Canad. J. Math. Vol. 62 (6), 2010 pp. 1387–1403
doi:10.4153/CJM-2010-061-4


# Homotopy Self-Equivalences of 4-manifolds with Free Fundamental Group

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*Abstract.* We calculate the group of homotopy classes of homotopy self-equivalences of 4-manifolds with free fundamental group and obtain a classification of such 4-manifolds up to $s$-cobordism.

## 1 Introduction

Let $M$ be a closed, connected, oriented, smooth or topological 4-manifold with a fixed base point $x_0 \in M$. We want to study the group of homotopy classes of homotopy self-equivalences of $M$, preserving both the given orientation on $M$ and the base-point. Let $\mathrm{Aut}_\bullet(M)$ denote the group of homotopy classes of such homotopy self-equivalences.

Let us start by fixing our notation. The fundamental group $\pi_1(M, x_0)$ will be denoted by $\pi$, and the higher homotopy groups $\pi_i(M, x_0)$ will be denoted by $\pi_i$. Let $\Lambda = \mathbb{Z}[\pi]$ denote the integral group ring of $\pi$. We will mean homology and cohomology with integral coefficients unless otherwise noted.

Let $B$ denote the 2-type of $M$; we may construct $B$ by adjoining cells of dimension at least 4 to kill the homotopy groups in dimensions $\geq 3$. The natural map $c\colon M \to B$ is given by the inclusion of $M$ into $B$. Hambleton and Kreck [5], defined a thickening $\mathrm{Aut}_\bullet(M, w_2)$ of $\mathrm{Aut}_\bullet(M)$ (see Section 3 for the definition) and established a commutative braid of exact sequences, valid for any closed, oriented, smooth or topological 4-manifold. The authors defined

$$\mathrm{Isom}[\pi, \pi_2, k_M, c_*[M]] := \{\phi \in \mathrm{Aut}_\bullet(B) |\ \phi_*(c_*[M]) = c_*[M]\}$$

and obtained an explicit formula when the fundamental group is finite of odd order.

***Theorem (Hambleton-Kreck)*** *Let $M^4$ be a connected, closed, oriented, smooth or topological manifold of dimension* 4. *If $\pi$ has odd order, then*

$$\mathrm{Aut}_\bullet(M, w_2) \cong KH_2(M; \mathbb{Z}/2) \rtimes \mathrm{Isom}[\pi, \pi_2, k_M, c_*[M]],$$

*where* $KH_2(M; \mathbb{Z}/2) := \ker(w_2\colon H_2(M; \mathbb{Z}/2) \to \mathbb{Z}/2)$.

We extend the above result to the case when $\pi$ is a free group. We are going to define an extension $\mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$ of $\mathrm{Isom}[\pi, \pi_2, k_M, c_*[M]]$ and prove the following result for 4-manifolds with free fundamental group.

Received by the editors May 14, 2008.
Published electronically July 6, 2010.
AMS subject classification: **57N13**, 55P10, 57R80.

***Theorem 1.1*** *Let $M$ be a connected, closed, oriented, smooth or topological manifold of dimension 4. If $\pi$ is a free group, then*

$$\mathrm{Aut}_\bullet(M, w_2) \cong (KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)) \rtimes \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M].$$

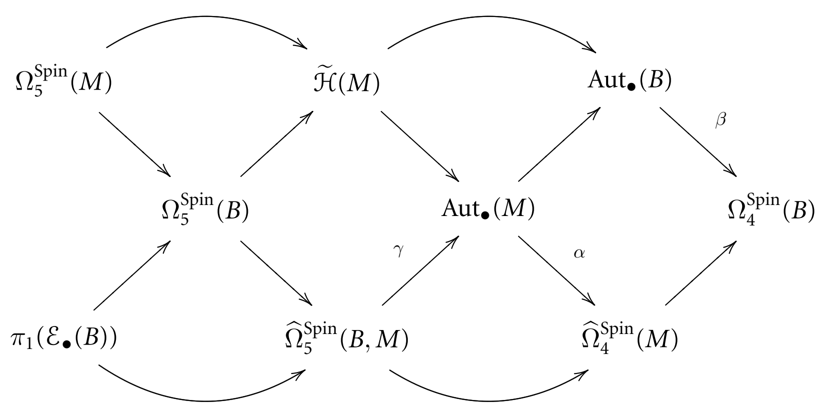
The last part of the paper deals with the classification of 4-manifolds up to $s$-cobordism. The geometric classification techniques, surgery, and $s$-cobordism theorem are not known to hold for free groups, so the most one can hope for at present is to obtain a classification up to $s$-cobordism. Based on the approach of [5], involving bordism techniques and the modified surgery theory of Kreck [10], we obtain the following result.

***Theorem 1.2*** *Let $M_1$ and $M_2$ be two closed, connected, oriented, topological 4-manifolds with free fundamental group and have the same Kirby-Siebenmann invariant. Then they are $s$-cobordant if and only if they have isometric quadratic 2-types.*

Theorem 1.2 is also stated in [3] and [8], but our line of argument is different, in the sense that our proof is primarily based on understanding homotopy self-equivalences.

## 2 Spin Case

For simplicity we start with spin manifolds. Throughout this section let $M$ be a spin manifold. To study the group $\mathrm{Aut}_\bullet(M)$, Hambleton and Kreck [5] constructed a braid of exact sequences




that is commutative up to sign. The sub-diagrams are all strictly commutative except for the two composites ending in $\mathrm{Aut}_\bullet(M)$, and valid for any closed, oriented, smooth or topological spin 4-manifold. Throughout this paper we always refer to [5] for the details of the definitions.

We will fix a lift $\nu_M\colon M \to B\,\mathrm{Spin}$ of the classifying map for the stable normal bundle of $M$. The Abelian group $\Omega_n^{\mathrm{Spin}}(M)$, with disjoint union as the group operation, denotes the singular bordism group of spin manifolds with a reference map to $M$. By imposing the requirement that the reference maps to $M$ must have degree zero, we obtain the modified bordism groups $\widehat{\Omega}_4^{\mathrm{Spin}}(M)$.

***Proposition 2.1*** *The relevant spin bordism groups of M are given as follows:*

$$\Omega_4^{\mathrm{Spin}}(M) \cong \mathbb{Z} \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z},$$
$$\Omega_5^{\mathrm{Spin}}(M) \cong H_1(M) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z}/2.$$

**Proof** This follows from the Atiyah–Hirzebruch spectral sequence, whose $E^2$-term is $H_p(M;\Omega_q^{\mathrm{Spin}}(*))$. The first differential $d_2\colon E^2_{p,q} \to E^2_{p-2,q+1}$ is given by the dual of $Sq^2$ (if $q = 1$) or that dual composed with reduction mod 2 (if $q = 0$), see [14, p. 751]. We substitute the values

$$\Omega_q^{\mathrm{Spin}}(*) = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0 \quad \text{for} \quad 0 \le q \le 5.$$

The differential for $(p,q) = (4,1)$ is dual to $Sq^2\colon H^2(M;\mathbb{Z}/2) \to H^4(M;\mathbb{Z}/2)$, which is zero, since $M$ is spin. We have a short exact sequence

$$0 \longrightarrow \Omega_4^{\mathrm{Spin}}(*) \oplus H_2(M;\mathbb{Z}/2) \longrightarrow F_{3,1} \longrightarrow H_3(M;\Omega_1^{\mathrm{Spin}}(*)) \longrightarrow 0$$

and $V \times S^1 \xrightarrow{f \circ p_1} F_{3,1}$ gives the splitting, where we consider an embedding $f\colon V \to M$ of a closed spin 3-manifold representing a generator of $H_3(M;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^r$, and $S^1$ is equipped with the non-trivial spin structure. Therefore, $\Omega_4^{\mathrm{Spin}}(M) \cong \mathbb{Z} \oplus H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \oplus \mathbb{Z}$. The result for $\Omega_5^{\mathrm{Spin}}(M)$ follows by similar arguments. ∎

In order to calculate the bordism groups of $B$, we need $H_i(B)$. We use the Serre spectral sequence of the fibration $\widetilde{B} \xrightarrow{p} B \to K(\pi,1)$, whose $E^2$-term is given by

$$E^2_{p,q} = H_p(K(\pi,1); H_q(\widetilde{B})),$$

the homology of $K(\pi,1)$ with local coefficients in the homology of $\widetilde{B}$. We need the homology of $\widetilde{B}$, but first recall a theorem of Whitehead.

