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The existence of unbounded solutions of asymmetric oscillations in the degenerate resonant case

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We prove the existence of unbounded solutions of the asymmetric oscillation in the case when each zero of the discriminative function is degenerate. This is the only case that has not been studied in the literature.

Keywords: Asymmetric oscillations; Unbounded solutions

1. Introduction

We are concerned with the asymmetric oscillation

$$x'' + ax^+ - bx^- = f(t), \tag{1.1}$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, a, b are two different positive constants, and $f(t)$ is a 2π-periodic function.

This equation models the suspension bridge [8] and has been widely studied. Fuçik [3] and Dancer [2] studied it in their investigations of boundary value problems associated to equations with 'jumping nonlinearities'. For recent developments, one can refer to [4, 5, 7, 16] and the references therein.

For Littlewood's boundedness problem of the asymmetric oscillation (1.1), the earliest contribution was due to Ortega [12]. In 1996, he considered the equation

$$x'' + ax^+ - bx^- = 1 + \varepsilon h(t),$$

where the smooth function $h(t)$ is 2π-periodic. He proved that if $|\varepsilon|$ is sufficiently small, then all solutions are bounded. That is, if $x(t)$ is a solution, then it is defined for all $t \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}}(|x(t)| + |x'(t)|) < +\infty.$$

This result is in contrast with the well-known phenomenon of linear resonance that occurs in the case $a = b = n^2$. For example, for any $\varepsilon \neq 0$, all solutions of

$$x'' + n^2 x = 1 + \varepsilon \cos(nt)$$

are unbounded.

For the asymmetric oscillation (1.1). Let $\omega_0 := \frac{1}{2}(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}})$.

If $\omega_0 \in \mathbb{R}\backslash\mathbb{Q}$, Ortega [**14**] in 2001 proved the boundedness of all solutions of equation (1.1) under the condition $\int_0^{2\pi} f(t)\mathrm{d}t \neq 0$.

Recently, Hu *et al.* [**6**] established an invariant curve theorem and applied it to equation (1.1), then they obtained the boundedness of all solutions with ω_0 satisfying the Diophantine condition, but without the assumption $\int_0^{2\pi} f(t)\mathrm{d}t \neq 0$.

Subsequently, we [**10**] also proved the boundedness of all solutions of equation (1.1) without the assumption $\int_0^{2\pi} f(t)\mathrm{d}t \neq 0$, but ω_0 is assumed to satisfy an approximation function condition.

If $\omega_0 \in \mathbb{Q}$, then there exist two positive integers m and n such that

$$\omega_0 = \frac{m}{n}. \tag{1.2}$$

Moreover, m and n are relatively prime.

Denote by $C(t)$ the solution of the 'homogeneous' equation

$$x'' + ax^+ - bx^- = 0$$

with the initial conditions $C(0) = 1$, $C'(0) = 0$. Then it is well known that $C(t) \in C^2(\mathbb{R})$ and can be given explicitly by the formula

$$C(t) = \begin{cases} \cos\sqrt{a}|t|, & 0 \leqslant |t| \leqslant \dfrac{\pi}{2\sqrt{a}}, \\ -\sqrt{\dfrac{a}{b}} \sin\sqrt{b}\left(|t| - \dfrac{\pi}{2\sqrt{a}}\right), & \dfrac{\pi}{2\sqrt{a}} < |t| \leqslant \dfrac{m}{n}\pi. \end{cases}$$

Denote the derivative of C by $S = C'$, then $S(t) \in C(\mathbb{R})$ and

(1) $C(-t) = C(t),\ \ S(-t) = -S(t)$;

(2) $C(t)$ and $S(t)$ are $2\pi\frac{m}{n}$-periodic functions;

(3) $S(t)^2 + a\,C^+(t)^2 + b\,C^-(t)^2 \equiv a$.

For a given 2π-periodic function $f(t)$. Let

$$\Phi_f(\theta) := \int_0^{2\pi} C\left(\frac{m}{n}\theta + mt\right) f(mt)\mathrm{d}t, \quad \theta \in \mathbb{R},$$

and

$$\mathcal{A}(f) := \{\theta \in \mathbb{R} :\ \Phi_f(\theta) = 0\}.$$

Then $\Phi_f(\theta)$ is a 2π-periodic function and its derivative is

$$\Phi'_f(\theta) = \frac{m}{n} \int_0^{2\pi} S\left(\frac{m}{n}\theta + mt\right) f(mt)\mathrm{d}t, \quad \theta \in \mathbb{R}.$$

On the one hand, if $\mathcal{A} = \emptyset$, Liu [**11**] in 1999 proved that all solutions of equation (1.1) are bounded. On the other hand, if $\mathcal{A} \neq \emptyset$ and all zeros of $\Phi_f(\theta)$ are non-degenerate, that is, $\Phi'_f(\theta) \neq 0$ for all $\theta \in \mathcal{A}$, Alonso and Ortega [**1**] in 1998 proved that there exists $R > 0$ such that every solution of (1.1) with

$$x(t_0)^2 + x'(t_0)^2 > R$$

for some $t_0 \in \mathbb{R}$ is unbounded. Particularly, the proof of this result implies that if there is a non-degenerate zero of $\Phi_f(\theta)$, then there exist unbounded solutions of equation (1.1). We remark that the references [**11**] and [**1**] assume that $\Phi_f(\theta) \not\equiv 0$.

In 1998, Ortega [**13**] proposed an example

$$x'' + 4x^+ - x^- = \lambda + \cos 4t, \quad \lambda \in \mathbb{R}. \tag{1.3}$$

In this example, $\omega_0 = 3/4$. Hence, the results of [**1**] and [**11**] can be applied. If $|\lambda| < 1/45$, then all solutions with large initial conditions are unbounded. If $|\lambda| > 1/45$, then all solutions are bounded.

However, when $|\lambda| = 1/45$, all zeros of $\Phi_f(\theta)$ are degenerate. Therefore, the references [**1**] and [**11**] can not be applied to this equation. In 2021, we [**9**] proved the existence of unbounded solutions of equation (1.3) with $\lambda = \pm 1/45$.

The main idea of [**9**] is as follows. First, the corresponding Poincaré map in action and angle variables can be expressed by

$$\begin{cases} \theta_1 = \theta_0 + 2m\pi + \dfrac{1}{r_0}\mu_1(\theta_0) + \dfrac{1}{r_0^2}k_1(\theta_0) + \dfrac{1}{r_0^3}h_1(\theta_0) + g_1(\theta_0, r_0), \\ r_1 = r_0 + \mu_2(\theta_0) + \dfrac{1}{r_0}k_2(\theta_0) + \dfrac{1}{r_0^2}h_2(\theta_0) + g_2(\theta_0, r_0). \end{cases}$$

Then in equation (1.3) with $\lambda = \pm\frac{1}{45}$, for all $\theta^* \in \mathcal{A}$, we have $\Phi_f(\theta^*) = \Phi'_f(\theta^*) = 0$. Thus, $\mu_1(\theta)$ has only degenerate zeros. However, in these two examples, the function $p(\theta, r) := \mu_1(\theta) + \frac{1}{r}k_1(\theta) + \frac{1}{r^2}h_1(\theta)$ has some non-degenerate zeros. Then an invariant set near the zero of $\mu_1(\theta)$ can be found, and each solution starting from this invariant set is unbounded.

