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NECESSARY CONDITIONS FOR OPTIMAL CONTROL OF ELLIPTIC SYSTEMS

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Abstract

In this paper, we consider the system governed *via* the coefficients of a semilinear elliptic equation and give the necessary conditions for optimal control. Furthermore, we obtain the necessary conditions for an optimal domain in a domain optimization problem.

1. Formulation of the problem

In this paper, we consider the system governed *via* the divergence component of a semilinear elliptic equation. Standard results of optimal control problems for systems governed by elliptic equations with distributed control can be found in [2, 3, 5, 10, 13, 15, 16]. Casas considers the system governed *via* the coefficients of a linear elliptic equation in [4] and gives the necessary conditions of optimal control by using convex analysis under the supposition that the phase spaces of both the control and the cost functional are convex. In this paper, the phase space of both the control and the cost functional may not be convex. We will give the necessary conditions of optimal control by using the convexification method and Ekeland's variational principle.

We first consider the following problem: there are two kinds of materials $\mathscr{A}$ and $\mathscr{B}$ (for example, material $\mathscr{A}$ may be oil and material $\mathscr{B}$ may be water). Let the temperatures of $\mathscr{A}$ and $\mathscr{B}$ be given by $y_{\mathscr{A}}$ and $y_{\mathscr{B}}$ respectively. The quantities $\Omega_{\mathscr{A}}$ and $\Omega_{\mathscr{B}}$ represent the domains occupied by $\mathscr{A}$ and $\mathscr{B}$. Assume $\Gamma_{\mathscr{A}}$ (the boundary of $\Omega_{\mathscr{A}}$) and $\Gamma_{\mathscr{B}}$ ((the boundary of $\Omega_{\mathscr{B}}$)$\backslash\Gamma_{\mathscr{A}}$) are smooth (see Figure 1). Then $y_{\mathscr{A}}$ and

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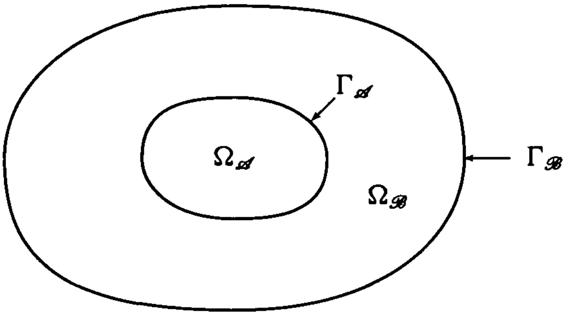

FIGURE 1.

$y_{\mathscr{B}}$ satisfy the equations

$$\begin{cases} -a\Delta y_{\mathscr{A}}(x) = f(x), & x \in \Omega_{\mathscr{A}}, \\ -b\Delta y_{\mathscr{B}}(x) = f(x), & x \in \Omega_{\mathscr{B}}, \\ y_{\mathscr{B}}\,|_{\Gamma_{\mathscr{B}}} = 0, & \\ a\dfrac{\partial}{\partial n} y_{\mathscr{A}}\,|_{\Gamma_{\mathscr{A}}} = b\dfrac{\partial}{\partial n} y_{\mathscr{B}}\,|_{\Gamma_{\mathscr{A}}}, & y_{\mathscr{A}}\,|_{\Gamma_{\mathscr{A}}} = y_{\mathscr{B}}\,|_{\Gamma_{\mathscr{A}}}, \end{cases} \tag{1.1}$$

where $b > a > 0$. We know that there exists a unique classical solution $y_{\mathscr{A}}$ and $y_{\mathscr{B}}$ for Problem (1.1) if $f(\cdot) \in C^{0,\alpha}(G)$ for $0 < \alpha < 1$. We now give a cost functional

$$F(\Omega_{\mathscr{A}}) = \int_{\Omega_{\mathscr{A}}} f^0(x, y_{\mathscr{A}}(x), \nabla y_{\mathscr{A}}(x))\,dx + \int_{\Omega_{\mathscr{B}}} f^0(x, y_{\mathscr{B}}(x), \nabla y_{\mathscr{B}}(x))\,dx \tag{1.2}$$

and a set

$$\Pi = \{\Omega_{\mathscr{A}} \mid \Omega_{\mathscr{A}} \subset G, |\Omega_{\mathscr{A}}| = 1, \Gamma_{\mathscr{A}} \text{ is smooth}\}, \tag{1.3}$$

where $|E| = \text{meas}\, E$, G is a fixed domain and $G = \Omega_{\mathscr{A}} \cup \Omega_{\mathscr{B}}$.

We can raise the following domain optimization problem.

PROBLEM D. Find a domain $\tilde{\Omega}_{\mathscr{A}} \in \Pi$, such that

$$F(\tilde{\Omega}_{\mathscr{A}}) = \inf\{F(\Omega_{\mathscr{A}}) \mid \Omega_{\mathscr{A}} \in \Pi\}. \tag{1.4}$$

If there exists a domain $\tilde{\Omega}_{\mathscr{A}}$ such that (1.4) holds, we say that the domain $\tilde{\Omega}_{\mathscr{A}}$ is an optimal domain.

We now introduce a function

$$u(x) = a\chi_{\Omega_{\mathscr{A}}}(x) + b\chi_{\Omega_{\mathscr{B}}}(x). \tag{1.5}$$

Problem (1.1) can then be written as

$$\begin{cases} -\operatorname{div}(u(x)\nabla y_u(x)) = f(x), & x \in G, \\ y_u|_{\Gamma_G} = 0, \end{cases} \tag{1.6}$$

that is,

$$y_u(x) = \begin{cases} y_{\mathscr{A}}(x), & x \in \Omega_{\mathscr{A}}, \\ y_{\mathscr{B}}(x), & x \in \Omega_{\mathscr{B}}. \end{cases} \tag{1.7}$$

The cost functional (1.2) becomes

$$J(u) = \int_G f^0(x, y_u(x), \nabla y_u(x))\, dx. \tag{1.8}$$

Let

$$\mathscr{W} = \{u(x) = a\chi_{\Omega_{\mathscr{A}}}(x) + b\chi_{\Omega_{\mathscr{B}}}(x) \mid \Omega_{\mathscr{A}} \in \Pi\}. \tag{1.9}$$

We can now raise the optimal control problem corresponding to Problem D.

PROBLEM C. Find a control $\bar{u} \in \mathscr{W}$, such that

$$J(\bar{u}) = \inf\{J(u) \mid u \in \mathscr{W}\}. \tag{1.10}$$

In this practical problem, the control variable is involved in the coefficient, the admissible control set $\mathscr{W}$ is not convex and the cost functional (1.8) may not be convex.

In this paper, we will discuss a more general system:

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, u(x))\frac{\partial}{\partial x_j} y(x)\right) = f(x, y(x), u(x)), \\ y|_{\Gamma_\Omega} = 0, \end{cases} \tag{1.11}$$

where Ω ($\subset R^n$) is a bounded domain with smooth boundary Γ_Ω, $u(x) \in U$ is a control function, $U \subset R^m$ is a bounded closed set and $\operatorname{co} U$ stands for the convex hull of the U.

We denote the set of all admissible controls by $\mathscr{U}_{\text{ad}}$, that is,

$$\mathscr{U}_{\text{ad}} = \{u(x) \in U \mid u(\cdot) \text{ is measurable on } \Omega\}. \tag{1.12}$$

If for any $u(\cdot) \in \mathscr{U}_{\text{ad}}$, $y(x) = y(x; u)$ is a solution of Problem (1.11), we can define the cost functional

$$J(u) = \int_\Omega f^0(x, y(x; u), \nabla y(x; u), u(x))\, dx. \tag{1.13}$$

Our optimal control problem can be stated as follows.

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PROBLEM E. Find a $\bar{u}(\cdot) \in \mathscr{U}_{\text{ad}}$, such that

$$J(\bar{u}(\cdot)) = \inf\{J(u(\cdot)) \mid u(\cdot) \in \mathscr{U}_{\text{ad}}\}, \tag{1.14}$$

where $J(u(\cdot))$ is given by (1.13).

Any admissible control $\bar{u}(\cdot)$ satisfying (1.14) is called an optimal control for Problem E; the corresponding state $\bar{y}(\cdot)$ is called an optimal state and the pair $(\bar{y}(\cdot), \bar{u}(\cdot))$ is referred to as an optimal pair.

2. Variation of convexification problems

Let us assume that

(P1) $a_{ij} : \Omega \times \operatorname{co} U \to R$ satisfies the following conditions.

(1) The quantity $a_{ij}(\cdot, u)$ is bounded measurable on Ω and $a_{ij}(x, \cdot)$ is Lipschitz continuous of rank K on co U.

(2) There exists a constant $\lambda > 0$, for any $(x, u) \in \Omega \times \operatorname{co} U$, such that

$$\sum_{i,j=1}^{n} a_{ij}(x, u)\eta_i\eta_j \geq \lambda|\eta|^2, \quad \forall \eta \in R^n.$$

We shall also assume that

(P2) $f : \Omega \times R \times \operatorname{co} U \to R$ is such that $f(\cdot, y, u)$ is measurable on Ω and that $f(x, \cdot, \cdot)$ and $f_y(x, \cdot, \cdot)$ are Lipschitz continuous of rank K on $R \times \operatorname{co} U$. There exists a constant $L > 0$, such that

$$-L \leq f_y(x, y, u) \leq 0, \quad \forall (x, y, u) \in \Omega \times R \times \operatorname{co} U,$$

where $f^0(\cdot, y, \zeta, u)$ is measurable on Ω and $f^0(x, \cdot, \cdot, \cdot)$, $f_y^0(x, \cdot, \cdot, \cdot)$ and $f_{\zeta_i}^0(x, \cdot, \cdot, \cdot)$ are Lipschitz continuous of rank K on $R \times R^n \times \operatorname{co} U$.