***Theorem 2.2*** ([16]) *Let $X$ be a CW complex and $\Gamma$ denote the Whitehead's quadratic functor, then there is a functorial Whitehead exact sequence*

$$\pi_4(\widetilde{X}) \longrightarrow H_4(\widetilde{X}) \longrightarrow \Gamma(\pi_2(\widetilde{X})) \longrightarrow \pi_3(\widetilde{X}) \longrightarrow H_3(\widetilde{X}) \longrightarrow 0.$$

We have $H_4(\widetilde{B}) \cong \Gamma(\pi_2)$, since $\pi_i(\widetilde{B}) = 0$ for $i > 3$. Hillman [8] proved that $\pi_2 \cong \Lambda^{\beta_2(M)}$, and by [1, Theorem 5] it follows that $\Gamma(\pi_2)$ is a free $\Lambda$-module whenever $\pi$ is a free group.

Let $X_0 = *$, $X_1 = K(\mathbb{Z},2), \dots, X_N = K(\mathbb{Z}^N,2), \dots$, where $\mathbb{Z}^N$ is the $N$-fold product of $\mathbb{Z}$. Note that $\pi_2$ is countable and consider the sequence of maps

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \cdots,$$

where $i_k$'s are inclusions. Observe that $\widetilde{B}$ is homotopy equivalent to the mapping telescope of the above sequence and we have (see [6, p. 312])

$$H_n(\widetilde{B}) \cong \varinjlim H_n(X_k),$$
$$H^n(\widetilde{B};\mathbb{Z}/2) \cong \varprojlim H^n(X_k;\mathbb{Z}/2).$$

***Proposition 2.3*** *Let $B$ denote the 2-type of a spin 4-manifold with free fundamental group. The homology groups of $B$ are given by*

$$H_i(B) \cong \begin{cases} H_i(M) & \text{if } i = 0, 1 \text{ or } 2, \\ 0 & \text{if } i = 3 \text{ or } 5, \\ \mathbb{Z} \otimes_\Lambda \Gamma(\pi_2(M)) & \text{if } i = 4. \end{cases}$$

**Proof** The result follows easily from the the Serre spectral sequence of the fibration $\widetilde{M} \to M \to K(\pi, 1)$. ■

***Proposition 2.4*** *Let $\Omega_*^{\mathrm{Spin}}(B)$ denote the singular bordism group of spin manifolds with a reference map to $B$. We have the following:*

$$\Omega_4^{\mathrm{Spin}}(B) \subset H_4(B) \oplus \mathbb{Z} \quad \textit{and} \quad \Omega_5^{\mathrm{Spin}}(B) \cong H_1(B)$$

**Proof** We use the same spectral sequence, whose $E^2$-term is $H_p(B;\Omega_q^{\mathrm{Spin}}(*))$. The commutative diagram

$$\begin{array}{ccc} H^2(\widetilde{B};\mathbb{Z}/2) & \xrightarrow{Sq^2} & H^4(\widetilde{B};\mathbb{Z}/2) \\ {\scriptstyle p^*}\uparrow & & {\scriptstyle p^*}\uparrow \\ H^2(B;\mathbb{Z}/2) & \xrightarrow{Sq^2} & H^4(B;\mathbb{Z}/2) \end{array}$$

implies that $Sq^2 \colon H^2(B;\mathbb{Z}/2) \to H^4(B;\mathbb{Z}/2)$ is injective. Hence $d_2 \colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)$ is surjective. Therefore, on the line $p+q = 4$, the only groups that survive to $E^\infty$ are $\mathbb{Z}$ in the $(0, 4)$ position and a subgroup of $H_4(B)$ in the $(4, 0)$ position.

For the line $p + q = 5$, consider the diagram

$$\begin{array}{ccccc} H^2(\widetilde{B};\mathbb{Z}/2) & \xrightarrow{Sq^2} & H^4(\widetilde{B};\mathbb{Z}/2) & \xrightarrow{Sq^2} & H^6(\widetilde{B};\mathbb{Z}/2) \\ {\scriptstyle p^*}\uparrow & & {\scriptstyle p^*}\uparrow & & {\scriptstyle p^*}\uparrow \\ H^2(B;\mathbb{Z}/2) & \xrightarrow{Sq^2} & H^4(B;\mathbb{Z}/2) & \xrightarrow{Sq^2} & H^6(B;\mathbb{Z}/2). \end{array}$$

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Let $\alpha \in H^4(B;\mathbb{Z}/2)$ such that $Sq^2(\alpha) = 0$ and $p^*(\alpha) = \beta$. There exists $\lambda \in H^2(\widetilde{B};\mathbb{Z}/2)$ such that $Sq^2(\lambda) = \beta$, since the above row is exact. Therefore the sequence

$$H^2(B;\mathbb{Z}/2) \xrightarrow{Sq^2} H^4(B;\mathbb{Z}/2) \xrightarrow{Sq^2} H^6(B;\mathbb{Z}/2)$$

is exact. By the surjectivity of $H_6(B;\mathbb{Z}) \to H_6(B;\mathbb{Z}/2)$, we can conclude that $d_2\colon H_6(B;\mathbb{Z}) \to H_4(B;\mathbb{Z}/2)$ is surjective onto the kernel of the differential $d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)$. Thus the only group that survives to $E_\infty$ is $H_1(B) = H_1(M)$ in the $(1,4)$ position. ∎

The map $\alpha\colon \operatorname{Aut}_\bullet(M) \to \Omega_4^{\mathrm{Spin}}(M)$ is defined by $\alpha(f) = [M, f] - [M, \mathrm{id}]$. An element $(W, F)$ of $\widehat{\Omega}_5^{\mathrm{Spin}}(B, M)$ is a 5-dimensional spin manifold with boundary $(W, \partial W)$, equipped with a reference map $F\colon W \to B$ such that $F|_{\partial W}$ factors through the classifying map $c\colon M \to B$ and that $F|_{\partial W}\colon \partial W \to M$ has degree zero.