Unfortunately, this method to find the invariant set depends on the property that the function $p(\theta, r)$ has non-degenerate zero, and cannot deal with the other cases, including that all zeros of $p(\theta, r)$ are degenerate, or $p(\theta, r)$ has no zero. In this paper, we will obtain the existence of unbounded solutions of equation (1.1) without considering the zero of $p(\theta, r)$. More precisely, we will prove

THEOREM 1.1. *Assume that the resonance condition* (1.2) *holds,* $f(t)$ *is a real analytic* 2π*-periodic function such that* $\Phi_f(\theta) \not\equiv 0$ *and* $\mathcal{A}(f) \neq \emptyset$*. Then equation* (1.1) *has unbounded solutions.*

REMARK 1.2. When $\Phi_f(\theta) \not\equiv 0$, if $\mathcal{A}(f) = \emptyset$, then all solutions of equation (1.1) are bounded by [**11**]. If $\mathcal{A}(f) \neq \emptyset$, then equation (1.1) has unbounded solutions by theorem 1.1. Therefore, theorem 1.1 together with the result of [**11**] completely solves Littlewood's boundedness problem for the asymmetric oscillation (1.1) in the resonance case under the assumption $\Phi_f(\theta) \not\equiv 0$.

In fact, according to the result of Alonso and Ortega in [1], here we only need to consider the situation that for all $\theta^* \in \mathcal{A}(f)$, $\Phi'_f(\theta^*) = 0$. The main idea for proving theorem 1.1 is similar to [15] and as follows. By means of a series of transformations, the original system is transformed into a normal form, for which the twist condition is violated. Then an invariant set will be found, and each solution starting from the invariant set is unbounded.

The rest of this paper is organized in 4 sections as follows. In § 2, we will give some examples to illustrate the main theorem. Section 3 is devoted to finding the transformations and the normal form (for which the twist condition is violated). Then in § 4, we will give some properties of the discriminative function $\Phi_f(\theta)$, which is crucial in this paper. Finally, the proof of the existence of unbounded solutions will be given in § 5.

2. Some remarks

We give several examples to illustrate theorem 1.1. The first two examples show that theorem 1.1 is applicable.

Example 2.1. For equation (1.3) with $\lambda = \pm\frac{1}{45}$, the discriminative function takes the form

$$\Phi_f(\theta) = \begin{cases} -\dfrac{4}{45} + \dfrac{4}{45}\cos 3\theta, & \lambda = \dfrac{1}{45}, \\ \dfrac{4}{45} + \dfrac{4}{45}\cos 3\theta, & \lambda = -\dfrac{1}{45}. \end{cases}$$

In view of theorem 1.1, this equation has unbounded solutions, which is consistent with the result of [9].

Example 2.2. Consider the equation

$$x'' + 4x^+ - x^- = \lambda_1 + \lambda_2 \cos 4t + \lambda_3 \sin 4t. \tag{2.1}$$

The discriminative function of this equation takes the form

$$\Phi_f(\theta) = -4\lambda_1 + \frac{4}{45}\lambda_2 \cos 3\theta - \frac{4}{45}\lambda_3 \sin 3\theta,$$

and the results of [1, 11] and theorem 1.1 can be applied.

- When $\lambda_2 = \lambda_3 = 0$, the discriminative function is of the form

$$\Phi_f(\theta) = -4\lambda_1.$$

  If $\lambda_1 \neq 0$, then all solutions are bounded. If $\lambda_1 = 0$, then equation (2.1) becomes

$$x'' + 4x^+ - x^- = 0,$$

  and all solutions are bounded. Thus, when $\lambda_2 = \lambda_3 = 0$, all solutions of (2.1) are bounded.

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- When $\lambda_2 = 0,\ \ \lambda_3 \neq 0$, the discriminative function is of the form

$$\Phi_f(\theta) = -4\lambda_1 - \frac{4}{45}\lambda_3 \sin 3\theta.$$

  If $\left|\frac{\lambda_1}{\lambda_3}\right| > \frac{1}{45}$, then all solutions are bounded. If $\left|\frac{\lambda_1}{\lambda_3}\right| < \frac{1}{45}$, then all solutions with large initial conditions are unbounded. If $\left|\frac{\lambda_1}{\lambda_3}\right| = \frac{1}{45}$, then equation (2.1) has unbounded solutions.

- When $\lambda_2 \neq 0$, the discriminative function is of the form

$$\Phi_f(\theta) = -4\lambda_1 + \frac{4}{45}\sqrt{\lambda_2^2 + \lambda_3^2}\cos(3\theta + \alpha), \quad \alpha = \arctan\frac{\lambda_3}{\lambda_2}.$$

  Thus, if $\left|\frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}\right| > \frac{1}{45}$, then all solutions are bounded. If $\left|\frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}\right| < \frac{1}{45}$, then all solutions with large initial conditions are unbounded. If $\left|\frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}\right| = \frac{1}{45}$, then equation (2.1) has unbounded solutions.

Finally, for equation (1.1), when $\Phi_f(\theta) \equiv 0$, there are no results which can be applied to determine the boundedness of its solutions. The following examples show that this situation can indeed happen.

Example 2.3. Consider the equation

$$x'' + ax^+ - bx^- = \cos(rnt),$$

where the resonance condition (1.2) holds, $a \neq b$ and r is a positive integer. Then the discriminative function is

$$\Phi_f(\theta) = \frac{2\sqrt{a}(b-a)n}{m(r^2n^2 - a)(r^2n^2 - b)}\cos\left(\frac{rn\pi}{2\sqrt{a}}\right)\cos(rm\theta), \quad r^2n^2 \neq a, b,$$

and it is easy to see that when $\frac{rn}{\sqrt{a}}$ is odd, we have $\Phi_f(\theta) \equiv 0$.

Example 2.4. Consider the equation

$$x'' + 4x^+ - x^- = \cos kt,$$

where k is a positive integer. Then the discriminative function is

$$\Phi_f(\theta) = \begin{cases} 0, & k = 1, 2, \\ \dfrac{-4}{(k^2-4)(k^2-1)}(1 + (-1)^k)\left(\cos\left(\dfrac{k\pi}{4} + \dfrac{3k}{4}\theta\right) + \cos\left(\dfrac{3k\pi}{4} + \dfrac{3k}{4}\theta\right)\right), & k \geqslant 3. \end{cases}$$

Thus, if $k = 4r$, $r = 1, 2, 3, \ldots$, then

$$\Phi_f(\theta) = \frac{(-1)^{r+1}4}{(4r^2 - 1)(16r^2 - 1)}\cos(3r\theta),$$

and if $k \neq 4r$, $r = 1, 2, 3, \ldots$, then $\Phi_f(\theta) \equiv 0$.

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3. Transformations

We make a series of canonical transformations to obtain a normal form for which the twist condition is violated.

3.1. Action and angle variables

Let $y = x'$, then equation (1.1) is equivalent to the following planar system

$$\begin{cases} x' = y, \\ y' = -ax^+ + bx^- + f(t). \end{cases} \tag{3.1}$$

The following result is standard.

LEMMA 3.1. *For any* $(x_0, y_0) \in \mathbb{R}^2$ *and* $t_0 \in \mathbb{R}$, *the unique solution* $z(t) = (x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$ *of* (3.1) *satisfying* $z(t_0) = (x_0, y_0)$ *exists on the whole* t*-axis.*

For $r > 0$, $\theta \,(\mathrm{mod}\ 2\pi)$, define the following generalized polar coordinates $\Gamma : (r, \theta) \to (x, y)$ by

$$\begin{cases} x = \rho r^{\frac{1}{2}} C(\dfrac{m}{n}\theta), \\ y = \rho r^{\frac{1}{2}} S(\dfrac{m}{n}\theta), \end{cases}$$

where $\rho := \sqrt{\frac{2n}{am}}$. It is easy to check that Γ is a symplectic transformation.