For any $M > 0$, there exists a function $F_M(\cdot) \in L^2(\Omega)$, such that

$$|f(x, y, u)| + |f_y(x, y, u)| + |f^0(x, y, \zeta, u)| + |f_y^0(x, y, \zeta, u)| \leq |F_M(x)|,$$
$$|y| \leq M, \quad \forall u \in \operatorname{co} U, \quad \forall \zeta \in R^n.$$

In deriving necessary conditions for optimal control, one needs to make certain perturbations for the control and the corresponding variations of the state and the cost functional need to be determined.

We first introduce a new control set

$$\mathscr{U} = \{u(x) \mid u(\cdot) : \Omega \to \operatorname{co} U \text{ is measurable}\}. \tag{2.1}$$

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DEFINITION 2.1. A function $y(\cdot)$ is called a generalized solution of Problem (1.11) if $y(\cdot) \in H_0^1(\Omega)$ and for any $\varphi \in H_0^1(\Omega)$ the following equality holds:

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x, u(x)) \frac{\partial}{\partial x_j} y(x) \frac{\partial}{\partial x_i} \varphi(x)\, dx = \int_\Omega f(x, y(x), u(x)) \varphi(x)\, dx. \tag{2.2}$$

We have the following lemmas.

LEMMA 2.1. *Let* (P1)–(P2) *hold. For any $u \in \mathcal{U}$, there exists a unique generalized solution $y(x) = y(x; u) \in H_0^1(\Omega)$ for Problem* (1.11), *and there exists a constant C being independent of u, such that*

$$\|y\|_{H^1(\Omega)} \le C. \tag{2.3}$$

PROOF. Step 1. We have

$$|f(x, y(x), u(x))| \le c|y(x)| + c_1(x).$$

In fact,

$$\begin{aligned} f(x, y(x), u(x)) &= f(x, y(x), u(x)) - f(x, 0, u(x)) + f(x, 0, u(x)) \\ &= \int_\Omega f_y(x, \tau y(x), u(x))\, d\tau y(x) + f(x, 0, u(x)). \end{aligned}$$

From this equality and condition (P2), we know that the inequality of Step 1 is true.
Step 2. For any $y \in H_0^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega \sum_{i,j=1}^n \bigg\{ & a_{ij}(x, u(x)) \frac{\partial}{\partial x_j} y(x) \frac{\partial}{\partial x_i} y(x)\, dx \\ & - f(x, y(x), u(x)) y(x) \bigg\} dx \ge c_2 \|y\|_{H^1(\Omega)} - c_3. \end{aligned}$$

In fact, for any $y \in H_0^1(\Omega)$,

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^n \left\{ a_{ij}(x, u(x)) \frac{\partial}{\partial x_j} y(x) \frac{\partial}{\partial x_i} y(x)\, dx - f(x, y(x), u(x)) y(x) \right\} dx \\ &\quad = \int_\Omega \sum_{i,j=1}^n \bigg\{ a_{ij}(x, u(x)) \frac{\partial}{\partial x_j} y(x) \frac{\partial}{\partial x_i} y(x)\, dx \\ &\qquad - \int_0^1 f_y(x, \tau y(x), u(x))\, d\tau y^2(x) - f(x, 0, u(x)) y(x) \bigg\} dx \\ &\quad \ge \lambda \|\nabla y\|_{L^2(\Omega)} - \varepsilon \|y\|_{L^2(\Omega)} - C(\varepsilon) \|f(\cdot, 0, u)\|_{L^2(\Omega)}. \end{aligned}$$

Since $\|\nabla y\|_{L^2(\Omega)} \geq C\|y\|_{L^2(\Omega)}$ for any $y \in H_0^1(\Omega)$, we can choose ε small enough so that

$$\lambda\|\nabla y\|_{L^2(\Omega)} - \varepsilon\|y\|_{L^2(\Omega)} \geq \frac{1}{2}\lambda\|\nabla y\|_{L^2(\Omega)} \geq c_2\|y\|_{H_0^1(\Omega)}.$$

Therefore Step 2. holds.

Secondly, from condition (P2), it is easy to see that

$$\int_\Omega \{f(x, y_1(x), u(x)) - f(x, y(x), u(x))y(x)\}\{y_1(x) - y_2(x)\}\,dx \leq 0.$$

According to Theorem 9.1 of Chapter 4 in [9], we know that there exists a function $y \in H_0^1(\Omega)$ which satisfies (2.2).

We can prove the uniqueness of this solution. In fact, suppose $y_1(x)$ and $y_2(x)$ are the generalized solutions of Problem (1.11). We then have

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x, u(x))\frac{\partial}{\partial x_j}(y_1(x) - y_2(x))\frac{\partial}{\partial x_i}\varphi(x)\,dx$$
$$= \int_\Omega [f(x, y_1(x), u(x)) - f(x, y_2(x), u(x))]\varphi(x)\,dx.$$

In particular, let $\varphi(x) = y_1(x) - y_2(x)$. We now have

$$\begin{aligned} 0 &\leq \lambda\|\nabla(y_1 - y_2)\|_{L^2(\Omega)} \\ &\leq \int_\Omega \sum_{i,j=1}^n a_{ij}(x, u(x))\frac{\partial}{\partial x_j}(y_1(x) - y_2(x))\frac{\partial}{\partial x_i}(y_1(x) - y_2(x))\,dx \\ &= \int_\Omega \int_0^1 f_y(x, y_2(x) + \tau(y_1(x) - y_2(x)), u(x))\,d\tau\{y_1(x) - y_2(x)\}^2\,dx \leq 0. \end{aligned}$$

Therefore $\|\nabla(y_1 - y_2)\|_{L^2(\Omega)} = 0$. That is, $\|y_1 - y_2\|_{H^1(\Omega)} = 0$, so $y_1(x) = y_2(x)$.

Using conditions (P1)–(P2), we have

$$\begin{aligned} \lambda\int_\Omega |\nabla y(x)|^2 dx &\leq \int_\Omega \sum_{i,j=1}^n a_{ij}(x, u(x))\frac{\partial}{\partial x_j}y(x)\frac{\partial}{\partial x_i}y(x)\,dx \\ &= \int_\Omega f(x, y(x), u(x))y(x)\,dx \\ &= \int_\Omega f(x, 0, u(x))y(x)\,dx + \int_\Omega\int_0^1 f_y(x, \tau y(x), u(x))\,d\tau y^2(x)\,dx \\ &\leq C\|f(\cdot, 0, u(\cdot))\|_{L^2(\Omega)}\|y\|_{L^2(\Omega)}. \end{aligned}$$

From Poincaré's inequality, we have

$$\lambda\|\nabla y(\cdot)\|^2_{L^2(\Omega)} \leq C\|F(\cdot)\|_{L^2(\Omega)}\|\nabla y(\cdot)\|_{L^2(\Omega)},$$

so the inequality (2.3) holds.

Let $\bar{u}, v \in \mathscr{U}$, $\bar{y}(\cdot) = y(\cdot, \bar{u})$ and $y_v(\cdot) = y(\cdot, v)$ be the generalized solutions of Problem (1.11) corresponding to $\bar{u}(\cdot)$ and $v(\cdot)$, respectively.

LEMMA 2.2. *Let* (P1)–(P2) *hold. We have the following estimate*

$$\|\bar{y}(\cdot) - y_v(\cdot)\|_{H^1(\Omega)} \le C\|\bar{u} - v\|_{L^\infty(\Omega)}, \tag{2.4}$$

where the constant C is independent of $\bar{u}(\cdot)$ and $v(\cdot)$.

PROOF. From (1.11), we have

$$\begin{aligned} &-\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x, v(x))\frac{\partial}{\partial x_j}(y_v(x) - \bar{y}(x))\right) \qquad (2.5)\\ &\quad= \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\big(a_{ij}(x, v(x)) - a_{ij}(x, \bar{u}(x))\big)\frac{\partial}{\partial x_j}\bar{y}(x)\\ &\qquad + [f(x, y_v(x), v(x)) - f(x, \bar{y}(x), \bar{u}(x))]\\ &\quad= \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\big(a_{ij}(x, v(x)) - a_{ij}(x, \bar{u}(x))\big)\frac{\partial}{\partial x_j}\bar{y}(x)\\ &\qquad + \int_0^1 f_y(x, \bar{y}(x) + \tau(y_v(x) - \bar{y}(x)), v(x))\,d\tau\,(y_v(x) - \bar{y}(x))\\ &\qquad + [f(x, \bar{y}(x), v(x)) - f(x, \bar{y}(x), \bar{u}(x))]. \end{aligned}$$

Multiplying (2.5) by $y_v(x) - \bar{y}(x)$ and integrating the resulting relation over Ω, we have

$$\begin{aligned} &\lambda\|\nabla(y_v - \bar{y})\|^2_{L^2(\Omega)}\\ &\quad\le \int_\Omega \sum_{i,j=1}^n [a_{ij}(x, \bar{u}(x)) - a_{ij}(x, v(x))]\frac{\partial}{\partial x_j}\bar{y}(x)\frac{\partial}{\partial x_i}(y_v(x) - \bar{y}(x))\,dx\\ &\qquad + \int_\Omega\int_0^1 f_y(x, \bar{y}(x) + \tau(y_v(x) - \bar{y}(x)), v(x))\,d\tau\,(y_v(x) - \bar{y}(x))^2\,dx\\ &\qquad + \int_\Omega [f(x, \bar{y}(x), v(x)) - f(x, \bar{y}(x), \bar{u}(x))](y_v(x) - \bar{y}(x))\,dx\\ &\quad\le K\|\bar{u} - v\|_{L^\infty(\Omega)}\|\nabla\bar{y}\|_{L^2(\Omega)}\|\nabla(y_v - \bar{y})\|_{L^2(\Omega)}\\ &\qquad + K\|\bar{u} - v\|_{L^\infty(\Omega)}\|y_v - \bar{y}\|_{L^2(\Omega)}\\ &\quad\le C\|\bar{u} - v\|_{L^\infty(\Omega)}\|\nabla(y_v - \bar{y})\|_{L^2(\Omega)}. \end{aligned}$$

We thus obtain the result of Lemma 2.2.