***Corollary 2.5*** *The group $\widehat{\Omega}_5^{\mathrm{Spin}}(B, M)$ is isomorphic to $H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ and it injects into $\operatorname{Aut}_\bullet(M)$. The image of $\alpha$ is equal to $H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$.*

**Proof** The map $\Omega_5^{\mathrm{Spin}}(M) \to \Omega_5^{\mathrm{Spin}}(B)$, which is composing with our reference map $c\colon M \to B$, maps the summand $H_1(M)$ isomorphically to $H_1(B)$ and $H_3(M;\mathbb{Z}/2) \oplus H_4(M;\mathbb{Z}/2)$ to zero. By the exactness of the braid, the map $\Omega_5^{\mathrm{Spin}}(B) \to \widehat{\Omega}_5^{\mathrm{Spin}}(B, M)$ is zero. Therefore

$$\begin{aligned}\widehat{\Omega}_5^{\mathrm{Spin}}(B, M)) &\cong \ker(\widehat{\Omega}_4^{\mathrm{Spin}}(M) \to \Omega_4^{\mathrm{Spin}}(B)) \\ &\cong H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2).\end{aligned}$$

The map $\widehat{\Omega}_5^{\mathrm{Spin}}(B, M) \to \widehat{\Omega}_4^{\mathrm{Spin}}(M)$ is injective, so by the commutativity of the braid the map $\pi_1(\mathcal{E}_\bullet(B)) \to \widehat{\Omega}_5^{\mathrm{Spin}}(B, M)$ is zero. Hence $\gamma\colon \widehat{\Omega}_5^{\mathrm{Spin}}(B, M) \to \operatorname{Aut}_\bullet(M)$ is injective.

The natural map $\Omega_4^{\mathrm{Spin}}(M) \to H_0(M)$ sends a spin 4-manifold to its signature. It follows that $\alpha(f) \in H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$. On the other hand, since both the map $\widehat{\Omega}_5^{\mathrm{Spin}}(B, M) \to \widehat{\Omega}_4^{\mathrm{Spin}}(M)$ and $\gamma$ are injective we have $H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \subseteq \operatorname{im}\alpha$. ∎

Let $\operatorname{Isom}[\pi, \pi_2]$ be the subgroup of $\operatorname{Aut}(\pi) \times \operatorname{Aut}(\pi_2)$ consisting of all those pairs $(\chi, \psi)$ for which $\psi(\eta a) = \chi(\eta)\psi(a)$ for all $\eta \in \pi$, $a \in \pi_2$. We have a split exact sequence [11, p. 31]

$$0 \longrightarrow H^2(\pi;\pi_2) \longrightarrow \operatorname{Aut}_\bullet(B) \underset{s}{\overset{(\pi_1,\pi_2)}{\rightleftarrows}} \operatorname{Isom}[\pi, \pi_2] \longrightarrow 1 .$$

In particular we have $\operatorname{Aut}_\bullet(B) = H^2(\pi;\pi_2) \rtimes \operatorname{Isom}[\pi, \pi_2]$. If $\pi$ is a free group, then $H^2(\pi;\pi_2) = 0$. Hence for $\pi$ a free group we have

$$\operatorname{Aut}_\bullet(B) \cong \operatorname{Isom}[\pi_1, \pi_2].$$

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Next, we will look for a relation between $c_*[M]$ and the cohomology intersection pairing $s_M$ on $M$. Let $\mathrm{Her}(H^2(B;\Lambda))$ be the group of Hermitian pairings on $H^2(B;\Lambda)$. Hillman [9] defined a homomorphism

$$B_{\pi_2}\colon H_4(B) \to \mathrm{Her}(H^2(B;\Lambda))$$

by $B_{\pi_2}(x)(u,v) = v(x \cap u) = (u \cup v)(x)$. The image of $c_*[M]$ is the cohomology intersection pairing on $M$. Moreover, by [9, Theorem 7] $B_{\pi_2}$ is an isomorphism whenever $\pi$ is a free group. Therefore $c_*[M]$ and $s_M$ uniquely determine each other.

Hambleton and Kreck [4] defined the quadratic 2-type of $M$ as the quadruple $[\pi, \pi_2, k_M, s_M]$. The isometries of the quadratic 2-type of $M$, which are denoted by $\mathrm{Isom}[\pi, \pi_2, k_M, s_M]$, consist of all pairs of isomorphisms

$$\chi\colon \pi \to \pi \quad \text{and} \quad \psi\colon \pi_2 \to \pi_2\,,$$

such that $\psi(gx) = \chi(g)\psi(x)$ for all $g \in \pi$ and $x \in \pi_2$, which preserve the $k$-invariant and the intersection form.

***Lemma 2.6*** $\ker(\beta\colon \mathrm{Aut}_\bullet(B) \to \Omega_4^{\mathrm{Spin}}(B)) = \mathrm{Isom}[\pi, \pi_2, s_M]$.

**Proof** If $\phi \in \mathrm{Aut}_\bullet(B)$ and $c\colon M \to B$ is the classifying map, then $\beta(\phi) := [M, \phi \circ c] - [M, c]$. The natural map $\Omega_4^{\mathrm{Spin}}(B) \to H_4(B)$ sends a bordism element to the image of its fundamental class. The image of $\beta(\phi)$ in $H_4(B)$ is zero when $\phi_*(c_*[M]) = c_*[M]$. Hence $\ker\beta$ is contained in the group of the isometries of the quadratic 2-type. On the other hand an element $\phi \in \mathrm{Isom}[\pi, \pi_2, s_M]$ will be $\phi \in \mathrm{Aut}_\bullet(B)$ such that $\phi_*(c_*[M]) = c_*[M]$, then clearly $\beta(\phi) = 0$. ■

***Lemma 2.7*** *For each $\phi \in \mathrm{Aut}_\bullet(B)$ such that $\phi_*(c_*[M]) = c_*[M]$, there is an $f \in \mathrm{Aut}_\bullet(M)$ such that $c \circ f \simeq \phi \circ c$.*

**Proof** First, let us assume that $H_2(M;\mathbb{Q}) \neq 0$. Since $\phi_*(c_*[M]) = c_*[M]$, there exists an $f \in \mathrm{Aut}_\bullet(M)$, such that $c \circ f \simeq \phi \circ c$ by [4, Lemma 1.3].

For the case $H_2(M;\mathbb{Q}) = 0$, we consider the homotopy equivalence $h\colon M \to \sharp_r(S^1 \times S^3)$ constructed in [2] which depends on a chosen isomorphism between $\pi$ and $*_r\mathbb{Z}$. Note that $\pi_1(\phi)$ induces an automorphism of $\pi$. Composing $\pi_1(\phi)$ with the previous isomorphism we get a different isomorphism between $\pi$ and $*_r\mathbb{Z}$. The same construction gives us another homotopy equivalence $h'\colon M \to \sharp_r(S^1 \times S^3)$. Since $h$ and $h'$ are degree 1 maps, we can construct an orientation preserving homotopy self equivalence of $M$ by $f := h \circ h'^{-1}\colon M \to M$. Now, it is easy to see that by construction $c \circ f \simeq \phi \circ c$. ■

***Corollary 2.8*** *The images of $\mathrm{Aut}_\bullet(M)$ and $\widetilde{\mathcal{H}}(M)$ in $\mathrm{Aut}_\bullet(B)$ are precisely equal to* $\mathrm{Isom}[\pi, \pi_2, s_M]$.