The Hamiltonian associated to the system (3.1) is expressed in Cartesian coordinates by

$$H(x, y, t) = \frac{1}{2}y^2 + \frac{a}{2}(x^+)^2 + \frac{b}{2}(x^-)^2 - f(t)x.$$

In the new coordinates (r, θ), it becomes

$$H(r, \theta, t) = \frac{n}{m}r - \rho r^{\frac{1}{2}} C\left(\frac{m}{n}\theta\right) f(t). \tag{3.2}$$

Thus, the system (3.1) is transformed into

$$\begin{cases} \theta' = H_r = \dfrac{n}{m} - \dfrac{1}{2}\rho r^{-\dfrac{1}{2}} C\left(\dfrac{m}{n}\theta\right) f(t), \\ r' = -H_\theta = \dfrac{m}{n}\rho r^{\dfrac{1}{2}} S\left(\dfrac{m}{n}\theta\right) f(t), \end{cases} \tag{3.3}$$

which is a semilinear system.

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3.2. A sublinear system

First, introduce a new time variable ϑ by $t = m\vartheta$ to eliminate the denominator of the linear part of (3.2). Then, the system (3.3) is transformed into

$$\begin{cases} \frac{\mathrm{d}\theta}{\mathrm{d}\vartheta} = \frac{\partial}{\partial r}\mathcal{H}(r, \theta, \vartheta), \\ \frac{\mathrm{d}r}{\mathrm{d}\vartheta} = -\frac{\partial}{\partial \theta}\mathcal{H}(r, \theta, \vartheta), \end{cases} \tag{3.4}$$

where

$$\mathcal{H}(r, \theta, \vartheta) = nr - m\rho r^{\frac{1}{2}} C\left(\frac{m}{n}\theta\right) f(m\vartheta). \tag{3.5}$$

For convenience, we can rewrite (3.5) and (3.4) respectively as

$$H(r, \theta, t) = nr - m\rho r^{\frac{1}{2}} C\left(\frac{m}{n}\theta\right) f(mt), \tag{3.6}$$

and

$$\begin{cases} \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial}{\partial r}H(r, \theta, t), \\ \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\partial}{\partial \theta}H(r, \theta, t). \end{cases}$$

Since m is a positive integer, then the new Hamiltonian $H(r,\, \theta,\, t)$ in (3.6) is 2π-periodic in θ and t.

Next we introduce a rotation transformation to eliminate the linear part of the Hamiltonian (3.6), which helps us to obtain a sublinear system.

Define the rotation transformation $\Phi_1 : (r_1,\, \theta_1,\, t) \to (r,\, \theta,\, t)$ by

$$\begin{cases} \theta = \theta_1 + nt, \\ r = r_1. \end{cases}$$

Then under Φ_1, the original semilinear system determined by the Hamiltonian (3.6) is transformed into a sublinear system given by the following Hamiltonian

$$H_1(r_1, \theta_1, t) = -m\rho r_1^{\frac{1}{2}} C\left(\frac{m}{n}\theta_1 + mt\right) f(mt). \tag{3.7}$$

It is worth to point out that the above transformations preserve the periodicity and boundedness of solutions. In fact, if $(r_1(t + 2\pi),\, \theta_1(t + 2\pi)) = (r_1(t),\, \theta_1(t))$, then for the Hamiltonian (3.5), $r(\vartheta + 2\pi) = r_1(\vartheta + 2\pi) = r_1(\vartheta) = r(\vartheta)$, and $\theta(\vartheta + 2\pi) = \theta_1(\vartheta + 2\pi) + n(\vartheta + 2\pi) = \theta_1(\vartheta) + n\vartheta + 2n\pi = \theta(\vartheta) + 2n\pi$. Since $t = m\vartheta$, for the Hamiltonian (3.2), we have $r(\frac{1}{m}t + 2\pi) = r(\frac{1}{m}t)$, and $\theta(\frac{1}{m}t + 2\pi) = \theta(\frac{1}{m}t) + 2n\pi$. Thus for the original system (3.1), $x(\frac{1}{m}t + 2\pi) = \rho r(\frac{1}{m}t + 2\pi)^{\frac{1}{2}} C(\frac{m}{n}\theta(\frac{1}{m}t + 2\pi)) = \rho r(\frac{1}{m}t)^{\frac{1}{2}} C(\frac{m}{n}\theta(\frac{1}{m}t) + 2m\pi) = x(\frac{1}{m}t)$, which leads to $x(t + 2\pi) = x(t)$. Thus, the periodicity is preserved. Similarly, it is easy to verify that the boundedness is also preserved.

3.3. The normal form without the twist condition

To reduce the power of term containing t in the Hamiltonian (3.7), we make the transformation $\Phi_2 : (r_2,\, \theta_2,\, t) \to (r_1,\, \theta_1,\, t)$ given by

$$\begin{cases} \theta_2 = \theta_1 + \dfrac{\partial S_2}{\partial r_2}(r_2, \theta_1, t), \\ r_1 = r_2 + \dfrac{\partial S_2}{\partial \theta_1}(r_2, \theta_1, t) \end{cases}$$

with the generating function $S_2(r_2,\, \theta_1,\, t)$ determined by

$$S_2(r_2, \theta_1, t) = -m\rho r_2^{\frac{1}{2}} \int_0^t [Cf]\left(\frac{m}{n}\theta_1\right) - C\left(\frac{m}{n}\theta_1 + ms\right) f(ms)\mathrm{d}s,$$

where

$$[Cf]\left(\frac{m}{n}\theta_1\right) = \frac{1}{2\pi}\int_0^{2\pi} C\left(\frac{m}{n}\theta_1 + mt\right) f(mt)\mathrm{d}t.$$

Under Φ_2, the Hamiltonian H_1 in (3.7) is transformed into

$$\begin{aligned} H_2(r_2, \theta_2, t) &= -m\rho\left(r_2 + \frac{\partial S_2}{\partial \theta_1}\right)^{\frac{1}{2}} C\left(\frac{m}{n}\theta_1 + mt\right) f(mt) + \frac{\partial S_2}{\partial t} \\ &= -m\rho r_2^{\frac{1}{2}} C\left(\frac{m}{n}\theta_1 + mt\right) f(mt) \\ &\quad - m\rho C\left(\frac{m}{n}\theta_1 + mt\right) f(mt) \int_0^1 \frac{1}{2}\left(r_2 + \mu\frac{\partial S_2}{\partial \theta_1}\right)^{-\frac{1}{2}} \frac{\partial S_2}{\partial \theta_1}\mathrm{d}\mu \\ &\quad + \frac{\partial S_2}{\partial t}. \end{aligned}$$

It is obvious that

$$-m\rho r_2^{\frac{1}{2}} C\left(\frac{m}{n}\theta_1 + mt\right) f(mt) + \frac{\partial S_2}{\partial t} = -m\rho r_2^{\frac{1}{2}} [Cf]\left(\frac{m}{n}\theta_1\right),$$

and thus,

$$\begin{aligned} H_2(r_2, \theta_2, t) = &- m\rho r_2^{\frac{1}{2}} [Cf]\left(\frac{m}{n}\theta_1\right) \\ &- m\rho C\left(\frac{m}{n}\theta_1 + mt\right) f(mt) \int_0^1 \frac{1}{2}\left(r_2 + \mu\frac{\partial S_2}{\partial \theta_1}\right)^{-\frac{1}{2}} \frac{\partial S_2}{\partial \theta_1}\mathrm{d}\mu. \end{aligned}$$

