SUMMABILITY OF THE HEINE AND NEUMANN SERIES OF LEGENDRE POLYNOMIALS

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With a holomorphic function $f(z)$ defined in a domain H which includes the closed interval $[-1, 1]$ we associate the Neumann series

$$(1) \qquad \sum_{n=0}^{\infty} a_n P_n(z), \qquad a_n = \frac{2n+1}{2\pi i} \oint_{\gamma} f(t) Q_n(t)\, dt,$$

where $P_n(z)$, $Q_n(t)$ are, respectively, the nth Legendre polynomials of the first and second kind and γ is a closed and rectifiable Jordan curve which includes $[-1, 1]$ in its interior and is included, together with its interior, in H.

It is known that the Neumann series (1) is convergent in the largest ellipse with foci ± 1 which does not include in its interior singular points of $f(z)$ (**3**, p. 322, §15.41). It is also known that the Heine series

$$(2) \qquad \sum_{n=0}^{\infty} (2n+1) P_n(z) Q_n(t)$$

is convergent to $(t-z)^{-1}$ for a fixed complex number t, not in $[-1, 1]$, for all z in the ellipse with foci ± 1 and passing through t (**3**, p. 321, §15.4).

In this paper we obtain some results on the domain in which the series (1) and (2) are summable by linear transformations. We use the notations, definitions, and conventions of (**1**).

THEOREM 1. *Let t be a complex number not in the closed interval* $[-1, 1]$. *Suppose that A, the infinite matrix $\|a_{nm}\|$ $(n, m = 0, 1, 2, \ldots)$, is efficient* (**1**, *Definition* 2.1) *in a certain domain D satisfying Conditions* (i), (ii), *and* (iii) *of* (**1**, *Definition* 1.1) *and that $D(t)$ is defined as the union of all convex domains that contain the disk $\{z\colon |z| < |t - (t^2-1)^{\frac{1}{2}}|^2\}$ and are included in D. Denote*

$$D_t^* = \bigcap_{0 \leqslant u < \infty} \tau(t, u) D(t)$$

and define

$$D_t \equiv \{\tfrac{1}{2}(w + w^{-1})\colon\ w \in D_t^* - \{z\colon |z| < 1\}\}.$$

Then the series (2) *is A-summable to $(t-z)^{-1}$, that is*

$$(3) \qquad \lim_{n\to\infty} \sum_{m=0}^{\infty} a_{nm} \sum_{r=0}^{m} (2r+1) P_r(z) Q_r(t) = (t-z)^{-1}$$

for all $z \in D_t$. For each closed and bounded set $H \subset D_t$, the series (2) *is uniformly A-summable to $(t-z)^{-1}$ in $z \in H$.*

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If D satisfied, in addition to the assumptions of Theorem 1, Condition (iv) of (**1**, Definition 1.1), then for each t not in $[-1, 1]$ we have $D(t) = D$. Thus (**1**, Theorem 1.1) is a special case of Theorem 1. R. E. Powell's Theorem 4.3 of the preceding paper (**2**) is also included in Theorem 1.

THEOREM 2. *Let $f(z)$ be a holomorphic function in some domain that includes the closed interval* $[-1, 1]$. *Suppose that A, the infinite matrix $\|a_{nm}\|$ $(n, m = 0, 1, 2, \ldots)$, is efficient in a certain domain D satisfying conditions* (i), (ii), *and* (iii) *of* (**1**, *Definition* 1.1). *Denote by $\Delta \equiv \Delta(f(z))$ the domain*

$$\Delta \equiv \bigcap_{t \in M^c} D_t.$$

If H is a closed and bounded set included in Δ, then the series (1) *is uniformly A-summable in $z \in H$ to $f(z)$; that is*

$$f(z) = \lim_{n \to \infty} \sum_{m=0}^{\infty} a_{nm} \sum_{k=0}^{m} a_k P_k(z) \qquad \textit{uniformly in } z \in H.$$

It is evident that (**1**, Theorem 2.1) is a special case of Theorem 2.

