2.16 that open cones have trivial Borel–Moore homology in degree 0. This implies that the row $q = -1$ of the spectral sequence is exact.



https://doi.org/10.1112/S0010437X0400123X Published online by Cambridge University Press

As for the row $q = 1$, it follows from the shape of a generator of $E^1_{2,1}$ that it maps to a generator of $E^1_{1,1}$ under the differential $d^1$. Then the Borel–Moore homology of $\Psi$ is trivial, which proves the claim. $\square$

*Remark* 2.18. A slight modification of Lemma 2.17 also allows us to conclude that strata with singular configurations that are union of a line and a fixed finite number $k$ of points have trivial Borel–Moore homology.

As before, we consider the projection to the configuration space and look at the fiber $\Psi$ over a fixed configuration $\{a_1, a_2, \dots a_k\} \cup l$. Let $h \geqslant 3$ be the maximal number of isolated points lying on the same line that appear in the previous strata of $\left|\Lambda(\bullet)\right|$. For any choice of distinct points $b_1, \dots, b_h$ on $l$, the union of the $(k+h)$-dimensional open simplices with vertices identified with $a_1, \dots, a_k, b_1, \dots, b_h, l$ is contained in $\Psi$. Indeed, $\Psi$ is the union of the open simplex $D$ with vertices $a_1, \dots, a_k, l$ and the union of such simplices for every choice of $b_1, \dots, b_h$. This means that $\Psi$ has a natural projection $\pi$ to $D$. Each point $p$ of $\Psi \setminus D$ is contained in exactly one open simplex with vertices $b_1, \dots, b_h, d$ for some $d \in D$; we pose $\pi(p) = d$. On $D \subset \Psi$ the projection $\pi$ coincides with the identity. If we look at the fibers of $\pi$, we see that they are homeomorphic to the fibers of the map $\Phi_4 \longrightarrow X_4$ studied in the proof of Lemma 2.17. Then we can apply the result found there, and conclude that the fibers of $\pi$ have trivial Borel–Moore homology. Then $\Psi$ also has trivial Borel–Moore homology, which implies the claim.

Next, we consider the case of the union of two rational curves, intersecting in one point. For simplicity, we state the result in the case of lines.

Lemma 2.19. *Suppose we have a variety $Z$ and the following families of configurations in $Z$:*

$X_1 = B(Z, 1)$;
$X_2 = B(Z, 2)$;
$X_3 = B(Z, 3)$;
$X_4 = B(Z, 4)$;
$X_5 = \{\text{lines on } Z\}$;
$X_6 = \{\{p, q, r, s, t\} \in B(Z, 5) : \ p, q \text{ lie on a line } l \subset Z, \ r, s \text{ lie on a line } m \subset Z, l \cap m = \{t\}\}$;
$X_7 = \{l \cup \{p\} : l \subset Z \text{ line}, p \notin l\}$;
$X_8 = \{l \cup \{p, q\} : l \subset Z \text{ line}, p, q \notin l, p \neq q\}$;
$X_9 = \{l \cup m : l, m \subset Z \text{ line}, \#(l \cap m) = 1\}$.

*Construct the cubical space $\Lambda(\bullet)$, its geometric realization and the filtration as in § 2.1. Then the space $\Phi_9$ has trivial Borel–Moore homology.*

*Proof.* Consider the fiber $\Psi$ of the projection $\Phi_9 \longrightarrow X_9$ over a configuration $l \cup m$, such that $l \cap m = \{t\}$. The maximal chains of inclusions of configurations we can construct are of the following forms:

(a) $\{p_1, p_2, p_3, p_4\} \subset l \subset l \cup \{q_1, q_2\} \subset l \cup m$, where $p_i \in l$, $q_j \in m \setminus \{t\}$;
(b) $\{q_1, q_2, q_3, q_4\} \subset m \subset m \cup \{p_1, p_2\} \subset l \cup m$, where $q_i \in m$, $p_j \in l \setminus \{t\}$;
(c) $\{p_1, p_2, t, q_1, q_2\} \subset l \cup \{q_1, q_2\} \subset l \cup m$, where $p_i \in l \setminus \{t\}$, $q_j \in m \setminus \{t\}$;
(d) $\{p_1, p_2, t, q_1, q_2\} \subset m \cup \{p_1, p_2\} \subset l \cup m$, where $p_i \in l \setminus \{t\}$, $q_j \in m \setminus \{t\}$.

The simplices constructed from chains of inclusions of types (c) and (d) are contained in the simplices arising from chains of types (a) and (b), so that we have to consider just these. Denote the union of closed simplices with vertices given by the configurations $p_1, p_2, p_3, p_4, q_1, q_2, l$ with $p_i \in l$

and $q_j \in m \setminus \{t\}$ by $U_1$. Analogously, denote the union of closed simplices with vertices given by the configurations $q_1, q_2, q_3, q_4, p_1, p_2, m$ by $U_2$. The fiber $\Psi$ is then the open cone (with vertex corresponding to the configuration $l \cup m$) over $U_1 \cup U_2$. Note that the intersection of $U_1$ and $U_2$ is given by the union of closed simplices with vertices $p_1, p_2, q_1, q_2, t$. It follows from the results of Remark 2.18 that the Borel–Moore homology of the open cone over $U_1 \cap U_2$ is trivial. The same can be said for the open cone over the union of simplices with four points of $l$ and two points of $m$ as vertices. The remaining part of $U_1$ is an open cone with vertex $l$ (over simplices with four points of $l$ and two points of $m$ as vertices). This space is clearly contractible, so that the open cone above it is contractible by Lemma 2.16. We can conclude exactly the same for $U_2$. Then the claim holds. □

Lemma 2.20. *Suppose $Z$ is the product of $\mathbb{C}$ and a variety $M$. Consider the following families of configurations in $Z$:*

$$\begin{aligned}
X_1 &= B(Z,1);\\
X_2 &= B(Z,2);\\
X_3 &= \{\{a,b,c\} \in B(Z,3) : a,b,c \in \mathbb{C}\times\{p\}, p \in M\};\\
X_4 &= \{\{a,b,c\} \in B(Z,3) : b,c \in \mathbb{C}\times\{p\}, p \in M, a \notin \mathbb{C}\times\{p\}\};\\
X_5 &= \{\{a,b,c,d\} \in B(Z,4) : a,b,c,d \in \mathbb{C}\times\{p\}\};\\
X_6 &= \{\{a,b,c,d\} \in B(Z,4) : a,b \in \mathbb{C}\times\{p\}, c,d \in \mathbb{C}\times\{q\}, p \neq q\}.
\end{aligned}$$

*Construct the cubical space $\Lambda(\bullet)$, its geometric realization and the filtration as in § 2.1. Then the space $\Phi_6$ has trivial Borel–Moore homology.*

*Proof.* The space $\Phi_6$ is an open non-orientable simplicial bundle over $X_6$. We study the Borel–Moore homology of $X_6$ in the system of coefficients locally isomorphic to $\mathbb{Q}$, with orientation induced by the orientation of the simplices. We look at the ordered situation. Let $Y = F(M,2) \times F(\mathbb{C},2) \times F(\mathbb{C},2)$. Every point $(p,q,a,b,c,d) \in Y$ gives an ordered configuration of points $((a,p),(b,p),(c,q),(d,q))$, and the twisted Borel–Moore homology of $X_6$ can be identified with the part of $\bar{H}_\bullet(Y;\mathbb{Q})$, which is:

- anti-invariant under the action of loops interchanging $a$ and $b$;
- anti-invariant under the action of loops interchanging $c$ and $d$;
- invariant under the action of loops interchanging $p$ and $q$, $a$ and $c$, $b$ and $d$.

It is clear that such homology classes cannot exist, because

$$\bar{H}_\bullet(Y;\mathbb{Q}) = \bar{H}_\bullet(F(M,2);\mathbb{Q}) \otimes \bar{H}_\bullet(F(\mathbb{C},2);\mathbb{Q}) \otimes \bar{H}_\bullet(F(\mathbb{C},2);\mathbb{Q})$$

and $\bar{H}_\bullet(F(\mathbb{C},2);\mathbb{Q})$ contains no classes that are anti-invariant with respect to the interchange of points (see Lemma 2.10). □

A variation of the situation of the above lemma is given, for instance, by the case of configurations of two triplets of collinear points in $\mathbb{P}^1 \times \mathbb{P}^1$. In that case we have to use the fact that there are no anti-invariant homological classes in $\bar{H}_\bullet(F(\mathbb{P}^1,3);\mathbb{Q})$.