Let $z(\cdot) \in H_0^1(\Omega)$ satisfy the following equation

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,\bar{u}(x))\frac{\partial}{\partial x_j}z(x)\right) \\ \quad = f_y(x,\bar{y}(x),\bar{u}(x))\,z(x) \\ \qquad + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,v(x)) - a_{ij}(x,\bar{u}(x)))\frac{\partial}{\partial x_j}\bar{y}(x) \\ \qquad + [f(x,\bar{y}(x),v(x)) - f(x,\bar{y}(x),\bar{u}(x))], \\ z\,|_{\Gamma_\Omega} = 0 \end{cases} \tag{2.6}$$

and let

$$z^0 = \int_\Omega \left\{ f_y^0(x,\bar{y}(x),\nabla\bar{y}(x),\bar{u}(x))z(x) + \sum_{i=1}^{n} f_{\zeta_i}^0(x,\bar{y}(x),\nabla\bar{y}(x),\bar{u}(x))\frac{\partial}{\partial x_i}z(x) \right.$$
$$\left. + f^0(x,\bar{y}(x),\nabla\bar{y}(x),v(x)) - f^0(x,\bar{y}(x),\nabla\bar{y}(x),\bar{u}(x)) \right\} dx. \tag{2.7}$$

REMARK 2.1. We see that the solutions $z(\cdot)$ of (2.6) and z^0 defined by (2.7) are dependent on the choice of $v, \bar{u} \in \mathscr{U}$. If $\bar{u}$ is fixed, then we can denote $z(\cdot) = z(\cdot, v)$ and $z^0 = z^0(v)$. Multiplying (2.6) by $z(x)$ and integrating the resulting relation over Ω, we have

$$\begin{aligned} \lambda\|\nabla z\|_{L^2(\Omega)}^2 &\le K\|\bar{u}-v\|_{L^\infty(\Omega)}\|\nabla\bar{y}\|_{L^2(\Omega)}\|\nabla z\|_{L^2(\Omega)} + K\|\bar{u}-v\|_{L^\infty(\Omega)}\|z\|_{L^2(\Omega)} \\ &\le C\|\bar{u}-v\|_{L^\infty(\Omega)}\|\nabla z\|_{L^2(\Omega)}. \end{aligned}$$

Thus, we have

$$\|z\|_{H^1(\Omega)} \le C\|\bar{u}-v\|_{L^\infty(\Omega)}. \tag{2.8}$$

THEOREM 2.1. *Suppose that conditions* (P1)–(P2) *hold. Then*

$$y_v(x) = \bar{y}(x) + z(x) + r(x), \tag{2.9}$$
$$J(v) = J(\bar{u}) + z^0 + r^0 \tag{2.10}$$

and

$$\|r\|_{H^1(\Omega)} = o\left(\|v-\bar{u}\|_{L^\infty(\Omega)}\right), \qquad |r^0| = o\left(\|v-\bar{u}\|_{L^\infty(\Omega)}\right). \tag{2.11}$$

PROOF. From (2.5) and (2.6), we obtain

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, v(x))\frac{\partial}{\partial x_j}(y_v(x) - \bar{y}(x) - z(x))\right) \\ \quad = \sum_{i,j=1}^{n}\left[\frac{\partial}{\partial x_i}(a_{ij}(x, v(x)) - a_{ij}(x, \bar{u}(x)))\frac{\partial}{\partial x_j}z(x)\right] \\ \qquad + \int_0^1 f_y(x, \bar{y}(x) + \tau(y_v(x) - \bar{y}(x)), v(x))d\tau(y_v(x) - \bar{y}(x) - z(x)) \\ \qquad + \int_0^1 [f_y(x, \bar{y}(x) + \tau(y_v(x) - \bar{y}(x)), v(x)) - f_y(x, \bar{y}(x), \bar{u}(x))]\, d\tau z(x), \\ (y_v - \bar{y} - z)\,|_{\Gamma_\Omega} = 0. \end{cases} \tag{2.12}$$

Multiplying (2.12) by $y_v(x) - \bar{y}(x) - z(x)$ and integrating the resulting relation over Ω, we have

$$\begin{aligned} &\lambda\|\nabla(y_v - \bar{y} - z)\|_{L^2(\Omega)}^2 \\ &\quad \le \int_\Omega \sum_{i,j=1}^{n}[a_{ij}(x, \bar{u}(x)) - a_{ij}(x, v(x))]\frac{\partial}{\partial x_j}z(x)\frac{\partial}{\partial x_i}(y_v(x) - \bar{y}(x) - z(x))\, dx \\ &\qquad + \int_\Omega\int_0^1 f_y(x, \bar{y}(x) + \tau(y_v(x) - \bar{y}(x)), v(x))d\tau(y_v(x) - \bar{y}(x) - z(x))^2\, dx \\ &\qquad + \int_\Omega\int_0^1 [f_y(x, \bar{y}(x) + \tau(y_v(x) - \bar{y}(x)), v(x)) - f_y(x, \bar{y}(x), \bar{u}(x))]\, d\tau \\ &\qquad \times z(x)(y_v(x) - \bar{y}(x) - z(x))\, dx \\ &\quad \le K\|\bar{u} - v\|_{L^\infty(\Omega)}\|\nabla z\|_{L^2(\Omega)}\|\nabla(y_v - \bar{y} - z)\|_{L^2(\Omega)} \\ &\qquad + K\|\bar{u} - v\|_{L^\infty(\Omega)}\|z\|_{L^2(\Omega)}\|y_v - \bar{y} - z\|_{L^2(\Omega)} \\ &\quad \le C\|\bar{u} - v\|_{L^\infty(\Omega)}^2\|\nabla(y_v - \bar{y} - z)\|_{L^2(\Omega)}. \end{aligned}$$

From this inequality, we obtain

$$\|r\|_{H^1(\Omega)} = \|y_v - \bar{y} - z\|_{H^1(\Omega)} = o(\|v - \bar{u}\|_{L^\infty(\Omega)}).$$

Furthermore, we may calculate directly

$$\begin{aligned} &J(v) - J(\bar{u}) \\ &\quad = \int_\Omega [f^0(x, y_v(x), \nabla y_v(x), v(x)) - f^0(x, \bar{y}(x), \nabla\bar{y}(x), \bar{u}(x))]\, dx \\ &\quad = \int_\Omega [f^0(x, y_v(x), \nabla y_v(x), v(x)) - f^0(x, \bar{y}(x), \nabla y_v(x), v(x))]\, dx \\ &\qquad + \int_\Omega [f^0(x, \bar{y}(x), \nabla y_v(x), v(x)) - f^0(x, \bar{y}(x), \nabla\bar{y}(x), v(x))]\, dx \end{aligned}$$

$$+\int_\Omega [f^0(x,\bar y(x),\nabla\bar y(x),v(x)) - f^0(x,\bar y(x),\nabla\bar y(x),\bar u(x))]\,dx$$

$$= \int_\Omega\int_0^1 f_y^0(x,\bar y(x)+\tau(y_v(x)-\bar y(x)),\nabla y_v(x),v(x))\,d\tau(y_v(x)-\bar y(x))\,dx$$

$$+\int_\Omega\sum_{i=1}^n\int_0^1 f^0_{\zeta_i}\Big(x,\bar y(x),\frac{\partial}{\partial x_1}\bar y(x),\dots,\frac{\partial}{\partial x_i}\bar y(x)$$

$$+\tau\,\frac{\partial}{\partial x_i}(y_v(x)-\bar y(x)),\dots,\frac{\partial}{\partial x_n}y_v(x),v(x)\Big)\,d\tau\frac{\partial}{\partial x_i}(y_v(x)-\bar y(x))\,dx$$

$$+\int_\Omega \big[f^0(x,\bar y(x),\nabla\bar y(x),v(x)) - f^0(x,\bar y(x),\nabla\bar y(x),\bar u(x))\big]\,dx.$$