Proof of Theorem 1. If $H \subset D_t$ is a closed and bounded set, then by (**1**, Lemma 6.2) the set

$$G(H, t) \equiv \{\mu(z, \phi)/\tau(t, u) \colon z \in H, 0 \leqslant \phi \leqslant \pi, 0 \leqslant u \leqslant \infty\}$$

is also closed and bounded. The assumption $z \in D_t$ implies that

$$z + (z^2 - 1)^{\frac{1}{2}} \in D_t,$$

that is

$$z + (z^2 - 1)^{\frac{1}{2}} \in \bigcap_{0 \leqslant u < \infty} \tau(t, u) D(t)$$

or $(z + (z^2 - 1)^{\frac{1}{2}})/\tau(t, u) \in D(t)$ for $u \geqslant 0$. We show now that

(4) $\qquad \mu(z, \phi)/\tau(t, u) \in D(t) \quad$ for $0 \leqslant \phi \leqslant \pi$ and $0 \leqslant u < \infty$.

For a given $u \geqslant 0$ we divide the proof of (4) into three cases:

(i) If $|(z + (z^2 - 1)^{\frac{1}{2}})/\tau(t, u)| < 1$, then $|z + (z^2 - 1)^{\frac{1}{2}}| > 1$ and $|z - (z^2 - 1)^{\frac{1}{2}}| < 1$ imply that

$$|\mu(z, \phi)/\tau(t, u)| \leqslant |(z + (z^2 - 1)^{\frac{1}{2}})/\tau(t, u)| < 1.$$

Hence by assumption (ii) of Definition 1.1 and the fact that $D(t)$ includes the unit disk $\{z \colon |z| < 1\}$, we have (4) for $0 \leqslant \phi \leqslant \pi$ in this case.

(ii) Suppose that $|(z + (z^2 - 1)^{\frac{1}{2}})/\tau(t, u)| = 1$. From $|z - (z^2 - 1)^{\frac{1}{2}}| < 1$ and

$$|\tau(t, u)| \geqslant |\tau(t, 0)| \equiv |t + (t^2 - 1)^{\frac{1}{2}}| > 1,$$

we have $|(z - (z^2 - 1)^{\frac{1}{2}})/\tau(t, u)| < 1$. Now since $D(t)$ includes the unit disk, we obtain the same result.

(iii) If $|(z + (z^2 - 1)^{\frac{1}{2}})/\tau(t, u)| > 1$, then from $(z + (z^2 - 1)^{\frac{1}{2}})/\tau(t, u) \in D(t)$ and the definition of $D(t)$, there is a convex domain that includes

$$(z + (z^2 - 1)^{\frac{1}{2}})/\tau(t, u)$$

and the disk $\{w: |w| < |t - (t^2 - 1)^{\frac{1}{2}}|^2\}$ and is included in D. Now

$$|(z - (z^2 - 1)^{\frac{1}{2}})/\tau(t, u)| = |\tau(t, u)(z - (z^2 - 1)^{\frac{1}{2}})|/|\tau(t, u)^2| < |\tau(t, 0)|^{-2} = |t - (t^2 - 1)^{\frac{1}{2}}|^2.$$

Thus this convex domain includes the segment with end points

$$(z + (z^2 - 1)^{\frac{1}{2}})/\tau(t, u)$$

and $(z - (z^2 - 1)^{\frac{1}{2}})/\tau(t, u)$. This proves again (4). Thus we proved that $G(H, t) \subset D(t) \subset D$. Now by **(1**, Lemma 7.2**)**, (3) is valid uniformly in $z \in H$.

Proof of Theorem 2. The proof is the same as the proof of Theorem 2.1 in **(1)**, but for the following three obvious changes: we now use the argument of the proof of Theorem 1 instead of the argument of the proof of **(1**, Theorem 1.1**)**; D is replaced by $D(t)$ up to (8.3) in **(1)** and **(**1, (8.3)**)** is replaced by

$$F \subset \Delta^* = \bigcap_{t \in M^c} D_t^* = \bigcap_{t \in M^c} \bigcap_{0 \leqslant u < \infty} \tau(t, u) D(t) \subset \bigcap_{t \in M^c} \bigcap_{0 \leqslant u < \infty} \tau(t, u) D = \bigcap_{t \in \tilde{Z}^c} tD.$$

References

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