## 3. Curves on a non-singular quadric

Any non-singular quadric surface is isomorphic to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. Such a surface is covered by two families of lines: the family of lines of the form $\mathbb{P}^1 \times \{q\}$ (which we call first ruling of the quadric) and that of lines of the form $\{p\} \times \mathbb{P}^1$ (second ruling). A curve on the Segre quadric is always given by the vanishing of a bihomogeneous polynomial in the two sets of variables $x_0, x_1$ and $y_0, y_1$. The bidegree $(n,m)$ of the polynomial has a geometrical interpretation as giving the number of points (counted with multiplicity) in the intersection of the curve with a general line of, respectively, the second and the first ruling. A curve on $\mathbb{P}^1 \times \mathbb{P}^1$ is said to be *of type* $(n, m)$ if it is defined by the vanishing of a polynomial of bidegree $(n, m)$.

The curves which are the intersection of the quadric with a cubic surface are the curves of type $(3, 3)$. This suggests that the space $C_0$ can be obtained as a quotient of the space of polynomials of bidegree $(3, 3)$ by the action of the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$. We denote that vector space by

$$V := \mathbb{C}[x_0, x_1, y_0, y_1]_{3,3} \cong \mathbb{C}^{16}.$$

In $V$ we can consider the discriminant locus $\Sigma$ of polynomials defining singular curves in $\mathbb{P}^1 \times \mathbb{P}^1$. The discriminant $\Sigma$ is closed in the Zariski topology, and is a 15-dimensional cone with the origin as vertex. It is also irreducible, because it is the cone over the dual variety of the Segre embedding of the product of two rational normal curves of degree 3. In other words, the projectivization of $\Sigma$ is the dual of the image of the map

$$\begin{array}{ccccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{v_3 \times v_3} & \mathbb{P}^3 \times \mathbb{P}^3 & \xrightarrow{\quad\sigma\quad} & \mathbb{P}(V) \\ ([x_0, x_1], [y_0, y_1]) & \longmapsto & & & [x_0^3 y_0^3, x_0^2 x_1 y_0^3, \ldots, x_1^3 y_1^3]. \end{array}$$

The locus of polynomials giving non-singular curves will be denoted by $X = V \setminus \Sigma$.

The automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$ is of the form $G \rtimes S_2$. The factor $S_2$ is generated by the involution $\upsilon$ interchanging the two rulings:

$$\begin{array}{rccc} \upsilon: & \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \\ & ([x_0, x_1], [y_0, y_1]) & \longmapsto & ([y_0, y_1], [x_0, x_1]). \end{array}$$

The connected component containing the identity of the automorphism group is the group

$$G := \mathrm{GL}(2) \times \mathrm{GL}(2) / \{\lambda I, \lambda^{-1} I\}_{\lambda \in \mathbb{C}^*},$$

and its action on $\mathbb{P}^1 \times \mathbb{P}^1$ is induced by the action of $\mathrm{GL}(2)$ on $\mathbb{P}^1$.

Being the quotient of a reductive group, $G$ is itself reductive. The action of $G$ on binary polynomials induces an action on $V$. The space $X$ is clearly invariant under this action of $G$, and all its points are $G$-stable. Thus there exists a geometric invariant theory quotient $X/G$, which is a double cover of $C_0$, the involution of the double cover being induced by $\upsilon$.

### 3.1 Generalized Leray–Hirsch theorem

The aim of this section is to prove that the action of $G$ on $X$ satisfies the hypotheses of Theorem 1.1.

We need first to compute the cohomology of $G$. Consider the map:

$$\begin{array}{rccc} \iota: & \mathbb{C}^* \times \mathrm{SL}(2) \times \mathrm{SL}(2) & \longrightarrow & G \\ & (\lambda, A, B) & \longmapsto & [(\lambda A, \lambda B)]. \end{array}$$

The map $\iota$ is an isogeny of connected algebraic groups. Its kernel is finite, hence $\iota$ induces an isomorphism of rational cohomology groups. As a consequence, the cohomology of $G$ is an exterior algebra on three independent generators: a generator $\xi$ of degree 1 and two generators $\eta_1, \eta_2$ of degree 3.

What we need to show is the surjectivity of the map (induced by the orbit inclusion $\rho: G \to X$)

$$\rho^*: H^i(X; \mathbb{Q}) \longrightarrow H^i(G; \mathbb{Q}).$$

We do this by studying the map

$$\begin{tikzcd} \psi : & H^i(X;\mathbb{Q}) \arrow[r] \arrow[d, leftrightarrow] & H^i(G;\mathbb{Q}) \arrow[r] & H^i(\mathrm{GL}(2)\times\mathrm{GL}(2);\mathbb{Q}) \arrow[d, leftrightarrow] \\ & \bar{H}_{31-i}(\Sigma;\mathbb{Q}) \arrow[rr] & & \bar{H}_{15-i}((D\times M)\cup(M\times D);\mathbb{Q}), \end{tikzcd}$$

where we have considered the embedding of $\mathrm{GL}(2)$ in the space $M$ of $2\times 2$ matrices, and written $D = M \setminus \mathrm{GL}(2)$ for the hypersurface defined by the vanishing of the determinant. Note that $H^3(G;\mathbb{Q})$ is isomorphic to $H^3(\mathrm{GL}(2)\times\mathrm{GL}(2);\mathbb{Q})$ and that $H^1(G;\mathbb{Q})$ can be identified with the part of $H^1(\mathrm{GL}(2)\times\mathrm{GL}(2);\mathbb{Q})$ that is invariant with respect to the interchange of the two factors of the product.

We only need to show that the generators of $H^\bullet(G;\mathbb{Q})$ are contained in the image of $\psi$. We will see that in this case we can write the generators of the cohomology groups of $X$ quite explicitly, by means of fundamental classes.

The cohomology of $\mathrm{GL}(2)\times\mathrm{GL}(2)$ is determined by the Borel–Moore homology of the discriminant $D\subset M$. We can compute it by looking at the desingularization

$$\tilde{D} = \{(p,A)\in\mathbb{P}^1\times M : Ap = 0\},$$
$$\mathbb{P}^1 \xleftarrow{\tau} \tilde{D} \longrightarrow D.$$

As a consequence, $\bar{H}_\bullet(\tilde{D};\mathbb{Q}) \cong \bar{H}_{\bullet-4}(\mathbb{P}^1;\mathbb{Q})$. Hence, the only non-trivial groups are $\bar{H}_6(\tilde{D};\mathbb{Q})$, which is generated by the fundamental class of $\tilde{D}$, and $\bar{H}_4(\tilde{D};\mathbb{Q})$, which is generated by the fundamental class of the preimage $\tilde{R}$ of a point in $\mathbb{P}^1$. As Borel–Moore homology is covariant for proper morphisms, there is a natural map $\bar{H}_\bullet(\tilde{D};\mathbb{Q}) \to \bar{H}_\bullet(D;\mathbb{Q})$, which must be an isomorphism in degrees 4,6 because in those cases the two groups have the same dimension. Thus, we know generators for $\bar{H}_\bullet(D;\mathbb{Q})$. The fundamental class of $D$ is a generator of degree 6, and the fundamental class of the image $R$ of $\tilde{R}$ is a generator of degree 4. In particular, we can choose $R$ to be the subvariety of matrices with only zeros in the first column.

We have natural projections

$$D\times M \xrightarrow{a_1} D \xleftarrow{a_2} M\times D$$

and natural immersions

$$\begin{array}{ccccc} D\times M & \xleftarrow{i_1} & D & \xrightarrow{i_2} & M\times D \\ (A,I) & \longleftarrow\!\!\shortmid & A & \longmapsto & (I,A). \end{array}$$

Then we have

$$\bar{H}_{14}((D\times M)\cup(M\times D);\mathbb{Q}) \cong \bar{H}_{14}(D\times M;\mathbb{Q}) \oplus \bar{H}_{14}(M\times D;\mathbb{Q}) \cong \mathbb{Q}\langle a_1^*([D]), a_2^*([D])\rangle,$$

and the part that comes from the cohomology of $G$ is generated by $a_1^*([D]) + a_2^*([D])$. Analogously, in degree 12 we again have

$$\bar{H}_{12}((D\times M)\cup(M\times D);\mathbb{Q}) \cong \bar{H}_{12}(D\times M;\mathbb{Q}) \oplus \bar{H}_{12}(M\times D;\mathbb{Q}) \cong \mathbb{Q}\langle a_1^*([R]), a_2^*([R])\rangle,$$

where the first isomorphism is a consequence of the fact that the two spaces have the same dimension.