Then by $\theta_2 = \theta_1 + \frac{\partial S_2}{\partial r_2}(r_2,\, \theta_1,\, t)$, we get

$$H_2(r_2, \theta_2, t) = -m\rho r_2^{\frac{1}{2}} [Cf]\left(\frac{m}{n}\theta_2\right) + P_1(\theta_2, t) + P_2(r_2, \theta_2, t), \tag{3.8}$$

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where

$$P_1(\theta_2,t) = -\frac{1}{4\pi}m^3n^{-1}\rho^2\int_0^{2\pi} S\left(\frac{m}{n}\theta_2+mt\right)f(mt)\int_0^t [Cf]\left(\frac{m}{n}\theta_2\right)$$
$$-C\left(\frac{m}{n}\theta_2+ms\right)f(ms)\mathrm{d}s\mathrm{d}t$$
$$+\frac{1}{2}m^3n^{-1}\rho^2C\left(\frac{m}{n}\theta_2+mt\right)f(mt)\int_0^t [Sf]\left(\frac{m}{n}\theta_2\right)$$
$$-S\left(\frac{m}{n}\theta_2+ms\right)f(ms)\mathrm{d}s,$$

and

$$P_2(r_2,\theta_2,t) = \frac{1}{4\pi}m^4n^{-2}\rho^2\int_0^{2\pi} S\left(\frac{m}{n}\theta_2+mt\right)f(mt)\int_0^t$$
$$\left(\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 S\left(\frac{m}{n}\theta_2+ms-\mu\frac{m}{n}\frac{\partial S_2}{\partial r_2}\right)\frac{\partial S_2}{\partial r_2}f(ms)\mathrm{d}\mu\mathrm{d}s\right.$$
$$\left.-\int_0^1 S\left(\frac{m}{n}\theta_2+ms-\mu\frac{m}{n}\frac{\partial S_2}{\partial r_2}\right)\frac{\partial S_2}{\partial r_2}f(ms)\mathrm{d}\mu\right)\mathrm{d}s\mathrm{d}t$$
$$-\frac{1}{2\pi}m^3n^{-2}\rho r_2^{\frac{1}{2}}\int_0^{2\pi}\int_0^1\int_0^1 S'\left(\frac{m}{n}\theta_2+mt-s\mu\frac{m}{n}\frac{\partial S_2}{\partial r_2}\right)\mu\left(\frac{\partial S_2}{\partial r_2}\right)^2 f(mt)\mathrm{d}s\mathrm{d}\mu\mathrm{d}t$$
$$-\frac{1}{2}m^4n^{-2}\rho^2C\left(\frac{m}{n}\theta_2+mt\right)f(mt)$$
$$\int_0^t\left(\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 S'\left(\frac{m}{n}\theta_2+ms-\mu\frac{m}{n}\frac{\partial S_2}{\partial r_2}\right)\frac{\partial S_2}{\partial r_2}f(ms)\mathrm{d}\mu\mathrm{d}s\right.$$
$$\left.-\int_0^1 S'\left(\frac{m}{n}\theta_2+ms-\mu\frac{m}{n}\frac{\partial S_2}{\partial r_2}\right)\frac{\partial S_2}{\partial r_2}f(ms)\mathrm{d}\mu\right)\mathrm{d}s$$
$$+\frac{1}{2}m^2n^{-1}\rho r_2^{-\frac{1}{2}}\int_0^1 S\left(\frac{m}{n}\theta_2+mt-\mu\frac{m}{n}\frac{\partial S_2}{\partial r_2}\right)f(mt)\frac{\partial S_2}{\partial r_2}\frac{\partial S_2}{\partial \theta_1}\mathrm{d}\mu$$
$$+\frac{1}{4}m\rho C\left(\frac{m}{n}\theta_2-\frac{m}{n}\frac{\partial S_2}{\partial r_2}+mt\right)f(mt)\int_0^1\int_0^1\left(r_2+s\mu\frac{\partial S_2}{\partial \theta_1}\right)^{-\frac{3}{2}}\mu\left(\frac{\partial S_2}{\partial \theta_1}\right)^2\mathrm{d}s\mathrm{d}\mu$$

with

$$[Sf]\left(\frac{m}{n}\theta\right) = \frac{1}{2\pi}\int_0^{2\pi} S\left(\frac{m}{n}\theta+mt\right)f(mt)\mathrm{d}t.$$

Then we have the following estimates, and the proof is elementary.

Lemma 3.2. *For r_2 large enough, θ_2, $t\in\mathbb{S}^1=\mathbb{R}/(2\pi\mathbb{Z})$, we have*

$$|\partial_{r_2}^k\partial_{\theta_1}^j S_2(r_2,\theta_1,t)|\leqslant Cr_2^{\frac{1}{2}-k},\quad k+j\leqslant 3,$$

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and

$$|\partial_{\theta_2}^{j} P_1(\theta_2, t)| \leqslant C, \quad j \leqslant 1,$$
$$|\partial_{r_2}^{k} \partial_{\theta_2}^{j} P_2(r_2, \theta_2, t)| \leqslant C r_2^{-\frac{1}{2}-k}, \quad k+j \leqslant 1,$$

where C is a constant larger than 1.

Finally, with the definitions of $\Phi_f(\theta)$ and $[Cf](\frac{m}{n}\theta)$, $[Sf](\frac{m}{n}\theta)$, the Hamiltonian $H_2(r_2, \theta_2, t)$ in (3.8) can be rewritten as

$$H(r, \theta, t) = -\frac{1}{2\pi} m\rho r^{\frac{1}{2}} \Phi_f(\theta) + P_1(\theta, t) + P_2(r, \theta, t), \tag{3.9}$$

and for r large enough, θ, $t \in \mathbb{S}^1$, one has

$$|\partial_{\theta}^{j} P_1(\theta, t)| \leqslant C, \quad j \leqslant 1,$$
$$|\partial_{r}^{k} \partial_{\theta}^{j} P_2(r, \theta, t)| \leqslant C r^{-\frac{1}{2}-k}, \quad k+j \leqslant 1,$$

where C is a constant larger than 1.

4. Some properties of $\Phi_f(\theta)$

We present several lemmas for the discriminative function $\Phi_f(\theta)$, which will be used in the proof of the existence of unbounded solutions.

First, we prove that under the assumptions of theorem 1.1, $\Phi_f(\theta)$ is an analytic function, and thus for any $\theta^* \in \mathcal{A}$, there exists an integer $k \geqslant 2$ such that $\Phi_f^{(k)}(\theta^*) \neq 0$.

LEMMA 4.1. *Under the assumptions of theorem 1.1, for any $\theta^* \in \mathcal{A}$, there exists an integer $k \geqslant 2$ such that $\Phi_f(\theta^*) = \Phi'_f(\theta^*) = \cdots = \Phi_f^{(k-1)}(\theta^*) = 0$, $\Phi_f^{(k)}(\theta^*) \neq 0$.*

Proof. Since f is real analytic and 2π-periodic in t, then it can be written as a uniformly convergent Fourier series

$$f(t) = \sum_{k \in \mathbb{Z}} f_k e^{ikt}, \quad t \in \mathbb{R},$$

where the Fourier coefficients

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} \mathrm{d}t, \quad k \in \mathbb{Z}.$$

Moreover, f can be analytically extended into a complex domain $\{t \in \mathbb{C} : |\mathrm{Im} t| \leqslant r\}$, with $r > 0$ a small constant, and we have $|f_k| \leqslant \|f\|_r e^{-|k|r}$, where $\|f\|_r = \sup_{|\mathrm{Im} t| \leqslant r} |f(t)|$.