**Proof** For each $[f] \in \mathrm{Aut}_\bullet(M)$, we have a base-point preserving homotopy self-equivalence $\phi_f\colon B \to B$ such that $c \circ f = \phi_f \circ c$. All we have to show is that $(\phi_f)_*(c_*[M]) = c_*[M]$. We have $(\phi_f)_*(c_*[M]) = (\phi_f \circ c)_*[M] = (c \circ f)_*[M] =$

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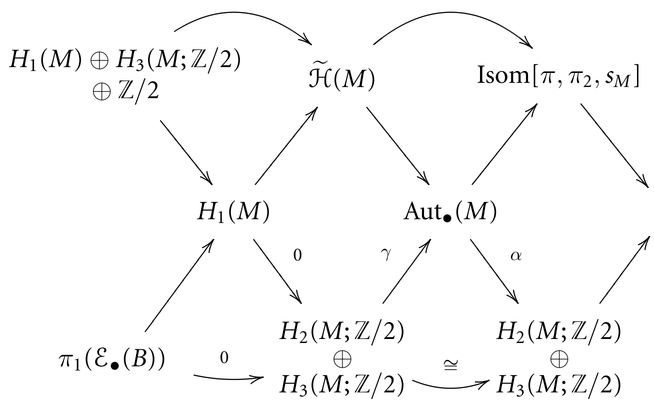
$c_*[M]$ since the fundamental class in $H_4(M)$ is preserved by an orientation preserving homotopy equivalence. We see that $\operatorname{im}(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B))$ is contained in $\mathrm{Isom}[\pi, \pi_2, s_M]$. The other inclusion follows from Lemma 2.7. The result for the image of $\widetilde{\mathcal{H}}(M)$ follows by the exactness of the braid and the fact that $\ker(\beta) = \mathrm{Isom}[\pi, \pi_2, s_M]$. ■

Here are the relevant terms of our braid diagram:




***Theorem 2.9*** *Let M be a connected, closed, oriented, smooth or topological spin manifold of dimension* 4. *If* $\pi$ *is a free group of rank r, then*

$$\mathrm{Aut}_\bullet(M) \cong (H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)) \rtimes \mathrm{Isom}[\pi, \pi_2, s_M].$$

**Proof** From the braid diagram, we have

$$\ker(\widetilde{\mathcal{H}}(M) \to \mathrm{Isom}[\pi, \pi_2, s_M]) \cong H_1(M),$$

so $\mathrm{Isom}[\pi, \pi_2, s_M] \cong \widetilde{\mathcal{H}}(M)/H_1$. This gives the splitting of the short exact sequence

$$0 \to K_1 \to \mathrm{Aut}_\bullet(M) \to \mathrm{Isom}[\pi, \pi_2, s_M] \to 1,$$

where $K_1 := \ker(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B))$. Hence it follows that

$$\mathrm{Aut}_\bullet(M) \cong K_1 \rtimes \mathrm{Isom}[\pi, \pi_2, s_M].$$

We already know that $\gamma$ is injective (Corollary 2.5). By the commutativity of the braid to show that it is actually an injective *homomorphism*, it is enough to show that $\alpha$ is a homomorphism on the image of $\gamma$. Let $\gamma(W, F) = f$ and $\gamma(W', F') = g$. Note that $\alpha(f \circ g) = \alpha(f) + f_*(\alpha(g))$. We have to show that $f_*(\alpha(g)) = \alpha(g)$. By Corollary 2.5, $\alpha(g) \in H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ and any element $f$ in the image of $\gamma$ is trivial in $\mathrm{Aut}_\bullet(B)$. Since $H_3(M;\mathbb{Z}/2) \cong H^1(M;\mathbb{Z}/2)$ and $c$ induces isomorphisms on $H_2(M;\mathbb{Z}/2)$ and $H^1(M;\mathbb{Z}/2)$, $f$ acts as the identity on $H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$.

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Now a diagram chase shows that $\gamma$ is a homomorphism. Therefore we have a short exact sequence of groups and homomorphisms

$$0 \to (H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)) \xrightarrow{\gamma} \mathrm{Aut}_\bullet(M) \to \mathrm{Isom}[\pi, \pi_2, s_M] \to 1.$$

Moreover, $K_1 = \mathrm{im}\,\gamma$ and $K_1$ is mapped isomorphically onto $H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ by the map $\alpha$. The conjugation action of $\mathrm{Isom}[\pi, \pi_2, s_M]$ on $K_1$ agrees with the induced action on homology under the identification $K_1 \cong H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$ via $\alpha$ (see [5]). It follows that

$$\mathrm{Aut}_\bullet(M) \cong (H_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)) \rtimes \mathrm{Isom}[\pi, \pi_2, s_M]. \qquad \blacksquare$$

***Example 2.10*** Let $M = S^1 \times S^3$. By the above theorem $\mathrm{Aut}_\bullet(M) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Note that these are orientation preserving homotopy self-equivalences. Define $\varphi\colon S^1 \times S^3 \to S^1 \times S^3$ by $\varphi(x, y) = (-x, y)$. If we compose orientation preserving self-equivalences with $\varphi$ we get also the orientation reversing homotopy self-equivalences. Therefore the based homotopy classes of based self homotopy equivalences of $S^1 \times S^3$ are isomorphic to $(\mathbb{Z}/2)^3$.

## 3 The Non-spin Case

When $w_2(M) \neq 0$, the bordism groups must be modified. The class $w_2$ gives a fibration and we can form the pullback

$$\begin{array}{ccc} B\langle w_2\rangle & \xrightarrow{j} & B \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle w_2} \\ BSO & \xrightarrow{w} & K(\mathbb{Z}/2, 2). \end{array}$$

The map $w = w_2(\gamma)$ pulls back the second Stiefel–Whitney class for the universal oriented vector bundle $\gamma$ over $BSO$. $B\langle w_2\rangle$ is called the normal 2-type of $M$ [10]. Let $\Omega_*(B\langle w_2\rangle)$ be bordism classes smooth manifolds equipped with a lift of the normal bundle. The spectral sequence used to compute $\Omega_*(B\langle w_2\rangle)$ has the same $E_2$-term as the one used above for $w_2 = 0$, but the differentials are twisted by $w_2$. In particular, $d_2$ is the dual of $Sq^2_w$, where $Sq^2_w(x) := Sq^2(x) + x \cup w_2$ (see [14, Section 2]).

There is a corresponding non-spin version of $\Omega^{\mathrm{Spin}}_*(M)$, namely the bordism groups $\Omega_*(M\langle w_2\rangle)$. The $E_2$-term of the spectral sequence is unchanged from the spin case, but the differentials are twisted by $w_2$ with the above formula for $Sq^2_w$. We choose a particular representative for the map $w_2$ such that $w_2 = w \circ \nu_M$. Next we define a suitable "thickening" of $\mathrm{Aut}_\bullet(M)$ for the non-spin case.

***Definition 3.1*** ([5]) Let $\mathrm{Aut}_\bullet(M, w_2)$ denote the set of equivalence classes of maps $\widehat{f}\colon M \to M\langle w_2\rangle$ such that

(i) $f := j \circ \widehat{f}$ is a base-point and orientation preserving homotopy equivalence, and
(ii) $\xi \circ \widehat{f} = \nu_M$.