From (2.9), we have

$$J(v)-J(\bar u)$$

$$= \int_\Omega\int_0^1 f_y^0(x,\bar y(x)+\tau(y_v(x)-\bar y(x)),\nabla y_v(x),v(x))\,d\tau z(x)\,dx$$

$$+\int_\Omega\sum_{i=1}^n\int_0^1 f^0_{\zeta_i}\Big(x,\bar y(x),\frac{\partial}{\partial x_1}\bar y(x),\dots,\frac{\partial}{\partial x_i}\bar y(x)$$

$$+\tau\,\frac{\partial}{\partial x_i}(y_v(x)-\bar y(x)),\dots,\frac{\partial}{\partial x_n}y_v(x),v(x)\Big)\,d\tau\frac{\partial}{\partial x_i}z(x)\,dx$$

$$+\int_\Omega\big[f^0(x,\bar y(x),\nabla\bar y(x),v(x)) - f^0(x,\bar y(x),\nabla\bar y(x),\bar u(x))\big]\,dx + o\big(\|v-\bar u\|_{L^\infty(\Omega)}\big)$$

$$= z^0 + \int_\Omega\int_0^1 [f_y^0(x,\bar y(x)+\tau(y_v(x)-\bar y(x)),\nabla y_v(x),v(x))$$

$$- f_y^0(x,\bar y(x),\nabla\bar y(x),\bar u(x))]d\tau z(x)\,dx$$

$$+\int_\Omega\sum_{i=1}^n\int_0^1\Big[f^0_{\zeta_i}\Big(x,\bar y(x),\frac{\partial}{\partial x_1}\bar y(x),\dots,\frac{\partial}{\partial x_i}\bar y(x)$$

$$+\tau\,\frac{\partial}{\partial x_i}(y_v(x)-\bar y(x)),\dots,\frac{\partial}{\partial x_n}y_v(x),v(x)\Big) - f^0_{\zeta_i}(x,\bar y(x),\nabla\bar y(x),\bar u(x))\Big]\,d\tau$$

$$\times\frac{\partial}{\partial x_i}z(x)\,dx + o\big(\|v-\bar u\|_{L^\infty(\Omega)}\big) = z^0 + o\big(\|v-\bar u\|_{L^\infty(\Omega)}\big).$$

Thus Theorem 2.1 is proved.

3. The case of U being the endpoints set of a cuboid

In this section, we suppose that U is a set of the end points of a cuboid and discuss the necessary conditions for the optimal control problem with a state constraint.

Now, suppose $Q \subset H_0^1(\Omega)$. For any $u(\cdot) \in \mathscr{U}_{ad}$, there exists a unique function $y_u(x) = y(x; u) \in H_0^1(\Omega)$, which is the generalized solution of (1.11). We can therefore talk about the state constraint

$$y_u(\cdot) \in Q.$$

We let

$$\mathscr{U}_{ad}^Q = \{u \in \mathscr{U}_{ad} \mid y(\cdot, u) \in Q\}.$$

Our optimal control problem can be stated as follows.

PROBLEM E_Q. Find a $\bar{u}(\cdot) \in \mathscr{U}_{ad}^Q$, such that

$$J(\bar{u}(\cdot)) = \inf\left\{J(u(\cdot)) \mid u(\cdot) \in \mathscr{U}_{ad}^Q\right\}, \tag{3.1}$$

where $J(u(\cdot))$ is given by (1.13).

Any admissible control $\bar{u}(\cdot)$ satisfying (3.1) is called an optimal control for Problem E_Q, the corresponding state $\bar{y}(\cdot)$ is called an optimal state and the pair $(\bar{y}(\cdot), \bar{u}(\cdot))$ is referred to as an optimal pair.

It is clear that Problem E is a special case of Problem E_Q.

We first raise the question: Suppose $\bar{u}(x)$ is an optimal control of Problem E and $u_d(x)$ is an optimal control of Problems (1.11), (1.13) and (2.1). Then is it true that $\bar{u}(x) = u_d(x)$?

EXAMPLE 3.1. We consider the system given by

$$\begin{cases} -\operatorname{div}(u(x)\nabla y(x)) = f(x), & x \in \Omega, \\ y_u \mid_{\Gamma_\Omega} = 0. \end{cases} \tag{3.2}$$

We set

$$\mathscr{U}_{ad} = \{u(x) \in U = \{a, b\} \mid u(\cdot) \text{ is measurable on } \Omega\}$$

and

$$J(u) = \int_\Omega \left\{(u(x) - \bar{u}(x))^2 (y_u(x) - y_d(x))^2 + (u(x) - u_d(x))^2\right\} dx,$$

where $0 < a < b$ and

$$u_d(x) = \begin{cases} a + (1/4)(b-a), & x \in E_1, \\ a + (3/4)(b-a), & x \in E_2. \end{cases}$$

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We have that $E_1 \cup E_2 = \Omega$, $E_1 \cap E_2 = \emptyset$, $y_d(x)$ is a solution of Problem (3.2) corresponding to u_d and

$$\bar{u}(x) = a\chi_{E_1}(x) + b\chi_{E_2}(x).$$

It is clear that

$$J(\bar{u}(\cdot)) = \inf\{J(u(\cdot)) \mid u(\cdot) \in \mathcal{U}_{\text{ad}}\}.$$

Let

$$\mathcal{U} = \{u(x) \in [a, b] \mid u(\cdot) \text{ is measurable on } \Omega\},$$

then

$$J(u_d(\cdot)) = \inf\{J(u(\cdot)) \mid u(\cdot) \in \mathcal{U}\}.$$

We thus have $\bar{u}(x) \neq u_d(x)$.

This example indicates that the optimal control of Problem E is not equal to the optimal control of the convexification problem for Problem E. It therefore implies that we can't use the convexification method alone to solve both Problems E and E_Q.

We shall now discuss Problem E_Q. Suppose that

(P3) Q is a closed and convex subset of $H_0^1(\Omega)$.

Let

$$d_Q(y) = \min\{\|y - q\|_{H^1(\Omega)} | q \in Q\} \quad \text{and} \quad d_U(u(x)) = \min_{v \in U} |v - u(x)|,$$

where $|u| = \left\{ \sum_{i=1}^n u_i^2 \right\}^{1/2}$ and $u = \{u_1, \dots, u_n\}$.

DEFINITION 3.1. Let Z be a Banach space. A set S is said to be finite codimensional in Z if there exists a point $z \in S$ such that $Z_0 \triangleq \text{span}(S - z)$ is a finite codimensional subspace of Z and $\overline{\text{co}}(S - z)$ has a nonempty interior in Z_0.

We have the following result.

THEOREM 3.1. *Let* (P1)–(P3) *hold,* $\bar{u}(\cdot) \in \mathcal{U}_{ad}^Q$ *be an optimal control of Problem* E_Q, $\bar{y}(\cdot) = y(\cdot, \bar{u})$ *be an optimal state and* Q *be finite codimensional in* $H_0^1(\Omega)$. *Then there exist* $\psi^0 \in [-1, 0]$, $\psi^1 \in [-1, 0]$, $\xi(x) = (\xi^1(x), \dots, \xi^m(x))$, $\varphi \in H^{-1}$, $\psi \in H_0^1(\Omega)$, *such that*

$$(\psi^0, \psi^1, \varphi) \neq 0, \tag{3.3}$$

$$\langle \varphi, q(\cdot) - \bar{y}(\cdot) \rangle \leq 0, \quad \forall q \in Q \tag{3.4}$$

and $\psi(\cdot)$ satisfies the following equation

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,\bar{u}(x))\frac{\partial}{\partial x_j}\psi(x)\right) \\ \quad = f_y(x,\bar{y}(x),\bar{u}(x))\psi(x) + \psi^0\Big[f_y^0(x,\bar{y}(x),\nabla\bar{y}(x),\bar{u}(x)) \\ \qquad - \sum_{i=1}^{n}\frac{\partial}{\partial x_i} f_{\xi_i}^0(x,\bar{y}(x),\nabla\bar{y}(x),\bar{u}(x))\Big] - \varphi, \\ \psi\mid_{\Gamma_\Omega} = 0 \end{cases} \tag{3.5}$$

and the maximum condition

$$K(x)\cdot\bar{u}(x) = \max_{v\in U} K(x)\cdot v, \quad a.e. \quad x\in\Omega \tag{3.6}$$

holds, where

$$\begin{aligned} H(x,u) = &-\sum_{i,j=1}^{n} a_{ij}(x,u)\frac{\partial}{\partial x_i}\bar{y}(x)\frac{\partial}{\partial x_j}\psi(x) \\ &+\psi(x)f(x,\bar{y}(x),u) + \psi^0 f^0(x,\bar{y}(x),\nabla\bar{y}(x),u), \end{aligned} \tag{3.7}$$

$$N(x)\in\partial_u H(x,\bar{u}(x)) \quad and \tag{3.8}$$

$$K(x) = N(x) + \psi^1\xi(x). \tag{3.9}$$

Here $\xi^i(x) = 1$ as $\bar{u}^i(x) = a^i$ and $\xi^i(x) = -1$ as $\bar{u}^i(x) = b^i$, in which $[a^i, b^i] =$ $\text{Proj}_{X_i}\,\text{co}\,U$, *$\bar{u}(x) = (\bar{u}^1(x), \dots, \bar{u}^m(x))$.*

