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ON THE ZEROS OF HILBERT SPACES OF ANALYTIC FUNCTIONS

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Abstract

An attempt is made to characterise the zeros of some Hilbert spaces of analytic functions means of their kernel functions. Results on the zeros of functions in D_ϕ and their uniqueness s are included, in particular we give an affirmative answer to a question of Shapiro and Shield

1. Introduction

Let U be the open unit disc and $H(U)$ the set of all analytic functions in Let $\phi(z) = \Sigma c_n z^n \in H(U)$, with $c_0 = 1$, $c_n > 0$ and

(1) $$c_n^2 \leqq c_{n-1} c_{n+1}.$$

For $f(z) = \Sigma a_n z^n \in H(U)$, define

$$\|f\|^2 = \Sigma \frac{1}{c_n} |a_n|^2$$

and let $D_\phi = \{f \in H(U) : \|f\|^2 < \infty\}$. Then D_ϕ is a Hilbert space under the inn product

$$(f, g) = \Sigma \frac{1}{c_n} a_n \bar{b}_n$$

where $g(z) = \Sigma b_n z^n$.

Let

$$D = \{f \in H(U) : f(0) = 0, \ \frac{1}{\pi} \int \int_U |f'(z)|^2 dxdy = \Sigma n |a_n|^2 < \infty\}.$$

Then D is also a Hilbert space under the inner product $(f, g) = \Sigma n a_n \bar{b}_n$.

The reproducing kernel for D_ϕ is $K_\xi(z) = \phi(\bar{\xi} z)$, $\xi \in U$. That is, for a $\xi \in U$,

$$f(\xi) = (f, K_\xi) \qquad \text{for} \qquad \text{all} \qquad f \in D_\phi.$$

The reproducing kernel for D is $K_\xi(z) = -\log(1-\bar{\xi}z)$. For simplicity we write K_n for K_ξ when $\xi = z_n$.

A set $\{z_n\}$ in U is called a set of uniqueness for a subspace $\mathcal{F}$ of $H(U)$ if $f \in \mathcal{F}$ and $f(z_n) = 0$ for all n imply $f \equiv 0$.

The following two results are due to Shapiro & Shields (1962).

THEOREM 1. *If $\{z_n\}$ is any sequence of points in U for which*

$$\sum \frac{1}{K_n(z_n)} < \infty \tag{2}$$

then there is an $f(\not\equiv 0) \in D_\phi$ vanishing at all these points (with a similar result for D).

THEOREM 2. *Let $h(t)$ be any continuous function with $h(0) = 0$, $h(t) > 0$ $(t > 0)$. Then there exists a set of uniqueness $\{z_n\}$ for D satisfying the condition*

$$\sum \frac{1}{-\log(1-|z_n|)} h(1-|z_n|) < \infty.$$

Shapiro & Shields (1962, page 224) observed that Theorem 2 holds for D_ϕ when $\phi(z) = (1-z)^{\alpha-1}$ for $0 \leqq \alpha < 1$. They raised the question whether this is true for all D_ϕ. We give this an affirmative answer in section 2. Examples of D_ϕ with zero sets violating (2) are also given.

2. Uniqueness sets and zeros of functions in D_ϕ

If (1) holds and $\phi(z) \in H(U)$ then as shown by Shapiro and Shields (1962, Lemma 6),

$$1 = c_0 \geqq c_1 \geqq c_2 \geqq \cdots.$$

The first lemma gives an estimate on the norm of a function with specified zeros.

LEMMA 1. *Let $z_1, z_2, \cdots z_n$ be n equally spaced points on the circle $|z| = r$, $0 < r < 1$. If $f \in D_\phi$ with $f(z_i) = 0$ $(i \leqq n)$ and $f(0) = 1$ then*

$$\|f\|^2 \geqq n/\phi(r^2).$$

PROOF. Without loss of generality take $z_1 = r$. Define $h(z)$ by $h = 1/n(K_1 + K_2 + \cdots + K_n)$. Then $(f, h) = 0$ and so

$$1 = (f, 1) = (f, 1-h) \leqq \|f\| \, \|1-h\|.$$

Further

$$\|1-h\|^2 = \sum_{m=1}^{\infty} c_{nm} r^{2nm} \leqq \frac{1}{n}\phi(r^2)$$

since c_k and r^{2k} decreases as k increases.