We can associate with $\Sigma$ the variety

$$\tilde{\Sigma} := \{(p,v)\in\mathbb{P}^1\times\mathbb{P}^1\times V : \text{the curve defined by } v=0 \text{ is singular at } p\},$$
$$\mathbb{P}^1\times\mathbb{P}^1 \xleftarrow{\nu} \tilde{\Sigma} \xrightarrow{\pi} \Sigma.$$

The map $\pi$ gives $\tilde{\Sigma}$ the structure of a $\mathbb{C}^{13}$-bundle over $\mathbb{P}^1\times\mathbb{P}^1$. Hence, $\tilde{\Sigma}$ is a desingularization of $\Sigma$ and is homotopy equivalent to $\mathbb{P}^1\times\mathbb{P}^1$. This ensures that the Borel–Moore homology group of degree 28 of $\tilde{\Sigma}$ is generated by the fundamental classes $[\tilde{D}_1],[\tilde{D}_2]$, where $\tilde{D}_1=\nu^{-1}(\{p_0\}\times\mathbb{P}^1)$, $\tilde{D}_2=\nu^{-1}(\mathbb{P}^1\times\{q_0\})$.

If we choose a point $q_1\in\mathbb{P}^1\setminus\{q_0\}$, the orbit inclusion defines a map

$$\begin{array}{rccc}\rho_1: & \tilde{D} & \longrightarrow & \tilde{\Sigma}\\ & (p,A) & \longmapsto & ((p,q_1),\rho(A,I)).\end{array}$$

The map $\rho_1$ is well defined, because $\rho(A,I)$ is the union of the line $\{p\}\times\mathbb{P}^1$, with multiplicity 3, and three lines of the other ruling. Hence, it is always singular at $(p,q_1)$.

Since $\rho_1^{-1}(\tilde{D}_1)=\tau^{-1}(\{p_0\}),\rho_1^{-1}(\tilde{D}_2)=\emptyset$, we have

$$\rho_1^*([\tilde{D}_1])=[\tau^{-1}(\{p_0\})],\quad \rho_1^*([\tilde{D}_2])=0.$$

Analogously, if we fix $p_1\neq p_0$, the map

$$\begin{array}{rccc}\rho_2: & \tilde{D} & \longrightarrow & \tilde{\Sigma}\\ & (p,A) & \longmapsto & ((p_1,q),\rho(I,A))\end{array}$$

satisfies $\rho_2^*([\tilde{D}_2])=[\tau^{-1}(\{q_0\})]$, $\rho_2^*([\tilde{D}_1])=0$.

We define $D_1$ and $D_2$ as the images of $\tilde{D}_1$ and $\tilde{D}_2$, respectively, under the map $\tilde{\Sigma}\to\Sigma$. The following diagrams are commutative.

$$\begin{tikzcd} D \arrow[r, "i_1"] & D\times M \arrow[r, "\rho"] & \Sigma \\ \tilde{D} \arrow[u] \arrow[rr, "\rho_1"] & & \tilde{\Sigma}, \arrow[u] \end{tikzcd}\qquad \begin{tikzcd} D \arrow[r, "i_2"] & M\times D \arrow[r, "\rho"] & \Sigma \\ \tilde{D} \arrow[u] \arrow[rr, "\rho_2"] & & \tilde{\Sigma} \arrow[u] \end{tikzcd}$$

This implies that we can use our knowledge about $\rho_1$ and $\rho_2$ to study the map induced by $\rho$ on Borel–Moore homology. Thus, we find

$$\begin{aligned}\rho^*([\Sigma])&=a_1^*([D])+a_2^*([D]),\\ \rho^*([D_1])&=a_1^*([R]),\\ \rho^*([D_2])&=a_2^*([R]),\end{aligned}$$

which is exactly what we wanted to prove.

### 3.2 Application of Gorinov–Vassiliev's method

We can associate with each $v\in V$ its zero locus on the Segre embedding of $\mathbb{P}^1\times\mathbb{P}^1$, which is a non-singular quadric $Q\subset\mathbb{P}^3$. For $v\neq 0$ we always get a curve of type $(3,3)$ on $Q$. Recall that $Q$ has two rulings, and that curves on $Q$ are classified by considering the number of points of intersection with a general line of, respectively, the second and the first ruling. We want to know what the singular locus of such a curve can be. The singular points of a curve are the union of the singular points of each irreducible component of it and the points that are pairwise intersection of components. Note that for each component, the number of singular points allowed is bounded by the arithmetic genus of the component. By writing all of the possible ways a (3,3)-curve can be decomposed, we get all of the possibilities in Table 1.

Note that for all configurations of singularities in the same class, the elements of $V$ that are singular at least at the chosen configurations always form a vector space, which has always the same codimension $c$. We write this codimension in the second column of the table.

Each item in Table 1 can be used to define a family of configurations on $Q$. For every $j=1,\dots,26$, we can define $X_j$ as the space of all configurations of type $(j)$. In this way, we get a sequence of subsets $X_1,X_2,\dots,X_{26}$, which satisfies conditions (i)–(iv) and (vi) in List 2.3. Conditions (v) and (vii) do not hold. This problem can be solved by enlarging the list.



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Table 1. Singular sets of $(3,3)$-curves.

| Type | $c$ | Description |
|---|---|---|
| 1 | 3 | One point |
| 2 | 6 | Two points |
| 3 | 7 | Three collinear points |
| 4 | 8 | A line |
| 5 | 9 | Three non-collinear points |
| 6 | 10 | Three collinear points plus a point not lying on the same line |
| 7 | 11 | Four points lying on a non-singular conic |
| 8 | 11 | Five points: two pairs of points lying each on a line of a different ruling, and the intersection of the two lines |
| 9 | 11 | A line plus another point |
| 10 | 12 | Four points, not lying on the same plane in $\mathbb{P}^3$ |
| 11 | 12 | A conic (possibly degenerating into two lines) |
| 12 | 13 | Five points: three points on a line of some ruling and two more points not coplanar with them |
| 13 | 14 | Five points: a point $p$ and four more points lying on a non-singular conic not passing through $p$. These points have to be distinct from the intersection of the conic with the two lines of the rulings of $Q$ that pass through $p$ |
| 14 | 14 | Six points: two points on a line of the first ruling, two on a line of the second ruling, the intersection of the two lines and an additional point, not lying on the same line of a ruling of $Q$ as any of the others |
| 15 | 14 | Six points: union of two triplets of collinear points (on lines of the same ruling). No two points of the configuration can lie on the same line of the other ruling |
| 16 | 14 | A line and two non-collinear points |
| 17 | 15 | Five points in general position (no two of them on a line of any ruling, no four on a conic) |
| 18 | 15 | Six points: two points lying on a line, three points lying on a non-singular conic and the intersection of the line and the conic. In this configuration only the first two points lie on a line contained in $Q$ |
| 19 | 15 | Six points: intersection of $Q$ with three concurrent lines in $\mathbb{P}^3$ |
| 20 | 15 | Seven points, intersection of components of the union of two lines of a ruling, one of the other and an irreducible curve of type $(2,1)$ or $(1,2)$ |
| 21 | 15 | Seven points, intersection of components of two lines of different rulings and two non-singular conics on $Q$ |
| 22 | 15 | Eight points, intersection of components of the union of two pairs of lines of different rulings and an irreducible conic |
| 23 | 15 | Nine points lying on six lines (three in every ruling) |
| 24 | 15 | A line and three points on a line of the same ruling |
| 25 | 15 | A conic (possibly reducible) plus an extra point |
| 26 | 16 | The whole of $Q$ |

In order for condition (vii) to be satisfied, it suffices to include all subconfigurations of finite configurations of Table 1. Condition (v) is more delicate. We have to consider all possible limit positions of configurations of singular points. For instance, points in a general position can become collinear, and conics of maximal rank can degenerate to the union of two lines.

In this way, we can construct a list of configurations that verify all the conditions in List 2.3. This list is really long, so we do not report it here. Luckily, most configurations give no contribution to the Borel–Moore homology of $\Sigma$. In particular, by Lemma 2.11, configurations with more than two points on a rational curve give no contribution and, by Lemma 2.10, the same holds for configurations with at least two points on a rational curve minus a point (which is $\cong \mathbb{C}$). Also, all configurations containing curves give no contribution, because we can apply Lemmas 2.17 and 2.19, or a slight modification of them.

We give below a list limited to the interesting configurations.

List 3.1.

(A) One point.

(B) Two points.

(C) Three points in general position.

(D) Four points in general position.

(E) Six points, intersection of $Q$ with three concurrent lines.

(F) Eight points, intersection of components of two pairs of lines for every ruling and a conic.