Given two maps $\widehat{f}, \widehat{g}\colon M \to M\langle w_2\rangle$ as above, we define

$$\widehat{f} \bullet \widehat{g}\colon M \to M\langle w_2\rangle$$

as the unique map from $M$ into the pull-back $M\langle w_2\rangle$ defined by the pair $f \circ g\colon M \to M$ and $\nu_M\colon M \to BSO$. It was proved in [5] that $\mathrm{Aut}_\bullet(M, w_2)$ is a group under this operation and there is a short exact sequence of groups

$$0 \longrightarrow H^1(M; \mathbb{Z}/2 \longrightarrow \mathrm{Aut}_\bullet(M, w_2) \longrightarrow \mathrm{Aut}_\bullet(M) \longrightarrow 1.$$

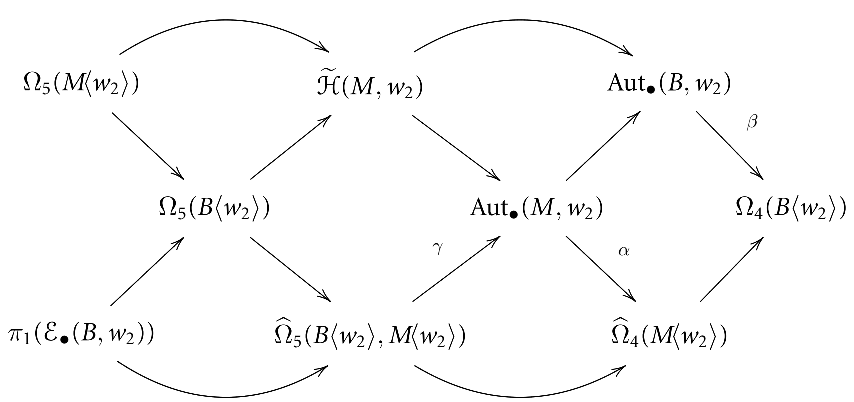
To define an analogous group $\mathrm{Aut}_\bullet(B, w_2)$ of self-equivalences, we should first state the following lemma from [5].

***Lemma 3.2*** *Given a base-point preserving map $f\colon M \to B$, there is a unique extension (up to base-point preserving homotopy) $\phi_f\colon B \to B$ such that $\phi_f \circ c = f$. If $f$ is a 3-equivalence, then $\phi_f$ is a homotopy equivalence. Moreover, if $w_2 \circ f = w_2$, then $w_2 \circ \phi_f = w_2$.*

***Definition 3.3*** ([5]) Let $\mathrm{Aut}_\bullet(B, w_2)$ denote the set of equivalence classes of maps $\widehat{f}\colon M \to B\langle w_2\rangle$ such that

(i) $f := j \circ \widehat{f}$ is a base-point preserving 3-equivalence, and
(ii) $\xi \circ \widehat{f} = \nu_M$.

***Theorem 3.4*** ([5]) *Let $M$ be a closed, oriented topological 4-manifold. Then there is a sign-commutative diagram of exact sequences*




*such that the two composites ending in* $\mathrm{Aut}_\bullet(M, w_2)$ *agree up to inversion, and the other sub-diagrams are strictly commutative.*

***Proposition 3.5*** *Let $B\langle w_2\rangle$ denote the normal 2-type of a 4-manifold $M$ with free fundamental group. Then we have*

$$\begin{aligned}\Omega_4(M\langle w_2\rangle) &\cong \mathbb{Z}\oplus H_2(M;\mathbb{Z}/2)\oplus H_3(M;\mathbb{Z}/2)\oplus\mathbb{Z},\\ \Omega_5(M\langle w_2\rangle) &\cong H_1(M)\oplus H_3(M;\mathbb{Z}/2)\oplus\mathbb{Z}/2,\\ \Omega_4(B\langle w_2\rangle) &\subset \mathbb{Z}\oplus\mathbb{Z}/2\oplus H_4(B),\\ \Omega_5(B\langle w_2\rangle) &\cong H_1(M).\end{aligned}$$

**Proof** We only need to compute the $d_2$ differentials. Since $M$ is orientable, $w_2$ is also the second Wu class of $M$. We have $Sq_w^2(x)=0$. Now, everything works exactly the same as in the spin case.

For the bordism groups of $B\langle w_2\rangle$, first consider the following commutative diagram

$$\begin{array}{ccc} H^2(\widetilde{B};\mathbb{Z}/2) & \xrightarrow{Sq_w^2} & H^4(\widetilde{B};\mathbb{Z}/2)\\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle p^*}\\ H^2(B;\mathbb{Z}/2) & \xrightarrow{Sq_w^2} & H^4(B;\mathbb{Z}/2).\end{array}$$

By the commutativity of the diagram, we have

$$\begin{aligned}\ker(Sq_w^2\colon H^2(B;\mathbb{Z}/2)\to H^4(B;\mathbb{Z}/2)) &\cong \langle w_2\rangle \cong \mathbb{Z}/2\\ &\cong \operatorname{coker}(d_2\colon H_4(B;\mathbb{Z}/2)\to H_2(B;\mathbb{Z}/2)).\end{aligned}$$

Since all the other differentials are zero, this gives the $\mathbb{Z}/2$ in the $E_{2,2}^{\infty}$ position. To see that $H_1(B)\cong H_1(M)$ is the only group on the line $p+q=5$ that survives to $E_\infty$, we use the following commutative diagram:

$$\begin{array}{ccccc} H^2(\widetilde{B};\mathbb{Z}/2) & \xrightarrow{Sq_w^2} & H^4(\widetilde{B};\mathbb{Z}/2) & \xrightarrow{Sq_w^2} & H^6(\widetilde{B};\mathbb{Z}/2)\\ {\scriptstyle p^*}\uparrow & & {\scriptstyle p^*}\uparrow & & {\scriptstyle p^*}\uparrow\\ H^2(B;\mathbb{Z}/2) & \xrightarrow{Sq_w^2} & H^4(B;\mathbb{Z}/2) & \xrightarrow{Sq_w^2} & H^6(B;\mathbb{Z}/2).\end{array}$$

We are going to show that the bottom row is exact. It is easy to see that

$$H^2(X_k;\mathbb{Z}/2)\xrightarrow{Sq_w^2} H^4(X_k;\mathbb{Z}/2)\xrightarrow{Sq_w^2} H^6(X_k;\mathbb{Z}/2)$$

is exact. We have $\{H^2(X_k;\mathbb{Z}/2),i_k^*\}$, $\{H^4(X_k;\mathbb{Z}/2),i_k^*\}$, and $\{H^6(X_k;\mathbb{Z}/2),i_k^*\}$, an inverse system of modules, where $i_k\colon X_{k-1}\to X_k$ is the inclusion map. Consider the commutative diagram with exact rows

$$\begin{array}{ccccc} H^2(X_k;\mathbb{Z}/2) & \xrightarrow{Sq_w^2} & H^4(X_k;\mathbb{Z}/2) & \xrightarrow{Sq_w^2} & H^6(X_k;\mathbb{Z}/2) \\ \downarrow{\scriptstyle i_k^*} & & \downarrow{\scriptstyle i_k^*} & & \downarrow{\scriptstyle i_k^*} \\ H^2(X_{k-1};\mathbb{Z}/2) & \xrightarrow{Sq_w^2} & H^4(X_{k-1};\mathbb{Z}/2) & \xrightarrow{Sq_w^2} & H^6(X_{k-1};\mathbb{Z}/2). \end{array}$$

Then the sequence

$$\varprojlim H^2(X_k;\mathbb{Z}/2) \xrightarrow{Sq_w^2} \varprojlim H^4(X_k;\mathbb{Z}/2) \xrightarrow{Sq_w^2} \varprojlim H^6(X_k;\mathbb{Z}/2)$$

is exact. Let $a \in H^2(B;\mathbb{Z}/2)$, then $Sq_w^2(a^2 + a \cup w_2) = 0$. Now, let $b \in H^4(B;\mathbb{Z}/2)$ such that $Sq_w^2(b) = 0$ and let $p^*(b) = y$, then $Sq_w^2(y) = 0$. There exists a $z \in H^2(\widetilde{B};\mathbb{Z}/2)$ such that $Sq_w^2(z) = y$. Then we also have a $c \in H^2(B;\mathbb{Z}/2)$ such that $p^*(c) = z$ and $Sq_w^2(c) = b$. Therefore the sequence

$$H^2(B;\mathbb{Z}/2) \xrightarrow{Sq_w^2} H^4(B;\mathbb{Z}/2) \xrightarrow{Sq_w^2} H^6(B;\mathbb{Z}/2)$$

is exact. Note also that $H_6(B) \to H_6(B;\mathbb{Z}/2)$ is surjective, hence $d_2\colon H_6(B) \to H_4(B;\mathbb{Z}/2)$ is onto the kernel of $d_2\colon H_4(B;\mathbb{Z}/2) \to H_2(B;\mathbb{Z}/2)$. ■