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LEMMA 2. *For any sequence r_k, $0 < r_k < 1$ there is a sequence of integers n such that*

$$\frac{2}{k} > \frac{\phi(r_k^2)}{n_k} \geqq \frac{1}{k}.$$

PROOF. Let $N_k = \{n : \phi(r_k^2) \geqq n/k, n \text{ integer}\}$. Since $\phi(r_k^2) > 1$, the set N_k is non-empty and is obviously bounded. Let $n_k = \max N_k$, then

$$\frac{\phi(r_k^2)}{n_k} \geqq \frac{1}{k} > \frac{\phi(r_k^2)}{n_k + 1} \geqq \frac{\phi(r_k^2)}{2n_k}.$$

COROLLARY. *If $h(t)$ is continuous and $h(0) = 0$, $h(t) > 0$, then there is a sequence r_n, $0 < r_n < 1$ such that*

$$\sum \frac{1}{\phi(r_n^2)} = \infty \quad \text{while} \quad \sum \frac{1}{\phi(r_n^2)} \cdot h(1 - r_n) < \infty.$$

PROOF. For each k, choose s_k, $0 < s_k < 1$ such that $h(1 - s_k) < \frac{1}{k^3}$. For this sequence choose a sequence of integers $\{n_k\}$ as in Lemma 2. Let $\{r_n\}$ be the sequence obtained by repeating n_k times each s_k.

We can now prove the following:

THEOREM 3. *Let $h(t)$ be any continuous function with $h(0) = 0$, $h(t) > 0$ $(t > 0)$. Then there exists a set of uniqueness $\{z_n\}$ for D_ϕ satisfying the condition*

$$\sum \frac{1}{K_n(z_n)} \cdot h(1 - |z_n|) \equiv \sum \frac{1}{\phi(|z_n|^2)} \cdot h(1 - |z_n|) < \infty.$$

PROOF. Choose r_k, $0 < r_k < 1$ such that $h(1 - r_k) < 1/k^3$ and then n_k as in Lemma 2. Now set

$$\{z_n\} = \bigcup_{k=1}^{\infty} \{z = r_k e^{2\pi m i/n_k} : m = 0, 1, \cdots, n_k - 1\}.$$

Then

$$\sum_n \frac{h(1 - |z_n|)}{\phi(|z_n|^2)} = \sum_k \frac{n_k h(1 - r_k)}{\phi(r_k^2)} \leqq \sum_k \frac{1}{k^2} < \infty.$$

If $f(\not\equiv 0) \in D_\phi$ with $f(z_n) = 0$, for all n, we can suppose for our purpose that $f(0) = 1$. Then by Lemma 1

$$\|f\|^2 \leqq n_k/\phi(r_k^2) > k/2, \qquad k = 1, 2, 3, \cdots.$$

Hence $\{z_n\}$ is a uniqueness set for D_ϕ.

For $\phi(z) = \Sigma z^n$, D_ϕ is just merely the Hardy space H^2 and $1/K_n(z_n) < \infty$ is then equivalent to the Blaschke condition which is necessary and sufficient for $\{z_n\}$ to be a zero set of H^2. Apart from this case, it is easy to show that for $\phi(z)$

with radius of convergence $R > 1$, $D_\phi \subseteq H(\bar{U})$ the class of functions analytic in a neighbourhood of the closed unit disc $\bar{U}$. Obviously here the non-trivial zero sets are just the finite subsets of U. As against that we have the following situation.

THEOREM 4. *If the Taylor coefficients c_n of ϕ satisfy $1/c_n = 0(n^k)$ for some positive integer k and $c_n \to 0$, then there exists an $f(\not\equiv 0) \in D_\phi$, whose zeros $\{z_n\}$ satisfy*

$$\sum \frac{1}{K_n(z_n)} = \infty .$$

PROOF. The function $h(t) = t\phi((1-t)^2) \to 0$ as $t \to 0$. To see this observe that N can be chosen for each n such that $nc_k < \frac{1}{2}$ for $k \geqq N$. Then

$$n\phi(r^2) \leqq n(c_0 + \cdots + c_{N-1}r^{2(N-1)}) + \tfrac{1}{2}r^{2N} \cdot \frac{1}{1-r^2} \leqq \frac{1}{1-r^2}$$

for r close to 1. The Corollary to Lemma 2 now applies to give $\{z_n\}$ on the unit interval such that

$$\sum (1 - z_n) < \infty \qquad \text{and} \qquad \sum \frac{1}{\phi(z_n^2)} = \infty .$$

By a result of Caughran (1969, Theorem 2) there exists an $f(\not\equiv 0) \in H(U)$ with bounded $f^{(k)}$ and vanishing at all the points z_n. Note that

$$\sum_n n^k \,|\, a_n \,| < \infty \qquad \text{if} \qquad \iint_U |f^{(k)}(z)|^2 \, dxdy < \infty$$

and since $1/c_n = 0(n^k)$, this implies that $f \in D_\phi$.

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