(G) The whole quadric.

A configuration is said to be *general* if no three points of it lie on the same line, and no four points of it lie on the same plane in $\mathbb{P}^3$.

### 3.3 Configurations of type (A)–(D)

The space $F_A$ is a $\mathbb{C}^3$-bundle over $X_A \cong \mathbb{P}^1 \times \mathbb{P}^1$. The space $F_B$ is a $\mathbb{C}^6 \times \mathring{\Delta}_1$-bundle over $X_B \cong B(\mathbb{P}^1 \times \mathbb{P}^1, 2)$. The space $F_C$ is a $\mathbb{C}^9 \times \mathring{\Delta}_2$-bundle over $X_C$, which has the same twisted Borel–Moore homology as $B(\mathbb{P}^1 \times \mathbb{P}^1, 3)$ (the non-general configurations form a space with trivial twisted Borel–Moore homology). Analogously, $F_D$ is a $\mathbb{C}^{12} \times \mathring{\Delta}_3$-bundle over $X_D$, which has the same twisted Borel–Moore homology as $B(\mathbb{P}^1 \times \mathbb{P}^1, 4)$.

Recall that the simplicial bundles are non-orientable, so that we have to consider the Borel–Moore homology with coefficients in the local system $\pm\mathbb{Q}$. Thus, Lemma 2.13 allows us to compute the Borel–Moore homology of all spaces $F_A$, $F_B$, $F_C$ and $F_D$.

### 3.4 Configurations of type (E)

Configurations of type (E) are the singular loci of (3,3)-curves, which are the union of three conics lying on $Q$. This gives three pairs of points on the Segre quadric, which are the intersection of it with three concurrent lines in $\mathbb{P}^3$. Then it is natural to consider this configuration space as a fiber bundle $X_E$ over $\mathbb{P}^3 \setminus Q$. The projection is given by mapping each configuration to the common point of the three lines. The fiber over a point $p \in \mathbb{P}^3 \setminus Q$ is the space of configurations of three lines through $p$, with all three not tangent to $Q$. It is isomorphic to the space $\tilde{B}(\mathbb{P}^2 \setminus C, 3)$ of configurations of three non-collinear points on $\mathbb{P}^2 \cong \mathbb{P}(T_p\mathbb{P}^3)$ minus an irreducible conic $C \subset \mathbb{P}^2$. Note that the orientation of the simplicial bundle over $X_E$ changes under the action of a non-trivial element of the fundamental group of $\mathbb{P}^2 \setminus C$, which is $S_2$. This means that we have to compute the Borel–Moore homology of $\tilde{B}(\mathbb{P}^2 \setminus C, 3)$ with a system of coefficients locally isomorphic to $\mathbb{Q}$, which changes its orientation when a point of the configuration moves along a loop in $\mathbb{P}^2 \setminus C$ with non-trivial homotopy class. We denote this local system by $W$.

We consider $\tilde{B}(\mathbb{P}^2 \setminus C, 3)$ as a quotient of the corresponding space of ordered configurations, $\tilde{F} = (\mathbb{P}^2 \setminus C)^3 \setminus \Delta$, where

$$\Delta = \{(x, y, z) \in (\mathbb{P}^2 \setminus C)^3 : \dim\langle x, y, z\rangle \leqslant 1\}.$$

By abuse of notation, we denote also the local system on $(\mathbb{P}^2 \setminus C)^3$ by $W$.

The closed immersion $\Delta \hookrightarrow (\mathbb{P}^2 \setminus C)^3$ induces the exact sequence in Borel–Moore homology

$$\cdots \to \bar{H}_{i+1}(\tilde{F}; W) \to \bar{H}_i(\Delta; W) \to \bar{H}_i((\mathbb{P}^2 \setminus C)^3; W) \to \bar{H}_i(\tilde{F}; W) \to \cdots \tag{2}$$

The map associating with a point $p \in (\mathbb{P}^2 \setminus C)$ the intersection with $C$ of the line polar to $p$ with respect to $C$, induces an isomorphism $(\mathbb{P}^2 \setminus C) \cong B(C, 2) \cong B(\mathbb{P}^1, 2)$. In particular, we have

$$\bar{H}_\bullet((\mathbb{P}^2 \setminus C)^3; W) \cong (\bar{H}_\bullet(B(\mathbb{P}^1, 2); \pm\mathbb{Q}))^{\otimes 3},$$

which implies that $\bar{H}_\bullet((\mathbb{P}^2 \setminus C)^3; W)$ is $\mathbb{Q}(3)$ in degree 6, and trivial in all other degrees. Note that the generator of $\bar{H}_6((\mathbb{P}^2 \setminus C)^3; W)$ is invariant under the action of $S_3$.

Next, we compute the Borel–Moore homology of $\Delta$ by considering the following filtration:

$$\Delta = \Delta_3 \supset \Delta_2 \supset \Delta_1,$$
$$\Delta_1 := \{(x, y, z) \in \Delta : x = y = z\}$$
$$\Delta_2 := \{(x, y, z) \in \Delta : \exists l \in C\check{}\,(x, y, z \in l)\}.$$

The first term $\Delta_1$ of the filtration is isomorphic to $\mathbb{P}^2 \setminus C$, and hence to $B(\mathbb{P}^1, 2)$, so that its Borel–Moore homology with coefficients in $W$ is $\mathbb{Q}(1)$ in degree 2.

Let us come back to the original construction, and consider ordered configurations of three lines with a common point $p$. Then the configurations in $\Delta_2 \setminus \Delta_1$ correspond to triples of lines lying on a plane tangent to $Q$. This means that the space is fibered over the family of such planes, which is parametrized by $Q\check{} \cap p\check{}$, a non-singular conic. In particular, $Q\check{} \cap p\check{}$ is simply connected, hence the only system of coefficients we can have there is the constant one.

The fiber over a plane $\Pi$ is given by all ordered triples of lines passing through $p$, lying in $\Pi$ and not tangent to $Q$, such that not all lines coincide. This is the space $(\mathbb{C}^3 \setminus \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 = z_2 = z_3\})$, which has Borel–Moore homology $\mathbb{Q}(1)$ in degree 3 and $\mathbb{Q}(3)$ in degree 6. All elements of $\bar{H}_\bullet(\{z_1 = z_2 = z_3\}; \mathbb{Q})$ are invariant with respect to the $S_3$-action. We can conclude that the Borel–Moore homology of $\Delta_2 \setminus \Delta_1$ has Poincaré–Serre polynomial $t^3u^{-2} + t^5u^{-4} + t^6u^{-6} + t^8u^{-8}$.

For later use, we consider the action induced by the involution $\upsilon$ interchanging the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ on $\bar{H}_8(\Delta_2 \setminus \Delta_1; \mathbb{Q})$. This Borel–Moore homology group is obtained as the tensor product of $\bar{H}_4(\mathbb{Q}\check{} \cap p\check{}; \mathbb{Q})$ and the Borel–Moore homology group of degree 6 of the fiber. Both factors are invariant under the action induced by $\upsilon$, hence the whole group is invariant under it. Note that the action of $\upsilon$ on the fiber can be seen as interchanging the two lines of intersection of the plane $\Pi$ and $Q$. This means that, if we consider the configuration of points of intersections of the three lines and $Q$, the action of $\upsilon$ interchanges three pairs of points.

The space $\Delta_3 \setminus \Delta_2$ is a fiber bundle over $\check{\mathbb{P}}^2 \setminus C\check{}$. The fiber is isomorphic to $(\mathbb{C}^*)^3 \setminus \delta$, $\delta := \{(x, y, z) \in (\mathbb{C}^*)^3 : x = y = z\}$. The local system $W'$ induced by $W$ coincides with that induced by the local system $T$ on $\mathbb{C}^*$ locally isomorphic to $\mathbb{Q}$ and changing its sign if the point moves along a loop in $\mathbb{C}^*$ whose homotopy class is an odd multiple of the generator of $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$.

Again, we have an exact sequence

$$\cdots \to \bar{H}_{i+1}((\mathbb{C}^*)^3 \setminus \delta; W') \to \bar{H}_i(\delta, T) \to \bar{H}_i((\mathbb{C}^*)^3, T^{\otimes 3}) \to \bar{H}_i((\mathbb{C}^*)^3 \setminus \delta; W') \to \cdots.$$

Since $\bar{H}_\bullet(\mathbb{C}^*, T)$ is trivial by Lemma 2.15, $\Delta_3 \setminus \Delta_2$ gives no contribution to the homology of $\Delta$.