Let $\widehat{c}\colon M \to B\langle w_2\rangle$ denote the map defined by the pair $(c\colon M \to B, \nu_M\colon M \to BSO)$. Consider the diagram

$$\begin{array}{ccc} M\langle w_2\rangle & \xrightarrow{c\circ j} & B \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle w_2} \\ BSO & \xrightarrow{w} & K(\mathbb{Z}/2,2). \end{array}$$

We have $(w_2 \circ c) \circ j = w_2 \circ j$ and since the pullback satisfies the universal property, there exists a map $\bar{c}\colon M\langle w_2\rangle \to B\langle w_2\rangle$. Let $\widehat{\mathrm{id}}\colon M \to M\langle w_2\rangle$ denote the map defined by the pair $(\mathrm{id}_M\colon M \to M, \nu_M\colon M \to BSO)$. Given $[\widehat{f}] \in \mathrm{Aut}_\bullet(M, w_2)$, we define $\alpha\colon \mathrm{Aut}_\bullet(M, w_2) \to \widehat{\Omega}_4(M\langle w_2\rangle)$ by $\alpha(\widehat{f}) = [M, \widehat{f}] - [M, \widehat{\mathrm{id}}_M]$, where the modified bordism groups are defined by letting the degree of a reference map $\widehat{g}\colon N^4 \to M\langle w_2\rangle$ be the ordinary degree of $g = j \circ \widehat{g}$. An element $(W, \widehat{F})$ of $\widehat{\Omega_5}(B\langle w_2\rangle, M\langle w_2\rangle)$ is a 5-dimensional manifold with boundary $(W, \partial W)$, equipped with a reference map $\widehat{F}\colon W \to B\langle w_2\rangle$ such that $\widehat{F}|_{\partial W}$ factors through $\bar{c}$.

***Corollary 3.6*** *The group*

$$\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) \cong KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$$

*and it injects into* $\mathrm{Aut}_\bullet(M, w_2)$. *The image of* $\alpha$ *is*

$$\operatorname{im}\alpha = KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2).$$

**Proof** As in the proof of Corollary 2.5, $\Omega_5(M\langle w_2\rangle) \to \Omega_5(B\langle w_2\rangle)$ is onto, and, by the exactness of the braid $\Omega_5(B\langle w_2\rangle) \to \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)$ is zero. Thus,

$$\begin{aligned}\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) &\cong \ker\big(\widehat{\Omega}_4(M\langle w_2\rangle) \to \Omega_4(B\langle w_2\rangle)\big)\\ &\cong KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2).\end{aligned}$$

The map $\pi_1(\mathcal{E}_\bullet(B, w_2)) \to \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)$ is zero by the commutativity of the braid. Therefore

$$\gamma\colon \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) \to \mathrm{Aut}_\bullet(M, w_2)$$

is injective. The natural map $\Omega_4(M\langle w_2\rangle) \to H_0(M)$ sends a 4-manifold to its signature. Since the class $w_2 \in H^2(M;\mathbb{Z}/2)$ is a characteristic element for the cup product form (mod 2), it is preserved by the induced map of a self-homotopy equivalence of $M$. Therefore, the image of $\mathrm{Aut}_\bullet(M, w_2)$ in $\Omega_4(M\langle w_2\rangle)$ lies in the subgroup $KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2)$. Since the map $\gamma$ is injective, we also have

$$KH_2(M;\mathbb{Z}/2) \oplus H_3(M;\mathbb{Z}/2) \subseteq \operatorname{im}\alpha. \qquad \blacksquare$$

Next, we are going to define a homomorphism

$$\widehat{j}\colon \mathrm{Aut}_\bullet(B, w_2) \to \mathrm{Aut}_\bullet(B).$$

For any $\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2)$, $f := j \circ \widehat{f}\colon M \to B$ is a 3-equivalence. There is a unique homotopy equivalence $\phi_f\colon B \to B$ such that $\phi_f \circ c \simeq f$. We define $\widehat{j}(\widehat{f}) := \phi_f$. Let $\widehat{g}$ be another element of $\mathrm{Aut}_\bullet(B, w_2)$, then $\widehat{f} \bullet \widehat{g}$ is defined by the pair $(\phi_f \circ \phi_g \circ c, \nu_M)$. Therefore $\widehat{j}(\widehat{f} \bullet \widehat{g}) = \phi_f \circ \phi_g$. Let

$$\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M] := \{\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2)\ |\phi_f \in \mathrm{Isom}[\pi, \pi_2, s_M]\}.$$

***Lemma 3.7*** **([12])** *There is a short exact sequence of groups*

$$0 \longrightarrow H^1(M;\mathbb{Z}/2) \longrightarrow \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M] \xrightarrow{\widehat{j}} \mathrm{Isom}[\pi, \pi_2, s_M] \longrightarrow 1.$$

***Corollary 3.8*** *The image of* $\mathrm{Aut}_\bullet(M, w_2)$ *in* $\mathrm{Aut}_\bullet(B, w_2)$ *is precisely equal to* $\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M]$.

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**Proof** Let $\widehat{f} \in \mathrm{Aut}_\bullet(M, w_2)$ and $\phi_{\widehat{f}}$ denote the image of $\widehat{f}$ in $\mathrm{Aut}_\bullet(B, w_2)$. Then $\widehat{j}(\phi_{\widehat{f}}) = \phi_f$ satisfies $\phi_f \circ c = c \circ f$ and $\phi_f$ preserves $c_*[M]$. Hence

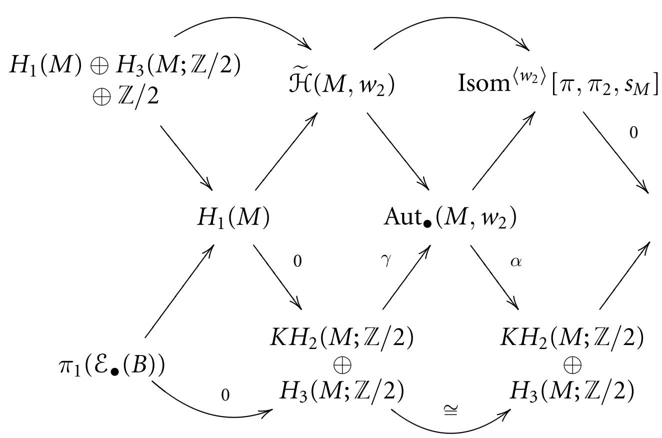
$$\phi_f \in \mathrm{Isom}[\pi, \pi_2, s_M].$$

Now suppose that $\phi \in \mathrm{Isom}[\pi, \pi_2, s_M]$, then there exists $f \in \mathrm{Aut}_\bullet(M)$ such that $\phi \circ f \simeq c \circ f$ (Lemma 2.7). We may assume that $\widehat{f} = (f, \nu_M) \in \mathrm{Aut}_\bullet(M, w_2)$ [5, Lemma 3.1]. Let $\phi_{\widehat{f}} \in \mathrm{Aut}_\bullet(B, w_2)$ denote the image of $\widehat{f}$; then we have $\widehat{j}(\phi_{\widehat{f}}) = \phi$. ∎

***Lemma 3.9*** *We have* $\ker(\beta\colon \mathrm{Aut}_\bullet(B, w_2) \to \Omega_4(B\langle w_2\rangle)) = \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M]$, *and the image of* $\widetilde{\mathcal{H}}(M, w_2)$ *in* $\mathrm{Aut}_\bullet(B, w_2)$ *is equal to* $\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M]$.