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Then it is obvious that the series

$$C\left(\frac{m}{n}\theta+mt\right)\left(\sum_{k\in\mathbb{Z}} f_k e^{ikmt}\right)=\sum_{k\in\mathbb{Z}} f_k C\left(\frac{m}{n}\theta+mt\right)e^{ikmt}$$
$$=\sum_{k\in\mathbb{Z}} f_k C\left(\frac{m}{n}\theta+mt\right)(\cos(kmt)+i\sin(kmt))$$

is also uniformly convergent to $C(\frac{m}{n}\theta+mt)f(mt)$. Thus,

$$\begin{aligned}\Phi_f(\theta)&=\int_0^{2\pi} C\left(\frac{m}{n}\theta+mt\right)f(mt)\mathrm{d}t\\&=\sum_{k\in\mathbb{Z}} f_k\int_0^{2\pi} C\left(\frac{m}{n}\theta+mt\right)\cos(kmt)\mathrm{d}t+i\sum_{k\in\mathbb{Z}} f_k\int_0^{2\pi} C\left(\frac{m}{n}\theta+mt\right)\sin(kmt)\mathrm{d}t\\&:=\sum_{k\in\mathbb{Z}} f_k\Phi_{ak}(\theta)+i\sum_{k\in\mathbb{Z}} f_k\Phi_{bk}(\theta),\end{aligned}$$

where

$$\Phi_{ak}(\theta)=\int_0^{2\pi} C\left(\frac{m}{n}\theta+mt\right)\cos(kmt)\mathrm{d}t,$$
$$\Phi_{bk}(\theta)=\int_0^{2\pi} C\left(\frac{m}{n}\theta+mt\right)\sin(kmt)\mathrm{d}t.$$

The periodicity of C and f yields that

$$\begin{aligned}\Phi_{ak}(\theta)&=\int_0^{2\pi} C\left(\frac{m}{n}\theta+mt\right)\cos(kmt)\mathrm{d}t=\int_0^{2\pi} C(mt)\cos\left(kmt-k\frac{m}{n}\theta\right)\mathrm{d}t\\&=\sum_{l=0}^{n-1}\int_{\frac{l}{m}(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}})}^{\frac{l+1}{m}(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}})} C(mt)\cos\left(kmt-k\frac{m}{n}\theta\right)\mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}&\int_{\frac{l}{m}(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}})}^{\frac{l+1}{m}(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}})} C(mt)\cos\left(kmt-k\frac{m}{n}\theta\right)\mathrm{d}t\\&=\int_{\frac{l}{m}(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}})}^{\frac{l}{m}(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}})+\frac{\pi}{2m\sqrt{a}}}(-1)^l\cos\left(\sqrt{a}mt-l\sqrt{\frac{a}{b}}\pi\right)\cos\left(kmt-k\frac{m}{n}\theta\right)\mathrm{d}t\\&\quad+\int_{\frac{l}{m}(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}})+\frac{\pi}{2m\sqrt{a}}}^{\frac{l}{m}(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}})+\frac{\pi}{2m\sqrt{a}}+\frac{\pi}{m\sqrt{b}}}(-1)^{l+1}\\&\qquad\sqrt{\frac{a}{b}}\sin\left(\sqrt{b}mt-(2l+1)\frac{\sqrt{b}}{2\sqrt{a}}\pi\right)\cos\left(kmt-k\frac{m}{n}\theta\right)\mathrm{d}t\\&\quad+\int_{\frac{l}{m}(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}})+\frac{\pi}{2m\sqrt{a}}+\frac{\pi}{m\sqrt{b}}}^{\frac{l+1}{m}(\frac{\pi}{\sqrt{a}}+\frac{\pi}{\sqrt{b}})}(-1)^{l+1}\cos\left(\sqrt{a}mt-(l+1)\sqrt{\frac{a}{b}}\pi\right)\cos\left(kmt-k\frac{m}{n}\theta\right)\mathrm{d}t.\end{aligned}$$

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By some calculations, when $k = \pm\sqrt{a}$, we get

$$\begin{aligned}\Phi_{ak}(\theta) &= \frac{1}{2m\sqrt{a}} \sin\left(\sqrt{a}\frac{m}{n}\theta\right) + \frac{\pi}{4m\sqrt{a}} \cos\left(\sqrt{a}\frac{m}{n}\theta\right) - \frac{\sqrt{a}}{m(b-a)} \sin\left(\sqrt{a}\frac{m}{n}\theta\right) \\ &\quad + \sum_{l=0}^{n-2} (-1)^{l+1} \frac{\pi}{2m\sqrt{a}} \cos\left((l+1)\sqrt{\frac{a}{b}}\pi - \sqrt{a}\frac{m}{n}\theta\right) \\ &\quad + (-1)^n \frac{1}{2m\sqrt{a}} \sin\left(n\sqrt{\frac{a}{b}}\pi - \sqrt{a}\frac{m}{n}\theta\right) \\ &\quad + (-1)^n \frac{\pi}{4m\sqrt{a}} \cos\left(n\sqrt{\frac{a}{b}}\pi - \sqrt{a}\frac{m}{n}\theta\right) \\ &\quad + (-1)^{n+1} \frac{\sqrt{a}}{m(b-a)} \sin\left(n\sqrt{\frac{a}{b}}\pi - \sqrt{a}\frac{m}{n}\theta\right).\end{aligned}$$

When $k = \pm\sqrt{b}$, similarly we have

$$\begin{aligned}\Phi_{ak}(\theta) &= -\frac{\sqrt{b}}{m(a-b)} \sin\left(\sqrt{b}\frac{m}{n}\theta\right) + \frac{\sqrt{b}}{m(a-b)} (-1)^{n-1} \sin\left(n\sqrt{\frac{b}{a}}\pi - \sqrt{b}\frac{m}{n}\theta\right) \\ &\quad + \sum_{l=0}^{n-1} (-1)^l \frac{\sqrt{a}\pi}{2mb} \sin\left(l\sqrt{\frac{b}{a}}\pi + \frac{\sqrt{b}\pi}{2\sqrt{a}} - \sqrt{b}\frac{m}{n}\theta\right).\end{aligned}$$

When $k \neq \pm\sqrt{a}, \pm\sqrt{b}$, we also obtain

$$\begin{aligned}\Phi_{ak}(\theta) = \sum_{l=0}^{n-1} \frac{\sqrt{a}(b-a)}{m(k^2-a)(k^2-b)} &\left(\cos\left(lk\left(\frac{\pi}{\sqrt{a}} + \frac{\pi}{\sqrt{b}}\right) + \frac{k\pi}{2\sqrt{a}} - k\frac{m}{n}\theta\right)\right. \\ &\left. + \cos\left(lk\left(\frac{\pi}{\sqrt{a}} + \frac{\pi}{\sqrt{b}}\right) + \frac{k\pi}{2\sqrt{a}} + \frac{k\pi}{\sqrt{b}} - k\frac{m}{n}\theta\right)\right).\end{aligned}$$