Let us consider the spectral sequence associated with the filtration $\Delta_i$. The only possibly non-trivial differential is that from $E_{1,2}$ to $E_{1,1}$, and this must be an isomorphism for dimensional reasons

($V \setminus \Sigma$ is affine of complex dimension 16). As a consequence, $\bar{H}_k(\Delta; W)^{S_3}$ is isomorphic to $\bar{H}_k(\Delta; W)$ and has Poincaré–Serre polynomial $t^5u^{-4} + t^6u^{-6} + t^8u^{-8}$.

From (2) we get that the Poincaré–Serre polynomial of $\bar{H}_k(\tilde{B}(\mathbb{P}^2 \setminus C, 3); W)$ is $t^6u^{-4} + t^9u^{-8}$.

Then, by Leray's theorem, the Borel–Moore homology of $X_E$ is the tensor product of the above Borel–Moore homology and that of $\mathbb{P}^3 \setminus Q$, which is $\mathbb{Q}(1)$ in degree 3 and $\mathbb{Q}(3)$ in degree 6. The local system induced on $\mathbb{P}^3 \setminus Q$ is indeed the constant one.

We consider the action induced by the involution $\upsilon$ on the Borel–Moore homology of $X_E$ and $F_E$. We are only interested in determining this action on the highest degree component of these homology groups. It is easy to find that $\bar{H}_3(\mathbb{P}^3 \setminus Q; \mathbb{Q})$ is anti-invariant with respect to the involution $\upsilon$, and $\bar{H}_6(\mathbb{P}^3 \setminus Q; \mathbb{Q})$ is invariant. The group $\bar{H}_{15}(X_E; \pm\mathbb{Q})$ is the tensor product of $\bar{H}_6(\mathbb{P}^3 \setminus Q; \mathbb{Q})$ and $\bar{H}_9(\tilde{B}(\mathbb{P}^2 \setminus C, 3); W)$. The latter group is isomorphic to $\bar{H}_8(\Delta_2 \setminus \Delta_1; \mathbb{Q})$, which we have already shown to be invariant. We also saw that the action of $\upsilon$ on $\Delta \setminus \Delta_1$ produces the interchange of an *odd* number of pairs of points, so that the action of $\upsilon$ on the generator of the Borel–Moore homology of the fiber of the bundle $\Phi_E \to X_E$ changes its orientation. Hence, the invariance of $\bar{H}_{15}(X_E; \pm\mathbb{Q})$ under $\upsilon$ implies the anti-invariance of both $\bar{H}_{20}(\Phi_E; \mathbb{Q})$ and $\bar{H}_{22}(F_E; \mathbb{Q})$.

### 3.5 Configurations of type (F)

The configurations of type (F) are the configurations of eight points which are the intersections of components of two pairs of lines for every ruling (say, $l_1, l_2, m_1, m_2$) and a conic $C$. The situation has to be sufficiently general to give exactly eight points of intersection.

First let us fix the conic. If it is singular, each pair of lines must be an element of $B(\mathbb{C}, 2)$, which has no Borel–Moore homology with twisted coefficients. This means that we have to consider only non-singular conics. On each such conic, the configuration is univocally determined by the intersection points of the four lines with it. Denote the space of two pairs of points in $\mathbb{P}^1$ by $\Psi$, i.e. it is the quotient of $F(\mathbb{P}^1, 4)$ by the relation $(z_1, z_2, z_3, z_4) \sim (z_2, z_1, z_3, z_4)$, $(z_1, z_2, z_3, z_4) \sim (z_1, z_2, z_4, z_3)$. The configuration space we have to consider is isomorphic to the product $(\check{\mathbb{P}}^3 \setminus Q^{\vee}) \times \Psi$.

The local system of coefficients is $\mathbb{Q}$ on $\check{\mathbb{P}}^3 \setminus Q^{\vee}$ and the rank one local system $R$ on $\Psi$ that changes its sign along the loops exchanging, respectively, only the first pair of points or only the second.

Lemma 3.2. *The Poincaré–Serre polynomial of the Borel–Moore homology of $F(\mathbb{P}^1, 4)$ is $2t^4u^{-2} + t^5u^{-4} + 2t^7u^{-6} + t^8u^{-8}$.*

*Proof.* The action of PGL(2) induces a quotient map $F(\mathbb{P}^1, 4) \to \mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Note that there is a natural action of $S_4$ on both $F(\mathbb{P}^1, 4)$ and $\mathcal{M}_{0,4}$, and that the quotient map is equivariant with respect to this action. The $S_4$-quotient of $F(\mathbb{P}^1, 4)$ is the space $\mathbb{P}U_{4,1}$ of non-zero square-free homogeneous quartic polynomials in two variables, up to scalar multiples. By [PS03] the map $H^\bullet(\mathbb{P}U_{4,1}; \mathbb{Q}) \to H^\bullet(\mathrm{PGL}(2); \mathbb{Q})$ is an isomorphism (it is induced by any orbit map). Hence, the pull-back of the orbit map $H^\bullet(F(\mathbb{P}^1, 4); \mathbb{Q}) \to H^\bullet(\mathrm{PGL}(2); \mathbb{Q})$ is also surjective.

We can apply the generalized Leray–Hirsch Theorem 1.1 and get that the rational cohomology of $F(\mathbb{P}^1, 4)$ is isomorphic to $H^\bullet(\mathcal{M}_{0,4}; \mathbb{Q}) \otimes_{\mathbb{Q}} H^\bullet(\mathrm{PGL}(2); \mathbb{Q})$. The cohomology of PGL(2) is $\mathbb{Q}$ in degree 0 and $\mathbb{Q}(-2)$ in degree 3. Then the claim follows from the fact that $F(\mathbb{P}^1, 4)$ is smooth of complex dimension 4, so that the cap product with the fundamental class $[F(\mathbb{P}^1, 4)]$ induces an isomorphism $\bar{H}_\bullet(F(\mathbb{P}^1, 4); \mathbb{Q}) \cong H^{8-\bullet}(F(\mathbb{P}^1, 4); \mathbb{Q})(4)$. □

Lemma 3.3. *The Poincaré–Serre polynomial of $\bar{H}_\bullet(\Psi; R)$ is $t^4u^{-2} + t^7u^{-6}$. Moreover, $\bar{H}_\bullet(\Psi; R)$ is invariant with respect to the $S_2$-action induced by the interchange of $(z_1, z_2, z_3, z_4)$ and $(z_3, z_4, z_1, z_2)$.*

*Proof.* We refer to the preceding lemma.

TABLE 2. Spectral sequence converging to the base of $F_G$.

| | | | | | | |
|---|---|---|---|---|---|---|
| 15 | | | | | $\mathbb{Q}(7)$ | |
| 14 | | | | | | $\mathbb{Q}(6)$ |
| 13 | | | | | | |
| 12 | | | | | $\mathbb{Q}(5)^2$ | |
| 11 | | | | | | $\mathbb{Q}(4)^2$ |
| 10 | | | | | | |
| 9 | | | | | $\mathbb{Q}(3)$ | |
| 8 | | | | | | $\mathbb{Q}(2)$ |
| 7 | | | $\mathbb{Q}(4)$ | $\mathbb{Q}(4)$ | | |
| 6 | | | | | | |
| 5 | | $\mathbb{Q}(3)^2$ | $\mathbb{Q}(3)^2$ | | | |
| 4 | | | | | | |
| 3 | $\mathbb{Q}(2)$ | $\mathbb{Q}(2)^2$ | $\mathbb{Q}(2)$ | | | |
| 2 | | | | | | |
| 1 | $\mathbb{Q}(1)^2$ | $\mathbb{Q}(1)^2$ | | | | |
| 0 | | | | | | |
| -1 | $\mathbb{Q}$ | | | | | |
| | A | B | C | D | E | F |

We can identify $\bar{H}_\bullet(\Psi; R)$ with the part of $\bar{H}_\bullet(F(\mathbb{P}^1, 4); \mathbb{Q})$ that is anti-invariant with respect to the action of the transpositions $(1,2)$ and $(3,4)$. By the equivariance of the quotient map, it is sufficient to determine the action on $\mathcal{M}_{0,4}$. The action of $S_4$ on $H^\bullet(\mathcal{M}_{0,4}; \mathbb{Q})$ factorizes via $S_3$, hence the action of any pair of commuting transpositions must coincide. The anti-invariant part of $H^\bullet(\mathcal{M}_{0,4}; \mathbb{Q})$ with respect to $(1,2) \in S_4$ is $\mathbb{Q}(-1)$ in degree 1. The actions of the two transpositions $(1,3)$ and $(2,4)$ on $\mathcal{M}_{0,4}$ coincide, and their product is the identity. If we pass from cohomology to Borel–Moore homology, this implies the claim. □

Lemma 3.3 implies that the Poincaré–Serre polynomial of the Borel–Moore homology of $X_F$ with coefficients in $\pm\mathbb{Q}$ is $t^7u^{-4} + 2t^{10}u^{-8} + t^{13}u^{-12}$.