PROOF. Now, for any $u(\cdot), v(\cdot)\in\mathscr{U}$, we define $d(u(\cdot), v(\cdot))$ by

$$d(u(\cdot),v(\cdot)) = \operatorname*{esssup}_{\Omega}|u(x)-v(x)|.$$

We know then that $(\mathscr{U}, d)$ is a complete metric space. Without loss of generality, we may assume that $J(\bar{u}) = 0$. For any $\varepsilon > 0$, we define $F_\varepsilon : \mathscr{U}\to R$ by

$$F_\varepsilon(u(\cdot)) = \left\{\left[(J(u(\cdot))+\varepsilon)^+\right]^2 + \left[\int_\Omega d_U(u(x))\,dx\right]^2 + [d_Q(y(\cdot;u))]^2\right\}^{1/2},$$

so that $F_\varepsilon : (\mathscr{U}, d)\to R$ is continuous. Furthermore, we have

$$F_\varepsilon(u(\cdot)) > 0, \quad \forall u(\cdot)\in\mathscr{U},$$
$$F_\varepsilon(\bar{u}(\cdot)) = \varepsilon \le \inf\{F_\varepsilon(u(\cdot)) \mid u(\cdot)\in\mathscr{U}\} + \varepsilon.$$

Hence, by Ekeland's variational principle (see [6]), we can find $u_\varepsilon(\cdot) \in \mathscr{U}$, such that

$$\begin{cases} d(\bar{u}(\cdot), u_\varepsilon(\cdot)) \le \sqrt{\varepsilon}, \quad F_\varepsilon(u_\varepsilon(\cdot)) \le F_\varepsilon(\bar{u}(\cdot)), \\ -\sqrt{\varepsilon} d(u(\cdot), u_\varepsilon(\cdot)) \le F_\varepsilon(u(\cdot)) - F_\varepsilon(u_\varepsilon(\cdot)), \quad \forall u(\cdot) \in \mathscr{U}. \end{cases} \tag{3.10}$$

Let $z_\varepsilon(\cdot) \in H_0^1(\Omega)$ satisfy the following relation

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x, u_\varepsilon(x)) \frac{\partial}{\partial x_j} z_\varepsilon(x) \\ \quad = f_y(x, y_\varepsilon(x), u_\varepsilon(x)) z_\varepsilon(x) \\ \quad\quad + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[(a_{ij}(x, u(x)) - a_{ij}(x, u_\varepsilon(x))) \frac{\partial}{\partial x_j} y_\varepsilon(x)\right] \\ \quad\quad + f(x, y_\varepsilon(x), u(x)) - f(x, y_\varepsilon(x), u_\varepsilon(x)), \\ z_\varepsilon |_{\Gamma_\Omega} = 0 \end{cases} \tag{3.11}$$

and let

$$\begin{aligned} z_\varepsilon^0 = \int_\Omega \Bigg\{ & f_y^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u_\varepsilon(x)) z_\varepsilon(x) \\ & + \sum_{i=1}^{n} f_{\zeta_i}^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u_\varepsilon(x)) \frac{\partial}{\partial x_i} z_\varepsilon(x) \\ & + f^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u(x)) - f^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u_\varepsilon(x)) \Bigg\} dx. \end{aligned} \tag{3.12}$$

We set

$$y_u(x) = y(x; u) \quad \text{and} \quad y_\varepsilon(x) = y(x; u_\varepsilon).$$

By Theorem 2.1, we have

$$\begin{cases} y_u(x) = y_\varepsilon(x) + z_\varepsilon(x) + r_\varepsilon(x), \\ J(u) = J(u_\varepsilon) + z_\varepsilon^0 + r_\varepsilon^0 \end{cases} \tag{3.13}$$

and

$$\|r_\varepsilon\|_{H^1(\Omega)} = o(\|u - u_\varepsilon\|_{L^\infty(\Omega)}), \qquad |r_\varepsilon^0| = o(\|u - u_\varepsilon\|_{L^\infty(\Omega)}).$$

From (3.10), we obtain that

$$-\sqrt{\varepsilon}\, d(u, u_\varepsilon) \le F_\varepsilon(u) - F_\varepsilon(u_\varepsilon) \tag{3.14}$$

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$$= \frac{1}{B_\varepsilon}[(J(u)+\varepsilon)^+ + (J(u_\varepsilon)+\varepsilon)^+][(J(u)+\varepsilon)^+ - (J(u_\varepsilon)+\varepsilon)^+]$$
$$+ \frac{1}{B_\varepsilon}\int_\Omega [d_U(u(x)) + d_U(u_\varepsilon(x))]\,dx \int_\Omega [d_U(u(x)) - d_U(u_\varepsilon(x))]\,dx$$
$$+ \frac{1}{B_\varepsilon}[d_Q(y(\cdot;u)) + d_Q(y(\cdot;u_\varepsilon))][d_Q(y(\cdot;u)) - d_Q(y(\cdot;u_\varepsilon))],$$

where $B_\varepsilon = F_\varepsilon(u) + F_\varepsilon(u_\varepsilon)$. We define

$$\begin{cases} \varphi_\varepsilon^0 = \dfrac{1}{F_\varepsilon(u_\varepsilon)}(J(u_\varepsilon)+\varepsilon)^+, \\ \varphi_\varepsilon^1 = \dfrac{1}{F_\varepsilon(u_\varepsilon)}\displaystyle\int_\Omega d_U(u_\varepsilon(x))\,dx, \\ \varphi_\varepsilon = \dfrac{1}{F_\varepsilon(u_\varepsilon)} d_Q(y(\cdot;u_\varepsilon))\partial d_Q(y(\cdot;u_\varepsilon)), \\ \xi_\varepsilon(x) = (\xi_\varepsilon^1(x), \dots, \xi_\varepsilon^m(x)), \end{cases} \tag{3.15}$$

in which $\xi_\varepsilon^i(x) = 1$ as $\bar{u}_\varepsilon^i(x) \in [a^i, a^i+\varepsilon]$ and $\xi_\varepsilon^i(x) = -1$ as $\bar{u}_\varepsilon^i(x) \in [b^i-\varepsilon, b^i]$, $[a^i, b^i] = \mathrm{Proj}_{X_i}\,\mathrm{co}\,U$, $\bar{u}_\varepsilon(x) = (\bar{u}_\varepsilon^1(x), \dots, \bar{u}_\varepsilon^m(x))$. From (3.14), we obtain that

$$-\sqrt{\varepsilon} d(u, u_\varepsilon) \le \varphi_\varepsilon^0 z_\varepsilon^0 + \varphi_\varepsilon^1 \int_\Omega \xi_\varepsilon(x)[u(x) - u_\varepsilon(x)]\,dx$$
$$+ \langle \varphi_\varepsilon, y_\varepsilon \rangle_{H^{-1},H^1} + o(\|u - u_\varepsilon\|_{L^\infty(\Omega)}). \tag{3.16}$$

Let $\psi_\varepsilon^0 = -\varphi_\varepsilon^0$, $\psi_\varepsilon^1 = -\varphi_\varepsilon^1$ and

$$\begin{cases} -\displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x, u_\varepsilon(x))\frac{\partial}{\partial x_j}\psi_\varepsilon(x)\right) \\ \quad = f_y(x, y_\varepsilon(x), u_\varepsilon(x))\psi_\varepsilon(x) + \psi_\varepsilon^0 \Bigg[f_y^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u_\varepsilon(x)) \\ \quad\quad - \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i} f_{\zeta_i}^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u_\varepsilon(x)) \Bigg] - \varphi_\varepsilon, \\ \psi_\varepsilon |_{\Gamma_\Omega} = 0. \end{cases} \tag{3.17}$$

From (3.11), (3.12) and (3.17), we have

$$\begin{aligned} \varphi_\varepsilon^0 z_\varepsilon^0 &= -\psi_\varepsilon^0 z_\varepsilon^0 \\ &= -\int_\Omega \psi_\varepsilon^0 \Bigg[f_y^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u_\varepsilon(x)) \\ &\quad - \sum_{i=1}^n \frac{\partial}{\partial x_i} f_{\zeta_i}^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u_\varepsilon(x)) \Bigg] z_\varepsilon(x)\,dx \end{aligned}$$

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$$
\begin{aligned}
&- \int_\Omega \psi_\varepsilon^0 \big[f^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u(x)) - f^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u_\varepsilon(x)) \big] dx \\
&= - \int_\Omega \Bigg\{ \sum_{i,j=1}^n a_{ij}(x, u_\varepsilon(x)) \frac{\partial}{\partial x_i} z_\varepsilon(x) \frac{\partial}{\partial x_j} \psi_\varepsilon(x) \\
&\quad - f_y(x, y_\varepsilon(x), u_\varepsilon(x)) \psi_\varepsilon(x) z_\varepsilon(x) \Bigg\} dx - \langle \varphi_\varepsilon, z_\varepsilon \rangle \\
&\quad - \int_\Omega \psi_\varepsilon^0 \big[f^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u(x)) - f^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u_\varepsilon(x)) \big] dx \\
&= \int_\Omega \Bigg\{ \sum_{i,j=1}^n [a_{ij}(x, u(x) - a_{ij}(x, u_\varepsilon(x))] \frac{\partial}{\partial x_i} z_\varepsilon(x) \frac{\partial}{\partial x_j} \psi_\varepsilon(x) \\
&\quad - \psi_\varepsilon(x) [f(x, y_\varepsilon(x), u) - f(x, y_\varepsilon(x), u_\varepsilon(x))] \\
&\quad - \psi_\varepsilon^0 [f^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u(x)) - f^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u_\varepsilon(x))] \Bigg\} dx \\
&\quad - \langle \varphi_\varepsilon, z_\varepsilon \rangle \\
&= - \int_\Omega [H(x, u(x), \varepsilon) - H(x, u_\varepsilon(x), \varepsilon)] \, dx - \langle \varphi_\varepsilon, z_\varepsilon \rangle, \qquad (3.18)
\end{aligned}
$$

where

$$
\begin{aligned}
H(x, u, \varepsilon) = &- \sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial}{\partial x_i} y_\varepsilon(x) \frac{\partial}{\partial x_j} \psi_\varepsilon(x) \\
&+ \psi_\varepsilon(x) f(x, y_\varepsilon(x), u) + \psi_\varepsilon^0 f^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u).
\end{aligned}
$$