Thus, for all $k \in \mathbb{Z}$, $\Phi_{ak}(\theta)$ can be analytically extended to $\tilde{\Phi}_{ak}(\theta)$ in $\{\theta \in \mathbb{C} : |\operatorname{Im}\theta| < r'\}$, with $r' \leqslant r\frac{n}{m}$, and it is easy to see that

$$\left|\tilde{\Phi}_{ak}(\theta)\right| \leqslant \begin{cases} \left(\dfrac{2+n\pi}{2m\sqrt{a}} + \dfrac{2\sqrt{a}}{m|b-a|}\right) e^{\sqrt{a}\frac{m}{n}r'}, & k = \pm\sqrt{a}; \\ \left(\dfrac{2\sqrt{b}}{m|b-a|} + \dfrac{\sqrt{a}n\pi}{2mb}\right) e^{\sqrt{b}\frac{m}{n}r'}, & k = \pm\sqrt{b}; \\ \dfrac{2\sqrt{a}|b-a|n}{m|k^2-a||k^2-b|} e^{|k|\frac{m}{n}r'}, & k \neq \pm\sqrt{a}, \pm\sqrt{b}. \end{cases}$$

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Similarly, for all $k \in \mathbb{Z}$, $\Phi_{bk}(\theta)$ can also be analytically extended to $\tilde{\Phi}_{bk}(\theta)$ in $\{\theta \in \mathbb{C} : |\operatorname{Im}\theta| < r'\}$, and

$$\left|\tilde{\Phi}_{bk}(\theta)\right| \leqslant \begin{cases} \left(\dfrac{2+n\pi}{2m\sqrt{a}} + \dfrac{2\sqrt{a}}{m|b-a|}\right) e^{\sqrt{a}\dfrac{m}{n}r'}, & k = \pm\sqrt{a}; \\ \left(\dfrac{2\sqrt{b}}{m|b-a|} + \dfrac{\sqrt{a}n\pi}{2mb}\right) e^{\sqrt{b}\dfrac{m}{n}r'}, & k = \pm\sqrt{b}; \\ \dfrac{2\sqrt{a}|b-a|n}{m|k^2-a||k^2-b|} e^{|k|\dfrac{m}{n}r'}, & k \neq \pm\sqrt{a}, \pm\sqrt{b}. \end{cases}$$

Then with $|f_k| \leqslant \|f\|_r e^{-|k|r}$, where $\|f\|_r = \sup_{|\operatorname{Im} t| \leqslant r} |f(t)|$, and since $r' \leqslant r\frac{n}{m}$, we have

$$|f_k\tilde{\Phi}_{ak}(\theta)|,\ |if_k\tilde{\Phi}_{bk}(\theta)| \leqslant \begin{cases} \|f\|_r \left(\dfrac{2+n\pi}{2m\sqrt{a}} + \dfrac{2\sqrt{a}}{m|b-a|}\right), & k = \pm\sqrt{a}; \\ \|f\|_r \left(\dfrac{2\sqrt{b}}{m|b-a|} + \dfrac{\sqrt{a}n\pi}{2mb}\right), & k = \pm\sqrt{b}; \\ \|f\|_r \dfrac{2\sqrt{a}|b-a|n}{m|k^2-a||k^2-b|}, & k \neq \pm\sqrt{a}, \pm\sqrt{b}. \end{cases}$$

By Weierstrass M-test, since the series $\sum_{\substack{k\in\mathbb{Z} \\ k\neq\pm\sqrt{a},\, \pm\sqrt{b}}} \frac{1}{|k^2-a||k^2-b|}$ is convergent, then in the domain $\{\theta \in \mathbb{C} : |\operatorname{Im}\theta| < r'\}$, the series

$$\sum_{k\in\mathbb{Z}} f_k\tilde{\Phi}_{ak}(\theta) + i\sum_{k\in\mathbb{Z}} f_k\tilde{\Phi}_{bk}(\theta)$$

uniformly converge to $\tilde{\Phi}_f(\theta)$, which is a complex extension of $\Phi_f(\theta)$. Since all $\tilde{\Phi}_{ak}(\theta)$, $\tilde{\Phi}_{bk}(\theta)$ are analytic, then by Weierstrass's Theorem, $\tilde{\Phi}_f(\theta)$ is also analytic in the domain $\{\theta \in \mathbb{C} : |\operatorname{Im}\theta| < r'\}$.

Finally, under the assumptions of theorem 1.1, for any $\theta^* \in \mathcal{A}$, we have $\tilde{\Phi}_f(\theta^*) = \Phi_f(\theta^*) = 0$, $\tilde{\Phi}'_f(\theta^*) = \Phi'_f(\theta^*) = 0$. Then with the isolation of zeros for analytic functions, for any $\theta^* \in \mathcal{A}$, there exists an integer $k \geqslant 2$ such that $\tilde{\Phi}_f(\theta^*) = \tilde{\Phi}'_f(\theta^*) = \cdots = \tilde{\Phi}_f^{(k-1)}(\theta^*) = 0$, $\tilde{\Phi}_f^{(k)}(\theta^*) \neq 0$. Thus, $\Phi_f(\theta^*) = \Phi'_f(\theta^*) = \cdots = \Phi_f^{(k-1)}(\theta^*) = 0$, $\Phi_f^{(k)}(\theta^*) \neq 0$. □

By lemma 4.1, choose some $\theta^* \in \mathcal{A}$. Without loss of generality, we can assume that $\Phi_f^{(k)}(\theta^*) > 0$, otherwise, make a time change $t \mapsto -t$.

In the following, for a fixed $\theta^* \in \mathcal{A}$ and the corresponding integer $k \geqslant 2$, some estimates of $\Phi_f(\theta)$ near θ^* are given.

Lemma 4.2. *Assume that there exist $\theta^* \in \mathbb{R}$ and $2 \leqslant k \in \mathbb{N}$ such that $\Phi_f(\theta^*) = \Phi'_f(\theta^*) = \cdots = \Phi_f^{(k-1)}(\theta^*) = 0$, $\Phi_f^{(k)}(\theta^*) > 0$. Then there exists $\delta_1 > 0$ such that for all $\theta : 0 < \theta - \theta^* \leqslant \delta_1$, one has $\Phi_f(\theta) > 0$, $\Phi'_f(\theta) > 0$.*

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Proof. This lemma can be easily proved by properties of the derivative, so we omit the details here. □

Let $\tau = \theta - \theta^*$. Then $\Phi_f(\theta) = \Phi_f(\tau + \theta^*)$, and we get the following lemma.

LEMMA 4.3. *Assume that there exists* $2 \leqslant k \in \mathbb{N}$ *such that* $\Phi_f(\theta^*) = \Phi'_f(\theta^*) = \cdots = \Phi_f^{(k-1)}(\theta^*) = 0$, $\Phi_f^{(k)}(\theta^*) > 0$. *Then there exists* $\delta_1 > 0$ *such that for all* $\tau : 0 < \tau \leqslant \delta_1$, *we have* $\Phi_f(\tau + \theta^*) > 0$, $\Phi'_f(\tau + \theta^*) > 0$.