Moreover, we can compute the action on $\bar{H}_\bullet(X_F; \pm\mathbb{Q})$ induced by the involution $\upsilon$ interchanging the two rulings of the quadric $Q$. Fix a configuration in $X_F$. Up to a choice of coordinates, we can assume that the action of $\upsilon$ permutes the two lines of the first ruling with the two of the second ruling in the configuration. This means that the action of $\upsilon$ on $\Psi$ can be identified with that interchanging $(z_1, z_2, z_3, z_4) \in F(\mathbb{P}^1, 4)$ with $(z_3, z_4, z_1, z_2)$. As a consequence, $\bar{H}_{13}(X_F; \pm\mathbb{Q})$ is invariant with respect to $\upsilon$. If we look at the configuration itself, three pairs of points are exchanged. Namely, the four points of intersection of the conic and the lines are exchanged in pairs, and also exactly two of the points of intersection between the lines are exchanged. Hence, the Borel–Moore homology of the fiber of the bundle $\Phi_F \to X_F$ is anti-invariant for $\upsilon$. We can conclude that the highest class in the Borel–Moore homology of $F_F$ (i.e. that of degree 22) is anti-invariant for the action of $\upsilon$.

### 3.6 Borel–Moore homology of $F_G$

By Proposition 2.7, the space $F_G$ is an open cone. The Borel–Moore homology of its base space can be computed by the spectral sequence in Table 2. The columns coincide with those of the main spectral sequence, but are shifted by twice the dimension of the fiber of the complex vector bundle we considered for each of them.

For dimensional reasons, the rows of indices 1, 3, 5 and 7 of this spectral sequence are exact. The reason is that if were not so, there would be non-trivial elements in the general spectral sequence,



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Table 3. Spectral sequence converging to $\bar{H}_k(\Sigma;\mathbb{Q})$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 29 | $\mathbb{Q}(15)$ | | | | | | |
| 28 | | | | | | | |
| 27 | $\mathbb{Q}(14)^2$ | | | | | | |
| 26 | | | | | | | |
| 25 | $\mathbb{Q}(13)$ | $\mathbb{Q}(13)^2$ | | | | | |
| 24 | | | | | | | |
| 23 | | $\mathbb{Q}(12)^2$ | | | | | |
| 22 | | | | | | | |
| 21 | | $\mathbb{Q}(11)^2$ | $\mathbb{Q}(11)$ | | | | |
| 20 | | | | | | | |
| 19 | | | $\mathbb{Q}(10)^2$ | | | | |
| 18 | | | | | | | |
| 17 | | | $\mathbb{Q}(9)$ | | $\mathbb{Q}(8)$ | | |
| 16 | | | | | | $\mathbb{Q}(7)$ | |
| 15 | | | | $\mathbb{Q}(8)$ | | | |
| 14 | | | | | $\mathbb{Q}(6)^2$ | | $\mathbb{Q}(7)+\mathbb{Q}(6)$ |
| 13 | | | | | | $\mathbb{Q}(5)^2$ | |
| 12 | | | | | | | |
| 11 | | | | | $\mathbb{Q}(4)$ | | $\mathbb{Q}(5)^2+\mathbb{Q}(4)^2$ |
| 10 | | | | | | $\mathbb{Q}(3)$ | |
| 9 | | | | | | | |
| 8 | | | | | | | $\mathbb{Q}(3)+\mathbb{Q}(2)$ |
| | A | B | C | D | E | F | G |

in a position such that they could not disappear. Hence, they would give non-trivial elements in $H^j(V\setminus\Sigma)\cong\bar{H}_{31-j}(\Sigma)$ for $j>16$, which is impossible because $V\setminus\Sigma$ is affine of dimension 16.

As a consequence, the spectral sequence here degenerates at $E^2$. Then Lemma 2.16 allows us to obtain the Borel–Moore homology of the open cone.

### 3.7 Spectral sequence

We now know all of the columns of the spectral sequence associated with the filtration

$$\mathrm{Fil}_A(|\mathcal{X}(\bullet)|)\subset\cdots\subset\mathrm{Fil}_G(|\mathcal{X}(\bullet)|).$$

Its $E^1$ term is represented in Table 3.

This spectral sequence degenerates at $E^1$. Indeed, the only possible non-trivial differentials are between the first four columns. We know *a priori* that $H^0(C_0;\mathbb{Q})$ has dimension 1. Then Theorem 1.1 implies that the cohomology of $X=V\setminus\Sigma$ must contain a copy of the cohomology of the group $G$. This is impossible if any of the differentials in columns (A)–(D) is non-zero.

We can compute the whole cohomology of $X$ from the Borel–Moore homology of $\Sigma$, using the isomorphism induced by the cap product with the fundamental class of the discriminant

$$\tilde{H}^\bullet(X;\mathbb{Q})\cong\bar{H}_{31-\bullet}(\Sigma;\mathbb{Q})(-d).$$

In view of the results in Section 3.1 on the rational cohomology of $G$, Theorem 1.1 implies that the Poincaré–Serre polynomial of $X/G$ is $1+t^5u^6+t^9(u^{16}+u^{18})$.

The quotient $X/G$ is a double cover of $C_0$, the $S_2$-action being generated by the involution $\upsilon$ interchanging the two rulings of the Segre quadric $Q$. The cohomology of $X$ is invariant with respect to this involution in the degrees $0,5$. The cohomology in degree 9 comes from the terms $E^1_{E,17}$, $E^1_{F,16}$ in the spectral sequence. During the computation of columns (E) and (F) we observed that these



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terms are anti-invariant with respect to the action of $S_2$ on $X$. Then $C_0$ has no cohomology in degree 9.

We can conclude that the rational cohomology of $C_0$ is $\mathbb{Q}$ in degree 0 and $\mathbb{Q}(-3)$ in degree 5, as we claimed in Theorem 1.2.

## 4. Curves on a quadric cone

### 4.1 The space $C_1$ as geometric quotient

The aim of this section is to realize $C_1$ as a geometric quotient, satisfying the hypotheses of Theorem 1.1. We perform it by regarding the elements of $C_1$ as curves of degree 6 on a quadric cone.

After a choice of coordinates, we can identify the quadric cone with $\mathbb{P}(1,1,2)$. Then we consider the vector space $\mathbb{C}[x,y,z]_6$, where $\deg x = \deg y = 1$, $\deg z = 2$.

This space has complex dimension 16. The polynomials defining singular curves form the *discriminant* $\Sigma$, which in this case has two irreducible components of dimension 15. One component, which we denote by $H$, is the locus of curves passing through the vertex of the cone. Such a curve is always singular and, in general, this singularity can be resolved by considering the proper transform in the blowing up of the vertex of the cone. The other component, which we denote by $S$, is the closure in $\mathbb{C}^{16}$ of the locus of curves that are singular in a point different from the vertex of the cone. More specifically, $S$ is the locus of curves such that the proper transform in the blowing up of the vertex of the cone is singular, or tangent to the exceptional locus. Note that $H$ is the hyperplane defined by the condition that the coefficient of $z^3$ is zero. The other component $S$ is the affine cone over the dual variety of the Veronese embedding of $\mathbb{P}(1,1,2)$ in $\mathbb{P}^{15}$.

Next, we find generators of the cohomology of $X$ in degree 0 and 2 which have an interpretation as fundamental classes of subvarieties of $S \cup H$. The whole cohomology of $X = \mathbb{C}^{16} \setminus (S \cup H)$ will be calculated in § 4.2.

We can compute the Borel–Moore homology of $S \cup H$ by considering the incidence correspondence

$$\mathcal{T} = \{(f,p) \in (S \cup H) \times \mathbb{P}(1,1,2) : f \text{ is singular at } p\}.$$

The incidence correspondence has a natural projection $\pi$ to the cone $\mathbb{P}(1,1,2)$. The fiber over the vertex $[0,0,1]$ is simply $\{[0,0,1]\} \times H$. If we restrict to $\pi^{-1}(\mathbb{P}(1,1,2) \setminus \{[0,0,1]\})$, $\pi$ is a complex vector bundle of rank 13. This allows us to compute the Borel–Moore homology of $\mathcal{T}$ and obtain $2t^{30}u^{-30} + t^{28}u^{-28}$ as its Poincaré–Serre polynomial.