Substituting (3.18) into (3.16), we obtain

$$
\begin{aligned}
-\sqrt{\varepsilon} d(u, u_\varepsilon) \le \int_\Omega \big\{ &- [H(x, u(x), \varepsilon) - H(x, u_\varepsilon(x), \varepsilon)] \\
&- \psi_\varepsilon^1 \xi_\varepsilon(x) [u(x) - u_\varepsilon(x)] \big\} dx + o(\|u - u_\varepsilon\|_{L^\infty(\Omega)}). \qquad (3.19)
\end{aligned}
$$

It is clear that

$$
(\psi_\varepsilon^0)^2 + (\psi_\varepsilon^1)^2 + \|\varphi_\varepsilon\|_{H^{-1}}^2 = 1.
$$

So there exist $\psi^0 \in [-1, 0]$, $\psi^1 \in [-1, 0]$, $\varphi_\varepsilon \in H^{-1}$ and a sequence of $\{\psi_{\varepsilon_i}^0, \psi_{\varepsilon_i}^1, \varphi_{\varepsilon_i}\}$, such that

$$
\{\psi_{\varepsilon_i}^0, \psi_{\varepsilon_i}^1, \varphi_{\varepsilon_i}\} \to \{\psi^0, \psi^1 \varphi\} \quad \text{weakly star.}
$$

Since $d(u_\varepsilon(\cdot), \bar{u}(\cdot)) \le \sqrt{\varepsilon}$, when $\varepsilon \to 0$, we have

$$
\|y_\varepsilon - \bar{y}\|_{H^1(\Omega)} \to 0.
$$

It is known that there exists $\psi \in H_0^1(\Omega)$, which satisfies

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,\bar{u}(x))\frac{\partial}{\partial x_j}\psi(x)\right) \\ \qquad = f_y(x,\bar{y}(x),\bar{u}(x))\psi(x) + \psi^0\Bigg[f_y^0(x,\bar{y}(x),\nabla\bar{y}(x),\bar{u}(x)) \\ \qquad\quad - \sum_{i=1}^{n}\frac{\partial}{\partial x_i} f_{\zeta_i}^0(x,\bar{y}(x),\nabla\bar{y}(x),\bar{u}(x))\Bigg] - \varphi, \\ \psi\mid_{\Gamma_\Omega} = 0. \end{cases}$$

Let

$$\begin{cases} H(x,u) = H_1(x,u) + H_2(x,u), \\ H_1(x,u) = -\sum_{i,j=1}^{n} a_{ij}(x,u)\frac{\partial}{\partial x_i}\bar{y}(x)\frac{\partial}{\partial x_j}\psi(x), \\ H_2(x,u) = \psi(x)f(x,\bar{y}(x),u) + \psi^0 f^0(x,\bar{y}(x),\nabla\bar{y}(x),u). \end{cases} \tag{3.20}$$

From (3.19), we obtain

$$\begin{aligned} -\sqrt{\varepsilon}d(u,u_\varepsilon) &- \int_\Omega \{[H(x,u(x)) - H(x,u_\varepsilon(x)] - [H(x,u(x),\varepsilon) - H(x,u_\varepsilon(x),\varepsilon)]\}\,dx \\ &\le \int_\Omega \{-[H(x,u(x)) - H(x,u_\varepsilon(x))] - \psi_\varepsilon^1\xi_\varepsilon(x)[u(x)-u_\varepsilon(x)]\}\,dx \\ &\quad + o(\|u-u_\varepsilon\|_{L^\infty(\Omega)}). \end{aligned} \tag{3.21}$$

We may therefore obtain

$$\int_\Omega \{[H(x,u(x)) - H(x,u_\varepsilon(x)] - [H(x,u(x),\varepsilon) - H(x,u_\varepsilon(x),\varepsilon)]\}\,dx \le C\Delta_\varepsilon d(u,u_\varepsilon),$$

where

$$\begin{aligned} \Delta_\varepsilon \triangleq \|\bar{y} - y_\varepsilon\|_{H^1(\Omega)} &+ |\psi^0 - \psi_\varepsilon^0| \\ &+ \left|\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,\bar{u}(x))\frac{\partial}{\partial x_i}\bar{y}(x)\frac{\partial}{\partial x_j}(\psi(x) - \psi_\varepsilon(x))\,dx\right|. \end{aligned}$$

It is easy to see that $\Delta_\varepsilon \to 0$ as $\varepsilon \to 0$. From (3.21), we have

$$\begin{aligned} -(\sqrt{\varepsilon} + C\Delta_\varepsilon)d(u,u_\varepsilon) \le \int_\Omega \{&-[H(x,u(x)) - H(x,u_\varepsilon(x))] \\ &- \psi_\varepsilon^1\xi_\varepsilon(x)[u(x)-u_\varepsilon(x)]\}\,dx + o(\|u-u_\varepsilon\|_{L^\infty(\Omega)}). \end{aligned} \tag{3.22}$$

For any $v \in \mathscr{U}$, let

$$u(x) = u_\varepsilon(x) + \rho(v(x) - u_\varepsilon(x)),$$

then $d(u, u_\varepsilon) = \rho d(v, u_\varepsilon)$. From (3.22), we have

$$-(\sqrt{\varepsilon} + C\Delta_\varepsilon)d(v, u_\varepsilon) \le \int_\Omega \{-N_\varepsilon(x) - \psi_\varepsilon^1 \xi_\varepsilon(x)\}[v(x) - u_\varepsilon(x)]\,dx, \tag{3.23}$$

where

$$N_\varepsilon(x) = \sum_{i,j=1}^n N_\varepsilon^{ij}(x)\frac{\partial}{\partial x_i}\bar{y}(x)\frac{\partial}{\partial x_j}\psi(x) + N_{2\varepsilon}(x),$$

$$N_\varepsilon^{ij}(x) \in \partial_u a_{ij}(x, u_\varepsilon(x)) \quad \text{and} \quad N_{2\varepsilon}(x) \in \partial_u H_2(x, u_\varepsilon(x)).$$

We shall now discuss the case as $\varepsilon \to 0$. We first want to prove that

$$\{\psi^0, \psi^1, \varphi\} \ne 0. \tag{3.24}$$

In fact, if $\psi^0 \ne 0$ or $\psi^1 \ne 0$ then (3.24) holds. We shall suppose that $\psi^0 = \psi^1 = 0$ and attempt to prove that $\varphi \ne 0$. In fact, according to the definition of the subdifferential, we have

$$\langle \partial d_Q(y_\varepsilon(\cdot)), q(\cdot) - y_\varepsilon(\cdot)\rangle \le 0, \quad \forall q(\cdot) \in Q,$$

which implies that

$$\langle \varphi_\varepsilon, q(\cdot) - y_\varepsilon(\cdot)\rangle \le 0, \quad \forall q(\cdot) \in Q.$$

Furthermore, we have

$$\delta_\varepsilon \triangleq \langle \varphi_\varepsilon, \bar{y} - y_\varepsilon(\cdot))\rangle \le \langle \varphi_\varepsilon, \bar{y} - q(\cdot)\rangle, \quad \forall q \in Q. \tag{3.25}$$

Since $d(u_\varepsilon(\cdot), \bar{u}(\cdot)) \le \sqrt{\varepsilon}$, when $\varepsilon \to 0$, we have

$$\|y_\varepsilon - \bar{y}\|_{H^1(\Omega)} \to 0.$$

Thus $\delta_\varepsilon^1 \to 0$ as $\varepsilon \to 0$. Since the set Q is finite codimensional in $H_0^1(\Omega)$, from [7, 12], we know that $\varphi \ne 0$, so (3.24) holds, that is, (3.3) holds. Furthermore, we obtain (3.4) from (3.25).

Set

$$I_i(u) = \int_\Omega H_i(x, u(x))\,dx, \quad i = 1, 2.$$

Then

$$\|N_\varepsilon^{ij}\|_{L^\infty(\Omega)} \le C, \quad \|N_{2\varepsilon}\|_{L^\infty(\Omega)} \le C,$$

so there exist $N^{ij} \in L^\infty(\Omega)$, $N_2 \in L^\infty(\Omega)$, such that

$$N_\varepsilon^{ij} \to N^{ij}, \quad N_{2\varepsilon} \to N_2 \quad \text{weakly star in } L^\infty(\Omega).$$

According to $\|u_\varepsilon - \bar{u}\|_{L^1(\Omega)} \to 0$ and from [6], we know that $N_2 \in \partial I_2(\bar{u})$, that is, $N_2(x) \in \partial_u H_2(x, \bar{u}(x))$. Furthermore, we have $N^{ij}(x) \in \partial_u a_{ij}(x, \bar{u}(x))$. We note that ξ_ε is independent of ε, so there exists $\xi(x)$ such that $\lim_{\varepsilon\to 0} \xi_\varepsilon(x) = \xi(x)$. Let

$$N(x) = \sum_{i,j=1}^{n} N^{ij}(x) \frac{\partial}{\partial x_i}\bar{y}(x) \frac{\partial}{\partial x_j}\psi(x) + N_2(x).$$

From (3.23), we have

$$0 \le \int_\Omega \{-N(x) - \psi^1 \xi(x)\}[v(x) - \bar{u}(x)]\, dx.$$

We shall now set

$$K(x) = N(x) + \psi^1 \xi(x).$$

Applying Fillipov's Lemma, we have

$$K(x) \cdot \bar{u}(x) = \max_{v \in coU} K(x) \cdot v = \max_{v \in U} K(x) \cdot v, \quad \text{a.e. } x \in \Omega.$$

Theorem 3.1 is thus proved.