LEMMA 4.4. *Assume that the function* $g(x)$ *is analytic at* $x = 0$, *and* $g^{(j)}(0) = 0$, $j = 0, 1, \ldots, k-1$, $g^{(k)}(0) > 0$. *Then there exists* $\delta_2 > 0$ *such that for all* $x : 0 \leqslant x \leqslant \delta_2$, *one has*

$$g(x) \in \left[\frac{c_1}{k!}x^k g^{(k)}(0),\ \frac{c_2}{k!}x^k g^{(k)}(0)\right],$$

where

$$c_1 = \frac{6k+10}{6k+11} < 1,\ c_2 = \frac{6k+12}{6k+11} > 1.$$

Proof. On the one hand, let

$$h_1(x) = g(x) - \frac{c_1}{k!}x^k g^{(k)}(0).$$

Then $h_1(0) = h'_1(0) = \cdots = h_1^{(k-1)}(0) = 0$, and $h_1^{(k)}(0) = (1 - c_1)g^{(k)}(0) = \frac{1}{6k+11} g^{(k)}(0) > 0$, so there exists $\delta_3 > 0$ such that for all $x : 0 \leqslant x \leqslant \delta_3$, we have $h_1(x) \geqslant 0$, which leads to

$$g(x) \geqslant \frac{c_1}{k!}x^k g^{(k)}(0).$$

On the other hand, let

$$h_2(x) = g(x) - \frac{c_2}{k!}x^k g^{(k)}(0).$$

Then $h_2(0) = h'_2(0) = \cdots = h_2^{(k-1)}(0) = 0$, and $h_2^{(k)}(0) = (1 - c_2)g^{(k)}(0) = \frac{-1}{6k+11} g^{(k)}(0) < 0$, so there exists $\delta_4 > 0$ such that for all $x : 0 \leqslant x \leqslant \delta_4$, we have $h_2(x) \leqslant 0$, which leads to

$$g(x) \leqslant \frac{c_2}{k!}x^k g^{(k)}(0).$$

Let $\delta_2 = \min\{\delta_3, \delta_4\} > 0$. Then for all $x : 0 \leqslant x \leqslant \delta_2$, we have

$$g(x) \in \left[\frac{c_1}{k!}x^k g^{(k)}(0),\ \frac{c_2}{k!}x^k g^{(k)}(0)\right],$$

where

$$c_1 = \frac{6k+10}{6k+11} < 1, \quad c_2 = \frac{6k+12}{6k+11} > 1.$$

□

Applying lemmas 4.1–4.4 to $\Phi_f(\tau+\theta^*)$ and $\Phi'_f(\tau+\theta^*)$. Then we can easily obtain the following result.

LEMMA 4.5. *Under the assumptions of theorem 1.1, there exists* $\delta > 0$ *such that for all* $\tau : 0 < \tau \leqslant \delta$*, we have* $\Phi_f(\tau+\theta^*) > 0$, $\Phi'_f(\tau+\theta^*) > 0$*, and*

$$\Phi_f(\tau+\theta^*) \in \left[\frac{c_1}{k!}\tau^k\Phi_f^{(k)}(\theta^*), \frac{c_2}{k!}\tau^k\Phi_f^{(k)}(\theta^*)\right],$$

$$\Phi'_f(\tau+\theta^*) \in \left[\frac{c_1}{(k-1)!}\tau^{k-1}\Phi_f^{(k)}(\theta^*), \frac{c_2}{(k-1)!}\tau^{k-1}\Phi_f^{(k)}(\theta^*)\right],$$

where $c_1 = \frac{6k+10}{6k+11} < 1,\ \ c_2 = \frac{6k+12}{6k+11} > 1$.

5. The existence of unbounded solutions

In this section, we prove that the Hamiltonian system with the Hamiltonian (3.9) has unbounded solutions.

The system with the Hamiltonian (3.9) is given by

$$\begin{cases} \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dfrac{\partial H}{\partial r} = -\dfrac{1}{4\pi}m\rho r^{-\frac{1}{2}}\Phi_f(\theta) + \partial_r P_2(r,\theta,t), \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = -\dfrac{\partial H}{\partial \theta} = \dfrac{1}{2\pi}m\rho r^{\frac{1}{2}}\Phi'_f(\theta) - \partial_\theta P_1(\theta,t) - \partial_\theta P_2(r,\theta,t). \end{cases} \tag{5.1}$$

Let $\tau = \theta - \theta^*$. Then the system (5.1) is transformed into

$$\begin{cases} \dfrac{d\tau}{\mathrm{d}t} = -\dfrac{1}{4\pi}m\rho r^{-\frac{1}{2}}\Phi_f(\tau+\theta^*) + \partial_r P_2(r,\tau+\theta^*,t), \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = \dfrac{1}{2\pi}m\rho r^{\frac{1}{2}}\Phi'_f(\tau+\theta^*) - \partial_\tau P_1(\tau+\theta^*,t) - \partial_\tau P_2(r,\tau+\theta^*,t). \end{cases} \tag{5.2}$$

For fixed $2 \leqslant k \in \mathbb{N}$ from lemma 4.1 and δ from lemma 4.5, choose $r^* \gg 1$ satisfying

(1) $r^* \gg \|f\|$;

(2) $2(r^*)^{-\frac{1}{3k}} < \delta$;

(3) $\frac{1}{3\pi}\frac{c_1}{(k-1)!}m\rho(\frac{r^*}{2})^{\frac{1}{2(2k-1)}}\Phi_f^{(k)}(\theta^*) \geqslant 1$, where $c_1 = \frac{6k+10}{6k+11}$.

Give an initial point $(r(0), \tau(0)) \in D := \left\{(r,\ \tau) : r^{-\frac{1}{2k-1}} \leqslant \tau \leqslant r^{-\frac{1}{3k}}\right\}$ and $r(0) \geqslant r^*$.

First, the second equation in (5.2) implies that $\frac{\mathrm{d}r}{\mathrm{d}t} = O(r^{\frac{1}{2}}+1)$, thus $r(t) \geqslant \frac{1}{2}r^*$ for any $t \in [0,\ 2\pi]$ by $r^* \gg \|f\|$. Also the first equation in (5.2) implies $\frac{\mathrm{d}\tau}{\mathrm{d}t} = O(r^{-\frac{1}{2}})$, hence $0 < \tau(t) \leqslant 2(r^*)^{-\frac{1}{3k}}$ for any $t \in [0,\ 2\pi]$. Thus, lemma 4.5 can be applied in the following.

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We claim that if the initial point $(r(0),\, \tau(0)) \in D$ and $r(0) \geqslant r^*$, then $(r(t),\, \tau(t)) \in D$ for any $t \in [0,\, 2\pi]$. Otherwise, let $t_1 := \sup\{t : (r(s),\, \tau(s)) \in D,\quad 0 \leqslant s \leqslant t\} < 2\pi$. It is obvious that $(r(t_1),\, \tau(t_1)) \in \partial D$, which leads to $r(t_1)^l \tau(t_1) = 1$ with $l = \frac{1}{2k-1}$ or $\frac{1}{3k}$.

By a direct computation, we get

$$\begin{aligned}
&(r(t)^l \tau(t))'|_{t=t_1} \\
&\quad = l r(t)^{l-1} r'(t)\tau(t) + r(t)^l \tau'(t)|_{t=t_1} \\
&\quad = l r(t)^{l-1}\tau(t) \left(\frac{1}{2\pi} m\rho r(t)^{\frac{1}{2}} \Phi_f'(\tau(t)+\theta^*) \right. \\
&\quad \left. -\partial_\tau P_1(\tau(t)+\theta^*, t) - \partial_\tau P_2(r(t), \tau(t)+\theta^*, t) \right) \Big|_{t=t_1} \\
&\qquad + r(t)^l \left(-\frac{1}{4\pi} m\rho r(t)^{-\frac{1}{2}} \Phi_f(\tau(t)+\theta^*) + \partial_r P_2(r(t), \tau(t)+\theta^*, t) \right) \Big|_{t=t_1} \\
&\quad := J_1 + J_2.
\end{aligned}$$