Although $\pi$ is no desingularization of $S \cup H$, it is a proper map, hence it induces a homomorphism of Borel–Moore homology groups. By the results in § 4.2 the map induced by $\pi$ in degree 28 and 30 must be an isomorphism. In particular, this implies that $\bar{H}_{28}(S \cup H; \mathbb{Q})$ is generated by the fundamental class of the locus $S'$ of curves with a singularity on a chosen line. Obviously $\bar{H}_{30}(S \cup H; \mathbb{Q})$ is generated by $[S]$ and $[H]$.

Automorphisms of the graded ring $\mathbb{C}[x,y,z]$ are of the form

$$\begin{cases} x & \mapsto & \alpha x + \beta y \\ y & \mapsto & \gamma x + \delta y \\ z & \mapsto & \epsilon z + q(x,y), \end{cases}$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are complex numbers such that $\epsilon(\alpha\delta - \beta\gamma) \neq 0$ and $q \in \mathbb{C}[x,y]_2$.

They form a group $G$ of dimension 8. By contracting the vector space $\mathbb{C}[x,y]_2 \cong \mathbb{C}^3$ to a point, we get that $G$ is homotopy equivalent to $\mathrm{GL}(2) \times \mathbb{C}^*$. This means that we can apply Theorem 1.1 to the action of $\mathrm{GL}(2) \times \mathbb{C}^*$ instead of the whole $G$. In order to be able to apply the theorem



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in § 4.2, we check that its hypotheses are satisfied. Namely, for each generator $\eta$ of degree $2r-1$ of the cohomology of $\mathrm{GL}(2)\times\mathbb{C}^*$, we want to define a subscheme $Y$ of the discriminant, of pure codimension $r$, whose fundamental class maps to a non-zero multiple of $\eta$ under the composition

$$\bar{H}_{2(16-r)}(Y)\to\bar{H}_{2(16-r)}(\Sigma)\xrightarrow{\sim}H^{2r-1}(X)\xrightarrow{\rho^*}H^{2r-1}(G),$$

where $\rho$ denotes any orbit inclusion of $G$ in $\mathbb{C}^{16}$.

The cohomology of the product $\mathrm{GL}(2)\times\mathbb{C}^*$ decomposes naturally into that of its subgroups $\{I\}\times\mathbb{C}^*$ and $\mathrm{GL}(2)\times\{1\}$. We can reduce the study of the orbit inclusion to that of the two maps

$$\rho_1:\mathbb{C}^*\longrightarrow X,\quad \rho_1(t)=\rho(I,t)$$

and

$$\rho_2:\mathrm{GL}(2)\longrightarrow X,\quad \rho(A)=\rho(A,1).$$

The map induced by $\rho_1$ on cohomology is

$$\bar{H}_{31-\bullet}(H\cup S)\cong H^\bullet(X;\mathbb{Q})\xrightarrow{\rho_1^*}H^\bullet(\mathbb{C}^*;\mathbb{Q}).$$

The cohomology of $\mathbb{C}^*$ is generated by the fundamental class of $\{0\}\subset\mathbb{C}$. If we extend $\rho_1$ to a map $\mathbb{C}\longrightarrow\mathbb{C}^{16}$, we find that 0 is mapped to a curve which is the union of six distinct lines through the vertex of the cone. This means that the preimage of $H$ coincides with 0, while the preimage of $S$ is empty. Hence, $\rho_1^*([H])$ is a non-zero multiple of $[0]$ (in fact, by direct computation, it is $3[0]$).

Next, we consider $\rho_2$. It induces the map

$$\bar{H}_{31-\bullet}(H\cup S)\cong H^\bullet(X;\mathbb{Q})\xrightarrow{\rho_2^*}H^\bullet(\mathrm{GL}(2);\mathbb{Q}).$$

Recall from § 3.1 that the cohomology of $\mathrm{GL}(2)$ is generated by the fundamental class of the complement $D$ of $\mathrm{GL}(2)$ in the space $M$ of $2\times2$ matrices and that of the subspace $R$ of matrices with only zeros on the first column. The Borel–Moore homology class $[D]\in\bar{H}_6(D;\mathbb{Q})$ corresponds to a class of degree 1 in the cohomology of $\mathrm{GL}(2)$ and the Borel–Moore homology class $[R]$ corresponds to a class in $H^3(\mathrm{GL}(2);\mathbb{Q})$.

We look at the extension of $\rho_2$ to $M\longrightarrow\mathbb{C}^{16}$. The elements in $D$ are mapped to curves that are the union of three non-singular conics, having the same tangent line in a common point. These are always elements of $S\setminus H$. If we choose $S'$ as the locus of curves singular at some point of the line $\{y=0\}\subset\mathbb{P}(1,1,2)$, then we have that the preimage of $S'$ is exactly $R$. These considerations imply the surjectivity of the orbit inclusion on cohomology, hence the hypotheses of Theorem 1.1 are established.

### 4.2 Application of Gorinov–Vassiliev's method

As a starting point for the application of Gorinov–Vassiliev's method, we study the possible singular loci of elements of $\Sigma$. This is achieved by considering all possible decompositions in irreducible components of a curve of degree 6 on $\mathbb{P}(1,1,2)$. Then we consider how many singular points can lie on each component, and where the pairwise intersection of components can lie. As a result, we get a list similar to that given in Table 1.

For two configurations of the same type, the linear subspace of $V$ of curves that are singular in one configuration and the linear subspace of curves that are singular in the other always have the same codimension. Hence, this codimension only depends on the type of the configuration.

The families of configurations defined by the possible singular loci do not satisfy the conditions in List 2.3. We have to refine it in order to have all finite subconfigurations of finite configurations included. In this way, we get a new list as follows.



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LIST 4.1 (Sequence of families of configurations that satisfy the conditions on List 2.3). The number within square brackets is the codimension of the space of polynomials that are singular at a chosen configuration of that type. We choose to only give the families such that this codimension is $\leqslant 7$ or $\geqslant 15$ here, because they are the only ones that are relevant in the sequel.

In this list, a configuration of points is said to be *general* if it does not contain the vertex, no two points of the configuration lie on the same line of the ruling of the cone and at most three points lie on the same conic contained in the cone.

(A) The vertex [1].

(B) A general point [3].

(C) The vertex and a general point [4].

- Two points distinct from the vertex, lying on the same line of the ruling [6].
- The vertex and two points on the same line of the ruling [6].

(D) Two general points [6].

- Three collinear points, distinct from the vertex [7].
- The vertex and three collinear points [7].
- Four collinear points, distinct from the vertex [7].
- The vertex and four collinear points [7].
- Five collinear points, distinct from the vertex [7].
- The vertex and five collinear points [7].
- A singular line [7].

(E) The vertex and two general points [7].

- [... omissis ...].
- A rational normal cubic [15].

(F) The intersection of components of the union of two lines and two conics, excluding the vertex [15].

(G) The intersection of components of the union of two lines and two conics [15].

(H) Six points, intersection of the cone with three concurrent lines [15].

- Three lines [15].
- Union of a line and a non-singular conic [15].

(I) The whole cone [16].

By the results in §§ 2.2 and 2.3, or an adaptation of them, the only configurations giving non-trivial contribution to the Borel–Moore homology are those indicated with (A)–(I). We only consider them.

*Case (A).* Obviously, $F_A \cong \mathbb{C}^{15}$. The only homology is in degree 30.

*Case (B).* The space $F_B$ is a $\mathbb{C}^{13}$-bundle over the cone minus its vertex.

*Case (C).* The space $F_C$ is a $\mathbb{C}^{10} \times \mathring{\Delta}_1$-bundle over the cone minus its vertex.

*Case (D).* The space $\Phi_D$ is a non-orientable bundle of open simplices of dimension 1 over the subspace of $B(\mathbb{C} \times \mathbb{P}^1, 2)$ consisting of points that are not on the same line. This has the same Borel–Moore homology with twisted coefficients of $B(\mathbb{C} \times \mathbb{P}^1, 2)$, which is non-trivial only in degree 6. The space $F_D$ is a $\mathbb{C}^{10}$-bundle over $\Phi_D$.

*Case (E).* The space $\Phi_E$ is a non-orientable bundle of open simplices of dimension 2 over the subspace of $B(\mathbb{C} \times \mathbb{P}^1, 2)$ consisting of points that are not on the same line. The space $F_E$ is a $\mathbb{C}^9$-bundle over $\Phi_E$.



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*Cases (F) and (G).* We can consider configurations of types (F) and (G) together. We have then that $\Phi_F \cup \Phi_G$ is a complex vector bundle of rank one over $F_F \cup F_G$.