**Proof** In the non-spin case, the map $\beta\colon \mathrm{Aut}_\bullet(B, w_2) \to \Omega_4(B\langle w_2\rangle)$ is defined by $\beta(\widehat{f}) = [M, \widehat{f}] - [M, \widehat{c}]$. Let $\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2)$ and suppose first that $\widehat{f} \in \ker \beta$, then $(j \circ \widehat{f})_*[M] = c_*[M]$. But since $(j \circ \widehat{f})$ is a 3-equivalence, there exists $\phi \in \mathrm{Aut}_\bullet(B)$ with $\phi \circ c = j \circ \widehat{f}$. So, $\phi_*(c_*[M]) = c_*[M]$, which means $\widehat{j}(\widehat{f}) = \phi \in \mathrm{Isom}[\pi, \pi_2, s_M]$. Therefore $\ker(\beta) \subseteq \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M]$. It is easy to see the other inclusion from the commutativity of the braid. The result about the image of $\widetilde{\mathcal{H}}(M, w_2)$ follows from the exactness of the braid [5, Lemma 2.7] and the fact that $\ker(\beta) = \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M]$. ∎

The relevant terms of our braid are now




**Proof of Theorem 1.1** We have a split short exact sequence

$$0 \longrightarrow \widehat{K_1} \longrightarrow \mathrm{Aut}_\bullet(M, w_2) \longrightarrow \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M] \longrightarrow 1,$$

where $\widehat{K_1} = \ker(\mathrm{Aut}_\bullet(M, w_2) \to \mathrm{Aut}_\bullet(B, w_2))$. Any element $\widehat{f}$ will act as an identity on $\mathrm{im}(\alpha) = KH_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$, so $\lambda$ is a homomorphism. Also $\widehat{K_1} \cong KH_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2)$, and the rest of the proof follows as in the spin case. ∎

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***Remark 3.10*** We have

$$H_2(M;\mathbb{Z}/2) \cong H_0(\pi; H_2(\widetilde{M};\mathbb{Z}/2)) \cong (\pi_2 \otimes \mathbb{Z}/2) \otimes_\Lambda \mathbb{Z}.$$

Therefore any element of $H_2(M;\mathbb{Z}/2)$ can be represented by a map $S^2 \to M$. Let $0 \neq x \in KH_2(M;\mathbb{Z}/2)$ and $\alpha\colon S^2 \to M$ corresponds to $x$ via the above isomorphism. Choose an embedding $D^4 \hookrightarrow M$ and shrink $\partial D^4$ to a point, to get a map $M \to M \vee S^4$. Now let $\eta\colon S^3 \to S^2$ be the Hopf map, $S\eta\colon S^4 \to S^3$ its suspension, and $\eta^2\colon S^4 \to S^2$ the composition $\eta^2 = \eta \circ S\eta$. Let $f$ be the composite map

$$M \longrightarrow M \vee S^4 \xrightarrow{\mathrm{id} \vee \eta^2} M \vee S^2 \xrightarrow{\mathrm{id} \vee \alpha} M$$

$f$ induces identities on $\pi_1$ and on $H_i(\widetilde{M})$, so $f$ is homologous to the $\mathrm{id}_M$, and hence it is a homotopy equivalence, but it is not homotopic to the identity, for $\gamma$ is injective.

To realize $H_3(M;\mathbb{Z}/2)$ as homotopy equivalences, first observe that $H_3(M) \cong H_3(\widetilde{M}) \otimes_\Lambda \mathbb{Z}$ and reduction mod 2 is onto, so by Hurewicz's theorem for any element of $H_3(M;\mathbb{Z}/2)$, there exists a map $\beta\colon S^3 \to M$. Now the following composite map

$$M \longrightarrow M \vee S^4 \xrightarrow{\mathrm{id} \vee S\eta} M \vee S^3 \xrightarrow{\mathrm{id} \vee \beta} M$$

is again a homotopy-equivalence.

## 4 S-Cobordism

In this section we are going to show that the *quadratic 2-type* with the *Kirby–Siebenmann invariant* determines a classification of topological 4-manifolds with free fundamental group, up to *s-cobordism*. Before we state our result, let us first recall that

$$\Omega_4^{STOP}(K(\pi,1)) \cong \Omega_4^{STOP}(*) \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

where $\pi \cong *_r\mathbb{Z}$. The isomorphism can be given by associating the pair $(\sigma(M), ks(M))$ with $M$, where $ks(M)$ is the *Kirby–Siebenmann invariant* of $M$. The latter invariant $\sigma(M)$ is the signature of the 4-manifold $M$. Recall that the signature of a closed, oriented 4-manifold $M$, $\sigma(M)$ is given by the signature of the usual intersection form

$$s_M^{\mathbb{Z}}\colon H_2(M) \otimes H_2(M) \to \mathbb{Z}.$$

Note that, when $\pi$ is a free group we have $H_2(M) \cong H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z}$ and

$$s_M \otimes_\Lambda \mathbb{Z}\colon (H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z}) \otimes (H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z}) \to \Lambda \otimes_\Lambda \mathbb{Z} \cong Z$$

is the integral intersection form $s_M^{\mathbb{Z}}$, since $H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z}$ is the largest quotient of $H_2(M;\Lambda)$ on which $\pi$ acts trivially. Therefore the signature of $M$ is determined by the formula

$$\sigma(M) = \sigma(s_M^{\mathbb{Z}}) = \sigma(s_M \otimes_\Lambda \mathbb{Z}).$$

Here is our main result for this section.

***Theorem 4.1*** *Let $M_1$ and $M_2$ be two closed, connected, oriented, topological 4-manifolds with free fundamental group and have the same Kirby-Siebenmann invariant. Then they are s-cobordant if and only if they have isometric quadratic 2-types.*

**The proof of Theorem 1.2** If $M_1$ and $M_2$ are $s$-cobordant, then the inclusion of $M_1$ into an $s$-cobordism between $M_1$ and $M_2$ and the homotopy inverse of the inclusion from $M_2$ is an orientation preserving homotopy equivalence and thus induces an isometry between the intersection forms. So, $M_1$ and $M_2$ have isometric *quadratic 2-types*. Suppose now that $M_1$ and $M_2$ have isometric *quadratic 2-types*. Then $M_1$ and $M_2$ have isomorphic equivariant intersection forms, and by the above arguments $\sigma(M_1) = \sigma(M_2)$. The hypotheses imply that we have a cobordism $W$ between $M_1$ and $M_2$ over $K(\pi, 1)$. We may assume that $W$ is connected.

Choose a handle decomposition of $W$. We can cancel all 0- and 5-handles. Further, we may assume, by low-dimensional surgery, $M_1 \hookrightarrow W$ is a 2 equivalence. So we can trade all 1-handles for 3-handles, and upside-down, all 4-handles for 2-handles. We end up with a handle decomposition of $W$ that only contains 2- and 3-handles, and view $W$ as

$$W = M_1 \times [0,1] \cup \{\text{2-handles}\} \cup \{\text{3-handles}\} \cup M_2 \times [-1,0],$$

which we split into two halves: on one side, $M_1$ and the 2-handles, on the other, $M_2$ and the 3-handles. Let $3/2$ be the level in $W$ that appears immediately after all 2-handles have been attached but before any 3-handle is attached.