REMARK 3.1. In Theorem 3.1, if $Q = H_0^1(\Omega)$, then $\varphi = 0$; If $U = \mathrm{co}U$, then $\psi^1 = 0$.

REMARK 3.2. In Theorem 3.1, we suppose that Q is finite codimensional in $H_0^1(\Omega)$. If Q is not finite codimensional in $H_0^1(\Omega)$, then Theorem 3.1 may be trivial.

EXAMPLE 3.2. We consider the system

$$\begin{cases} -\operatorname{div}[(1 + u(x))\nabla y(x)] = u(x), & x \in \Omega, \\ y_u \mid_{\Gamma_\Omega} = 0, \end{cases} \tag{3.26}$$

$$\mathcal{U}_{\text{ad}} = \{u(x) \in [-1, 1] \mid u(\cdot) \text{ is measurable on } \Omega\}$$

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and we give a functional

$$J(u) = \int_\Omega \{y_u(x) + u(x)\}\, dx.$$

Let $u_0(x) = 0$. Then $y_0(\cdot) = 0$ is the solution of (3.26) corresponding to $u = u_0$. We shall now suppose that $Q = \{y_0(\cdot) = 0\}$. It is clear that $\bar{u}(x) = 0$ is the optimal control and $\bar{y}(x) = 0$ is the optimal state. Suppose Theorem 3.1 holds. Then there exists $(\psi^0, \psi(\cdot)) \neq 0$, where $\psi(\cdot)$ satisfies

$$\begin{cases} -\Delta\psi(x) = \psi^0 + \varphi, & x \in \Omega, \\ \psi \mid_{\Gamma_\Omega} = 0, \end{cases} \tag{3.27}$$

$$H(x, u) = \psi(x)u + \psi^0 u$$

and

$$K(x) = \psi(x) + \psi^0,$$

such that

$$0 = \max_{u\in[-1,1]} [\psi(x) + \psi^0]u. \tag{3.28}$$

From (3.28), we obtain $\psi(x) = -\psi^0 =$ constant. From (3.27), we have $\psi(x) = -\psi^0 = 0$, which contradicts the assumption that $(\psi^0, \psi(\cdot)) \neq 0$.

Example 3.2 indicates that if Q is not finite codimensional in $H_0^1(\Omega)$, then it may be that $K(x) = 0$, that is, Theorem 3.1 may be trivial. Therefore, the condition that Q is finite codimensional in $H_0^1(\Omega)$ is necessary.

We will now discuss the equivalent constraint problem. We define the equivalent constraint to mean that for any $u \in \mathscr{U}_{ad}$, $\int_\Omega u_i(x)\, dx = 1$. Let

$$\mathscr{W}_{ad} = \left\{ u \in \mathscr{U}_{ad} \,\middle|\, \int_\Omega u_i(x)\, dx = 1, \quad i = 1, \dots, m \right\}.$$

PROBLEM E_1. Find $\bar{u} \in \mathscr{W}_{ad}$, such that

$$J(\bar{u}) = \inf\{J(u) \mid u \in \mathscr{W}_{ad}\}.$$

To solve Problem E_1, we introduce a new state equation

$$y_1(u) = \int_\Omega u(x)\, dx \in R^m$$

and thus obtain a new system

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, u(x))\frac{\partial}{\partial x_j} y(x)\right) = f(x, y(x), u(x)), \\ y_1(u) = \int_\Omega u(x)dx \in R^m, \\ y|_{\Gamma_\Omega} = 0, \end{cases}$$

which has a solution $(y, y_1) \in H_0^1(\Omega) \times R^m$ for any $u \in \mathscr{U}$. Let

$$Q_1 = H_0^1(\Omega) \times (1, \dots, 1).$$

Problem E_1 then becomes the problem of finding $\bar{u} \in \mathscr{U}_{ad}$, such that $(y, y_1) \in Q_1$ and

$$J(\bar{u}) = \inf\{J(u) \mid u \in \mathscr{U}_{ad}\}.$$

It is clear that Q_1 is finite codimensional in $H_0^1(\Omega) \times R^m$. We have the following variation equation

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, \bar{u}(x))\frac{\partial}{\partial x_j} z(x)\right) = f_y(x, \bar{y}(x), \bar{u}(x))z(x) \\ \qquad + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[(a_{ij}(x, v(x)) - a_{ij}(x, \bar{u}(x)))\frac{\partial}{\partial x_j}\bar{y}(x)\right] \\ \qquad + [f(x, \bar{y}(x), v(x)) - f(x, \bar{y}(x), \bar{u}(x))], \\ z_1 = \int_\Omega (v(x) - \bar{u}(x))\, dx, \\ z|_{\Gamma_\Omega} = 0, \end{cases}$$

and the following conjugate equation

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, \bar{u}(x))\frac{\partial}{\partial x_j} \psi(x)\right) \\ \qquad = f_y(x, \bar{y}(x), \bar{u}(x))\psi(x) + \psi^0\left[f_y^0(x, \bar{y}(x), \nabla\bar{y}(x), \bar{u}(x)) \right. \\ \qquad\quad \left. - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_{\xi_i}^0(x, \bar{y}(x), \nabla\bar{y}(x), \bar{u}(x))\right], \\ \psi_1 = \mu, \\ \psi|_{\Gamma_\Omega} = 0, \end{cases}$$

where $\mu = (\nu_1, \dots, \nu_m)$ for $\nu_i \in [-1, 1]$, $i = 1, \dots, m$. The Hamiltonian function is

$$H(x, u) = -\sum_{i,j=1}^{n} a_{ij}(x, u)\frac{\partial}{\partial x_i}\bar{y}(x)\frac{\partial}{\partial x_j}\psi(x)$$

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$$+\psi(x)f(x,\bar{y}(x),u)+\psi^0 f^0(x,\bar{y}(x),\nabla\bar{y}(x),u)+\langle\mu,u\rangle.$$

THEOREM 3.2. *Let* (P1)–(P2) *hold,* $\bar{u}(\cdot)\in\mathscr{W}_{\text{ad}}$ *be an optimal control of Problem* E_1 *and* $\bar{y}(\cdot)=y(\cdot,\bar{u})$ *be an optimal state. Then there exist* $\psi^i\in[-1,0]$, $i=0,1$, $\mu=(\nu_1,\cdots,\nu_m)$, $\nu_i\in[-1,1]$, $i=1,\ldots,m$, $\xi(x)$ *given by Theorem* 3.1 *and* $\psi\in H_0^1(\Omega)$ *satisfying* (3.5), *such that*

$$(\psi^0)^2+(\psi^1)^2+\|\mu\|^2=1.$$

Let

$$H(x,u)=-\sum_{i,j=1}^{n}a_{ij}(x,u)\frac{\partial}{\partial x_i}\bar{y}(x)\frac{\partial}{\partial x_j}\psi(x)$$
$$+\psi(x)f(x,\bar{y}(x),u)+\psi^0 f^0(x,\bar{y}(x),\nabla\bar{y}(x),u)+\langle\mu,u\rangle,$$
$$N(x)\in\partial_u H(x,\bar{u}(x))$$

and

$$K(x)=N(x)+\psi^1\xi(x).$$

We have that

$$K(x)\cdot\bar{u}(x)=\max_{v\in U}K(x)\cdot v,\quad a.e.\ x\in\Omega.$$

We shall now return to Problem D which was raised in Section 1 and shall describe the optimal domain using the result of Theorem 3.2.

Suppose $\Omega_{\mathscr{A}}\in\Pi$ (given by (1.3)) is an optimal domain. This implies that the function

$$\bar{u}(x)=a\chi_{\Omega_{\mathscr{A}}}(x)+b\chi_{\Omega_{\mathscr{B}}}(x)$$

is an optimal control of Problem C. Thus the optimal state $\bar{y}(x)$ satisfies:

$$\begin{cases}-\operatorname{div}(\bar{u}(x)\nabla\bar{y}(x))=f(x), & x\in G,\\ \bar{y}\,|_{\Gamma_G}=0.\end{cases}$$

From Theorem 3.2, we know that there exist $\psi^i\in[-1,0]$, $i=0,1$, $\nu\in[-1,1]$, $\xi(x)=\chi_{\Omega_{\mathscr{A}}}(x)-\chi_{\Omega_{\mathscr{B}}}(x)$ and $\psi\in H_0^1(\Omega)$, such that

$$(\psi^0)^2+(\psi^1)^2+(\nu)^2=1,$$

$$\begin{cases}-\operatorname{div}(\bar{u}(x)\nabla\psi(x))=\psi^0\left[f_y^0(x,\bar{y}(x),\nabla\bar{y}(x))-\displaystyle\sum_{i=1}^{n}\frac{\partial}{\partial x_i}f_{\xi_i}^0(x,\bar{y}(x),\nabla\bar{y}(x))\right],\\ \psi|_{\Gamma_G}=0.\end{cases}$$