We claim that the Borel–Moore homology of $F_F \cup F_G$ is trivial. Let us consider the fiber $\Psi$ of its projection to $X_F$ (which is canonically isomorphic to $X_G$). Then $\Psi$ is a simplex with vertices $t, a_1, a_2, a_3, a_4, a_5, a_6$, where $\{a_i\} \in X_F$ and $t$ is the top of the cone. We have to consider the closed simplex $S$, minus all external faces containing $t$. Denote the union of such external faces by $B$. Observe that both $B$ and $S$ can be contracted to the vertex $t$. Then $\bar{H}_\bullet(\Psi;\mathbb{Q}) = H_\bullet(S, B;\mathbb{Q}) = 0$, which yields the claim.

*Case (H).* The space $F_H$ is a $\mathbb{C} \times \mathring{\Delta}_5$-bundle over $X_H$. We claim that $X_H$ has no Borel–Moore homology in the system of coefficients induced by $F_H$.

Each configuration in $X_H$ is determined by three concurrent lines in $\mathbb{P}^3$. This implies that $X_H$ is fibered over the complement of the cone in $\mathbb{P}^3$. The fiber over a point $p$ is the space of configurations of three distinct lines through $p$, subject to certain conditions. Namely, the lines cannot be tangent to the cone, any such triple must span $\mathbb{P}^3$, and no two of the lines can span a plane passing through the vertex of the cone. The condition of being tangent to the cone defines the union of two lines $l, m$ inside $\mathbb{P}^2 \cong \mathbb{P}(T_p\mathbb{P}^3)$. Then what we need is to compute the Borel–Moore homology of the subset $B(\mathbb{C} \times \mathbb{C}^*, 3) \setminus S$, where $S$ is the subset of the configurations in which exactly two points of the configuration lie on a line through $l \cap m$, or the three points are collinear (in this case, we can distinguish whether or not the line on which they lie passes through $l \cap m$).

The system of coefficients we have to consider changes its sign every time that a point moves along a loop around one of the lines $l$ or $m$. This system of coefficients is induced by a system $T'$ on $\mathbb{C} \times \mathbb{C}^*$ and Lemma 2.15 implies that $\bar{H}_\bullet(\mathbb{C} \times \mathbb{C}^*; T')$ is trivial. Hence, $B(\mathbb{C} \times \mathbb{C}^*, 3)$ has no Borel–Moore homology in the system of coefficients we are interested in. However, $S$ also has trivial Borel–Moore homology in the chosen system of coefficients, because each of its three disjoint substrata has trivial Borel–Moore homology in the system of coefficients induced by $T'$. This is again a consequence of Lemma 2.15. Hence, $B(\mathbb{C} \times \mathbb{C}^*, 3)$ has trivial Borel–Moore homology in the system of coefficients we are interested in.

*Case (I).* By Proposition 2.7(3), the space $F_I$ is an open cone over $\mathrm{Fil}_G \left|\Lambda(\bullet)\right|$. The differentials between columns (B) and (C), and between columns (D) and (E) of the spectral sequence converging to the Borel–Moore homology of the base of the cone must be isomorphisms for dimensional reasons. This implies that $F_I$ has trivial Borel–Moore homology.

Putting information for all configurations together, we have that the spectral sequence converging to the Borel–Moore homology of $\Sigma = S \cup H$ has $E^1$ term as in Table 4. Columns corresponding to configurations (F)–(I) have been omitted because those configurations give trivial contributions.

The differential $E^1_{C,26} \to E^1_{B,26}$ is trivial because $H^\bullet(X;\mathbb{Q})$ must contain a copy of $H^\bullet(G;\mathbb{Q})$. This means that this spectral sequence degenerates at $E^1$. As a consequence, $H^\bullet(X;\mathbb{Q}) = H^\bullet(G;\mathbb{Q})$. Theorem 1.1 gives that $C_1 = X/G$ has the rational cohomology of a point. Theorem 1.3 is now established.

## 5. Hyperelliptic locus

The moduli space $\mathcal{H}_g$ of smooth hyperelliptic curves of genus $g \geqslant 2$ always has the cohomology of a point. We have that $\mathcal{H}_g$ coincides with the moduli space of configurations of $2g + 2$ distinct points on $\mathbb{P}^1$, which is, in turn, the quotient of the space of binary polynomials of degree $2g + 2$ without double roots, for the action of $\mathrm{GL}(2)$. This is a special case of a moduli space of hypersurfaces. It is shown in [PS03] that in the hypersurface case the hypotheses of Theorem 1.1 are always satisfied. Thus, all we need to prove is that the cohomology of the space of polynomials without double roots is generated by the elements mapped to the generators of the cohomology of $\mathrm{GL}(2)$.

TABLE 4. Spectral sequence converging to $\bar{H}_k(\Sigma;\mathbb{Q})$.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 30 | | | | | |
| 29 | $\mathbb{Q}(15)$ | | | | |
| 28 | | $\mathbb{Q}(15)$ | | | |
| 27 | | | | | |
| 26 | | $\mathbb{Q}(14)$ | $\mathbb{Q}(14)$ | | |
| 25 | | | | | |
| 24 | | | $\mathbb{Q}(13)$ | | |
| 23 | | | | $\mathbb{Q}(13)$ | |
| 22 | | | | | |
| 21 | | | | | $\mathbb{Q}(12)$ |

LEMMA 5.1. *Let $V = \mathbb{C}[x,y]_d$ be the vector space of homogeneous binary polynomials of degree $d \geqslant 4$ and let $\Delta \subset V$ be the discriminant, i.e. the locus of polynomials with multiple roots. Then $V \setminus \Delta$ has Poincaré–Serre polynomial $1 + tu^2 + t^3u^4 + t^4u^6$.*

*Proof.* We apply Gorinov–Vassiliev's method and compute the Borel–Moore homology of $\Delta$. For every $v \in \Delta$, we denote the locus in $\mathbb{P}^1$ that is the projectivization of the multiple roots of $v$ by $K_v$. Denote by $X_k$ the family of all configurations of $k$ distinct points in $\mathbb{P}^1$. Then $X_1, \dots, X_{[d/2]}, \{\mathbb{P}^1\}$ satisfy the conditions in List 2.3. The dimension $d_i$ of $L(x)$ for $x \in X_i$ is always $d + 1 - 2i$. This means that we can construct a geometrical resolution of $\Delta$. Note that in the filtration only the first two terms have non-trivial Borel–Moore homology, because $X_k \cong B(\mathbb{P}^1, k)$ has trivial twisted Borel–Moore homology for $k \geqslant 3$.

Clearly, the spectral sequence for the Borel–Moore homology of $\Delta$ degenerates at $E_1$. This implies the claim. □

## ACKNOWLEDGEMENTS

I would like to thank my advisor Joseph Steenbrink for proposing the subject and for useful discussions and suggestions. I would also like to thank the referee for his careful job.

## REFERENCES

Che52 C. Chevalley, *The Betti numbers of the exceptional simple Lie groups*, Proceedings of the International Congress of Mathematicians (Cambridge, MA, 1950), vol. 2 (American Mathematical Society, Providence, RI, 1952), 21–24.

Del71 P. Deligne, *Théorie de Hodge. II*, Publ. Math. Inst. Hautes Études Sci. **40** (1971), 5–57.

Fab90 C. Faber, *Chow rings of moduli spaces of curves. II. Some results on the Chow ring of $\overline{\mathcal{M}}_4$*, Ann. of Math. (2) **132** (1990), 421–449.

Ful84 W. Fulton, *Intersection theory*, Ergeb. Math. Grenzgeb. (3), vol. 2 (Springer, Berlin, 1984).

Gor01 A. Gorinov, *Real homology groups of the space of nonsingular curves of degree 5 in* $\mathbf{CP}^2$. Preprint (2001), arXiv:math.AT/0105108.

Har77 R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52 (Springer, Heidelberg, 1977).

HZ86 J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), 457–485.

PS03 C. A. M. Peters and J. H. M. Steenbrink, *Degeneration of the Leray spectral sequence for certain geometric quotients*, Moscow Math. J. **3** (2003), 1085–1095, 1201.

Vas92 V. A. Vassiliev, *Complements of discriminants of smooth maps: topology and applications*, Translations of Mathematical Monographs, vol. 98 (American Mathematical Society, Providence, RI, 1992).

Vas99 V. A. Vassiliev, *How to calculate the homology of spaces of nonsingular algebraic projective hypersurfaces*, Proc. Steklov Inst. Math. **225** (1999), 121–140.

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