We will cut $W$ into two halves, then glue them back after sticking in an $h$-cobordism of $M_{3/2}$. This cut and reglue procedure will create a new cobordism from $M_1$ to $M_2$. If we choose the right $h$-cobordism, then the 3-handles from the upper half will cancel the 2-handles from the lower half. This means that the newly created cobordism between $M_1$ and $M_2$ will have no homology relative to its boundaries, and so it will indeed be an $h$-cobordism from $M_1$ to $M_2$. Finally, note that the Whitehead group $Wh(\pi)$ is trivial for $\pi \cong *_r\mathbb{Z}$ ([13]), hence in this case being $s$-cobordant is equivalent to being $h$-cobordant.

From the lower half of $W$ we have $M_{3/2} \approx M_1 \sharp m(S^2 \times S^2)$, while from the upper half we have $M_{3/2} \approx M_2 \sharp m(S^2 \times S^2)$, see for example [10, Corollary 3]. Hence we have a homeomorphism

$$\zeta \colon M_2 \sharp m(S^2 \times S^2) \xrightarrow{\approx} M_1 \sharp m(S^2 \times S^2).$$

Let $B(M_i)$ denote the 2-types of $M_i$ and $c_i \colon M_i \to B(M_i)$ corresponding 3-equivalences for $i = 1, 2$. Since $M_1$ and $M_2$ have isometric quadratic 2-types, we have the following isomorphisms

$$\chi \colon \pi_1(M_1) \to \pi_1(M_2) \quad \text{and} \quad \psi \colon \pi_2(M_1) \to \pi_2(M_2)$$

such that

$$s_{M_2}(\psi(x), \psi(y)) = \chi_*(s_{M_1}(x, y)).$$

We can construct a homotopy equivalence $\theta\colon B(M_1) \to B(M_2)$ such that $\pi_2(\theta) \circ \pi_2(c_1) = \pi_2(c_2) \circ \psi$ and $\theta_\sharp(s_{M_2}) = s_{M_1}$. Now let

$$M := M_1 \sharp m(S^2 \times S^2) \quad \text{and} \quad M' := M_2 \sharp m(S^2 \times S^2)$$

such that the quadratic 2-type of $M$ is

$$[\pi, \pi_2, s_M] := [\pi_1(M_1), \pi_2(M_1) \oplus \Lambda^{2m}, s_{M_1} \oplus H(\Lambda^m)],$$

where $H(\Lambda^m)$ is the hyperbolic form on $\Lambda^m \oplus (\Lambda^m)^*$. Next, note that

$$(\pi_1(\zeta) \circ \chi, \pi_2(\zeta) \circ (\psi \oplus \mathrm{id})) = (\mathrm{id}, \pi_2(\zeta) \circ (\psi \oplus \mathrm{id}))$$

gives us an element in $\mathrm{Isom}[\pi, \pi_2, s_M]$ since it is the composition of isometries. Let $B := B(M)$ denote the 2-type of $M$. Remember that we have $\mathrm{Aut}_\bullet(B) \cong \mathrm{Isom}[\pi, \pi_2]$. Therefore we can find a $\phi \in \mathrm{Aut}_\bullet(B)$ such that

$$\pi_1(\phi) = \mathrm{id} \quad \text{and} \quad \pi_2(\phi) = \pi_2(\zeta) \circ (\psi \oplus \mathrm{id}).$$

We can choose, by Lemma 3.7, $\widehat{f} \in \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M]$ such that $\widehat{j}(\widehat{f}) = \phi$. There exists $(W, \widehat{F}) \in \widetilde{\mathcal{H}}(M, w_2)$ that maps to $\widehat{f}$, *i.e.*, $\widehat{F}\colon W \to B\langle w_2 \rangle$ and $F|_{\partial_2 W} = \widehat{f}$. We have a commutative diagram of exact sequences (see [5, Lemma 4.1])

$$\begin{array}{ccccc}
\tilde{L}_6(\mathbb{Z}[\pi_1]) & = & \tilde{L}_6(\mathbb{Z}[\pi_1]) & & \\
\downarrow & & \downarrow & & \\
\mathcal{S}(M \times I, \partial) & \longrightarrow & \mathcal{H}(M) & \longrightarrow & \mathrm{Aut}_\bullet(M, w_2) \\
\downarrow & & \downarrow & & \\
\mathcal{T}(M \times I, \partial) & \longrightarrow & \widetilde{\mathcal{H}}(M, w_2) & \twoheadrightarrow & \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M] \\
\downarrow & & \downarrow & & \\
L_5(\mathbb{Z}[\pi_1]) & = & L_5(\mathbb{Z}[\pi_1]), & &
\end{array}$$

where the left-hand vertical sequence is from Wall's surgery exact sequence [15, Chapter 10]. To obtain the right-hand vertical sequence we use the modified surgery theory of [10].

The group $\mathcal{H}(M)$ consists of oriented $h$-cobordisms $W^5$ from $M$ to $M$, under the equivalence relation induced by $h$-cobordism relative to the boundary. The group structure on $\mathcal{T}(M \times I, \partial)$ is defined as for $\widetilde{\mathcal{H}}(M, w_2)$. The map $\mathcal{T}(M \times I, \partial) \to \widetilde{\mathcal{H}}(M, w_2)$ takes $F\colon (W, \partial W) \to (M \times I, \partial)$ to $(W, \widehat{F}) \in \widetilde{\mathcal{H}}(M, w_2)$, where $\widehat{F} = \widehat{p_1} \circ F$. Let $\sigma_5 \in L_5(\mathbb{Z}[\pi_1])$ be the image of $(W, \widehat{F})$, the map $\mathcal{T}(M \times I, \partial) \to L_5(\mathbb{Z}[\pi_1])$ is onto

[7, Lemma 6.9]. Let $(W', F') \in \mathcal{T}(M \times I, \partial)$ maps to $\sigma_5$ and let $(W', \widehat{F'}) \in \widetilde{\mathcal{H}}(M, w_2)$ be the image of $(W', F')$. Consider the difference of these elements in $\widetilde{\mathcal{H}}(M, w_2)$,

$$(W'', \widehat{F''}) := (W', \widehat{F'}) \bullet (-W, \hat{f}^{-1} \bullet \widehat{F}) \in \widetilde{\mathcal{H}}(M, w_2).$$

The element $(W'', \widehat{F''}) \in \widetilde{\mathcal{H}}(M, w_2)$ maps to $0 \in L_5(\mathbb{Z}[\pi_1])$. By the exactness of the right-hand vertical sequence there exists an $h$-cobordism $T$ of $M$ which maps to $(W'', \widehat{F''})$. Let $f$ denote the induced homotopy self equivalence of $M$. By construction we have $c \circ f \simeq \phi \circ c$ where $c \circ f = j \circ \widehat{f}$. Note that $\pi_2(\zeta^{-1} \circ f) = \psi \oplus \mathrm{id}$ and also $\zeta^{-1} \circ f$ gives us a self-equivalence of $M_{3/2}$. Now, if we put the $s$-cobordism $T$ in between two halves of $W$, then the 3-handles from the upper half cancel the 2-handles from the lower half. ∎

**Acknowledgments** The author would like to thank his supervisor Ian Hambleton for many helpful comments and Jonathan A. Hillman for taking the time to answer all his questions. The results in this paper are part of author's Ph. D. thesis [12].

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