Let $K(x) = -\nabla\bar{y}(x)\cdot\nabla\psi(x) + \psi^1\xi(x) + \nu$. Then

$$\bar{u}(x) = \begin{cases} b, & \text{if} \quad K(x) > 0, \\ a, & \text{if} \quad K(x) < 0, \end{cases}$$

that is,

$$\{x \in G \mid K(x) < 0\} \subset \Omega_{\mathscr{A}}, \qquad \{x \in G \mid K(x) > 0\} \subset \Omega_{\mathscr{B}}.$$

4. The case of U being a closed set

In this section, we consider the case of U being a general closed set. Let

$$d_U(u) = \inf\{\|v - u\|_{L^\infty(\Omega)} \mid v(\cdot) \in \mathscr{U}_{\text{ad}}\}.$$

THEOREM 4.1. *Let* (P1)–(P3) *hold,* $\bar{u}(\cdot) \in \mathscr{U}_{\text{ad}}^Q$ *be an optimal control of Problem* E_Q *and* $\bar{y}(\cdot) = y(\cdot, \bar{u})$ *be an optimal state. Let* Q *be finite codimensional in* $H_0^1(\Omega)$. *Then there exist* $\psi^0 \in [-1, 0]$, $\varphi \in H^{-1}$, $\xi \in L^\infty(\Omega)^*$ *and* $\psi \in H_0^1(\Omega)$, *such that*

$$(\psi^0, \xi, \varphi) \neq 0, \tag{4.1}$$

$$\langle\varphi, q(\cdot) - \bar{y}(\cdot)\rangle \leq 0, \quad \forall q \in Q, \tag{4.2}$$

$\psi(\cdot)$ *satisfies* (3.5) *and the following variational inequality holds*:

$$0 \leq \int_\Omega N(x)(\bar{u}(x) - v(x))dx + \langle\xi, \bar{u} - v\rangle, \quad \forall v \in \mathscr{U}_{\text{ad}}, \tag{4.3}$$

where $N(x)$ *is given by* (3.7)–(3.8).

PROOF. This proof is similar to the proof of Theorem 3.1. Without loss of generality, we may assume that $J(\bar{u}) = 0$. For any $\varepsilon > 0$, we define $F_\varepsilon : \mathscr{U} \to R$ by

$$F_\varepsilon(u(\cdot)) = \{[(J(u(\cdot)) + \varepsilon)^+]^2 + [d_U(u)]^2 + [d_Q(y(\cdot; u))]^2\}^{1/2}. \tag{4.4}$$

By Ekeland's variational principle (see [6]), we can find $u_\varepsilon(\cdot) \in \mathscr{U}$, such that

$$\begin{cases} d(\bar{u}(\cdot), u_\varepsilon(\cdot)) \leq \sqrt{\varepsilon}, \quad F_\varepsilon(u_\varepsilon(\cdot)) \leq F_\varepsilon(\bar{u}(\cdot)), \\ -\sqrt{\varepsilon}d(u(\cdot), u_\varepsilon(\cdot)) \leq F_\varepsilon(u(\cdot)) - F_\varepsilon(u_\varepsilon(\cdot)), \quad \forall u(\cdot) \in \mathscr{U}. \end{cases} \tag{4.5}$$

Let $z_\varepsilon(\cdot) \in H_0^1(\Omega)$ satisfy (3.11) and let z_ε^0 be given by (3.12). Set

$$y_u(x) = y(x; u) \quad \text{and} \quad y_\varepsilon(x) = y(x; u_\varepsilon).$$

By Theorem 2.1, we have

$$\begin{cases} y_u(x) = y_\varepsilon(x) + z_\varepsilon(x) + r_\varepsilon(x), \\ J(u) = J(u_\varepsilon) + z_\varepsilon^0 + r_\varepsilon^0 \end{cases} \tag{4.6}$$

and

$$\|r_\varepsilon\|_{H^1(\Omega)} = o(\|u - u_\varepsilon\|_{L^\infty(\Omega)}), \qquad |r_\varepsilon^0| = o(\|u - u_\varepsilon\|_{L^\infty(\Omega)}). \tag{4.7}$$

From (4.5), we obtain

$$\begin{aligned} -\sqrt{\varepsilon} d(u, u_\varepsilon) &\le F_\varepsilon(u) - F_\varepsilon(u_\varepsilon) \\ &= \frac{1}{B_\varepsilon}[(J(u)+\varepsilon)^+ + (J(u_\varepsilon)+\varepsilon)^+][(J(u)+\varepsilon)^+ - (J(u_\varepsilon)+\varepsilon)^+] \\ &\quad + \frac{1}{B_\varepsilon}[d_U(u) + d_U(u_\varepsilon)][d_U(u) - d_U(u_\varepsilon)] \\ &\quad + \frac{1}{B_\varepsilon}[d_Q(y(\cdot;u)) + d_Q(y(\cdot;u_\varepsilon))][d_Q(y(\cdot;u)) - d_Q(y(\cdot;u_\varepsilon))], \end{aligned} \tag{4.8}$$

where $B_\varepsilon = F_\varepsilon(u) + F_\varepsilon(u_\varepsilon)$. We define

$$\begin{cases} \varphi_\varepsilon^0 = \dfrac{1}{F_\varepsilon(u_\varepsilon)}(J(u_\varepsilon)+\varepsilon)^+, \\ \xi_\varepsilon = \dfrac{1}{F_\varepsilon(u_\varepsilon)} d_U(u_\varepsilon)\partial d_U(u_\varepsilon), \\ \varphi_\varepsilon = \dfrac{1}{F_\varepsilon(u_\varepsilon)} d_Q(y(\cdot;u_\varepsilon))\partial d_Q(y(\cdot;u_\varepsilon)). \end{cases} \tag{4.9}$$

It is clear that

$$(\psi_\varepsilon^0)^2 + (\|\xi_\varepsilon\|_{(L^\infty(\Omega))^*})^2 + (\|\varphi_\varepsilon\|_{H^{-1}})^2 = 1.$$

From (4.8), we obtain that

$$-\sqrt{\varepsilon} d(u, u_\varepsilon) \le \varphi_\varepsilon^0 z_\varepsilon^0 + \langle \xi_\varepsilon, u - u_\varepsilon \rangle + o(\|u - u_\varepsilon\|_{L^\infty(\Omega)}). \tag{4.10}$$

Let $\psi_\varepsilon^0 = -\varphi_\varepsilon^0$ and suppose that $\psi_\varepsilon(x)$ satisfies (3.17). Then from (3.11), (3.12) and (3.17), we have that

$$\varphi_\varepsilon^0 z_\varepsilon^0 = -\int_\Omega [H(x, u(x), \varepsilon) - H(x, u_\varepsilon(x), \varepsilon)]\, dx, \tag{4.11}$$

where

$$H(x, u, \varepsilon) = -\sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial}{\partial x_i} y_\varepsilon(x) \frac{\partial}{\partial x_j} \psi_\varepsilon(x)$$

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$$+ \psi_\varepsilon(x) f(x, y_\varepsilon(x), u) + \psi_\varepsilon^0 f^0(x, y_\varepsilon(x), \nabla y_\varepsilon(x), u).$$

Substituting (4.11) into (4.10), we obtain

$$-\sqrt{\varepsilon} d(u, u_\varepsilon) \leq - \int_\Omega [H(x, u(x), \varepsilon) - H(x, u_\varepsilon(x), \varepsilon)]\, dx \\ + \langle \xi_\varepsilon, u - u_\varepsilon \rangle + o(\|u - u_\varepsilon\|_{L^\infty(\Omega)}). \qquad (4.12)$$

In a manner similar to that used for Theorem 3.1 we have

$$-(\sqrt{\varepsilon} + C\Delta_\varepsilon) d(u, u_\varepsilon) \leq - \int_\Omega [H(x, u(x)) - H(x, u_\varepsilon(x))]\, dx \\ + \langle \xi_\varepsilon, u - u_\varepsilon \rangle + o(\|u - u_\varepsilon\|_{L^\infty(\Omega)}). \qquad (4.13)$$

For any $v \in \mathscr{U}$, let

$$u(x) = u_\varepsilon(x) + \rho(v(x) - u_\varepsilon(x)),$$

then $d(u, u_\varepsilon) = \rho d(v, u_\varepsilon)$. From (4.13), we have

$$-(\sqrt{\varepsilon} + C\Delta_\varepsilon) d(v, u_\varepsilon) \leq \int_\Omega (-N_\varepsilon(x))[v(x) - u_\varepsilon(x)]dx + \langle \xi_\varepsilon, v - u_\varepsilon \rangle, \qquad (4.14)$$

where $N_\varepsilon(x) \in \partial_u H(x, u_\varepsilon(x))$. In a manner similar to that used for Theorem 3.1, there exist $\psi^0 \in [-1, 0]$, $\xi \in (L^\infty(\Omega))^*$, $\varphi \in H^{-1}$, such that $\psi_\varepsilon^0 \to \psi^0$ and

$$\{\xi_\varepsilon, \varphi_\varepsilon\} \to \{\xi, \varphi\} \quad \text{weakly star},$$

such that (4.1) and (4.2) hold. From the proof of Theorem 3.1, we know that there exists $N(x) \in \partial_u H(x, \bar{u}(x))$ such that

$$0 \leq \int_\Omega N(x)[\bar{u}(x) - v(x)]dx + \langle \xi, \bar{u} - v \rangle.$$

Theorem 4.1 is thus proved.

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