Since $r(t_1)^l \tau(t_1) = 1$, $r(t) \geqslant \frac{1}{2} r^*$ and $0 < \tau(t) \leqslant 2(r^*)^{-\frac{1}{3k}} < \delta$ for $t \in [0,\, 2\pi]$, then we get

$$\begin{aligned}
J_1 &= l r(t_1)^{l-1}\tau(t_1) \left(\frac{1}{2\pi} m\rho r(t_1)^{\frac{1}{2}} \Phi_f'(\tau(t_1)+\theta^*) - \partial_\tau P_1(\tau(t) \right. \\
&\quad \left. +\theta^*, t)|_{t=t_1} - \partial_\tau P_2(r(t), \tau(t)+\theta^*, t)|_{t=t_1} \right) \\
&= \frac{1}{2\pi} m\rho l r(t_1)^{-\frac{1}{2}} \Phi_f'(\tau(t_1)+\theta^*) + O(r(t_1)^{-1}),
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= r(t_1)^l \left(-\frac{1}{4\pi} m\rho r(t_1)^{-\frac{1}{2}} \Phi_f(\tau(t_1)+\theta^*) + \partial_r P_2(r(t), \tau(t)+\theta^*, t)|_{t=t_1} \right) \\
&= -\frac{1}{4\pi} m\rho r(t_1)^{-\frac{1}{2}} r(t_1)^l \Phi_f(\tau(t_1)+\theta^*) + O\left(r(t_1)^{-\frac{3}{2}+l} \right).
\end{aligned}$$

Now it is a position to apply lemma 4.5 to J_1 and J_2. On the one hand, if $l = \frac{1}{3k}$, then we have

$$\begin{aligned}
J_1 + J_2 &= \frac{1}{2\pi} m\rho l r(t_1)^{-\frac{1}{2}} \Phi_f'(\tau(t_1)+\theta^*) + O(r(t_1)^{-1}) \\
&\quad - \frac{1}{4\pi} m\rho r(t_1)^{-\frac{1}{2}} r(t_1)^l \Phi_f(\tau(t_1)+\theta^*) + O\left(r(t_1)^{-\frac{3}{2}+l} \right) \\
&\leqslant \frac{1}{2\pi} m\rho l r(t_1)^{-\frac{1}{2}} \frac{c_2}{(k-1)!} \tau(t_1)^{k-1} \Phi_f^{(k)}(\theta^*) + O(r(t_1)^{-1}) \\
&\quad - \frac{1}{4\pi} m\rho r(t_1)^{-\frac{1}{2}} r(t_1)^l \frac{c_1}{k!} \tau(t_1)^k \Phi_f^{(k)}(\theta^*) + O\left(r(t_1)^{-\frac{3}{2}+l} \right)
\end{aligned}$$

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$$= \frac{1}{2\pi} m\rho l r(t_1)^{-\frac{1}{2}} \frac{c_2}{(k-1)!} r(t_1)^{-l(k-1)} \Phi_f^{(k)}(\theta^*) + O(r(t_1)^{-1})$$
$$- \frac{1}{4\pi} m\rho r(t_1)^{-\frac{1}{2}} r(t_1)^l \frac{c_1}{k!} r(t_1)^{-lk} \Phi_f^{(k)}(\theta^*) + O\left(r(t_1)^{-\frac{3}{2}+l}\right)$$
$$= \frac{1}{2\pi} \frac{1}{(k-1)!} m\rho \left(c_2 l - \frac{c_1}{2k}\right) r(t_1)^{-\frac{1}{2}-l(k-1)} \Phi_f^{(k)}(\theta^*)$$
$$+ O(r(t_1)^{-1}) + O\left(r(t_1)^{-\frac{3}{2}+l}\right).$$

Since $c_1 = \frac{6k+10}{6k+11}$, $c_2 = \frac{6k+12}{6k+11}$ and $l = \frac{1}{3k}$, $k \geqslant 2$, then $c_2 l - \frac{c_1}{2k} < 0$, $-\frac{3}{2} + l < -1 < -\frac{1}{2} - l(k-1)$, which lead to

$$J_1 + J_2 = \frac{1}{2\pi} \frac{1}{(k-1)!} m\rho \left(c_2 l - \frac{c_1}{2k}\right) r(t_1)^{-\frac{1}{2}-l(k-1)} \Phi_f^{(k)}(\theta^*)(1 + o(1)) < 0.$$

That is, if $l = \frac{1}{3k}$, then $(r(t)^l \tau(t))'|_{t=t_1} < 0$. Therefore, there exists $t_2 > t_1$ such that $r(t)^{\frac{1}{3k}} \tau(t) \leqslant 1$ for $t \in [t_1, t_2]$, which contradicts the definition of t_1. Thus for all $t \in [0, 2\pi]$, we have $\tau(t) \leqslant r(t)^{-\frac{1}{3k}}$.

On the other hand, if $l = \frac{1}{2k-1}$, then we have

$$J_1 + J_2 = \frac{1}{2\pi} m\rho l r(t_1)^{-\frac{1}{2}} \Phi'_f(\tau(t_1) + \theta^*) + O(r(t_1)^{-1})$$
$$- \frac{1}{4\pi} m\rho r(t_1)^{-\frac{1}{2}} r(t_1)^l \Phi_f(\tau(t_1) + \theta^*) + O\left(r(t_1)^{-\frac{3}{2}+l}\right)$$
$$\geqslant \frac{1}{2\pi} m\rho l r(t_1)^{-\frac{1}{2}} \frac{c_1}{(k-1)!} \tau(t_1)^{k-1} \Phi_f^{(k)}(\theta^*) + O(r(t_1)^{-1})$$
$$- \frac{1}{4\pi} m\rho r(t_1)^{-\frac{1}{2}} r(t_1)^l \frac{c_2}{k!} \tau(t_1)^k \Phi_f^{(k)}(\theta^*) + O\left(r(t_1)^{-\frac{3}{2}+l}\right)$$
$$= \frac{1}{2\pi} m\rho l r(t_1)^{-\frac{1}{2}} \frac{c_1}{(k-1)!} r(t_1)^{-l(k-1)} \Phi_f^{(k)}(\theta^*) + O(r(t_1)^{-1})$$
$$- \frac{1}{4\pi} m\rho r(t_1)^{-\frac{1}{2}} r(t_1)^l \frac{c_2}{k!} r(t_1)^{-lk} \Phi_f^{(k)}(\theta^*) + O\left(r(t_1)^{-\frac{3}{2}+l}\right)$$
$$= \frac{1}{2\pi} \frac{1}{(k-1)!} m\rho \left(c_1 l - \frac{c_2}{2k}\right) r(t_1)^{-\frac{1}{2}-l(k-1)} \Phi_f^{(k)}(\theta^*)$$
$$+ O(r(t_1)^{-1}) + O\left(r(t_1)^{-\frac{3}{2}+l}\right).$$

Since $c_1 = \frac{6k+10}{6k+11}$, $c_2 = \frac{6k+12}{6k+11}$ and $l = \frac{1}{2k-1}$, $k \geqslant 2$, then $c_1 l - \frac{c_2}{2k} > 0$, $-\frac{3}{2} + l < -1 < -\frac{1}{2} - l(k-1)$, which lead to

$$J_1 + J_2 = \frac{1}{2\pi} \frac{1}{(k-1)!} m\rho \left(c_1 l - \frac{c_2}{2k}\right) r(t_1)^{-\frac{1}{2}-l(k-1)} \Phi_f^{(k)}(\theta^*)(1 + o(1)) > 0.$$

That is, if $l = \frac{1}{2k-1}$, then $(r(t)^l \tau(t))'|_{t=t_1} > 0$. Therefore, there exists $t_2 > t_1$ such that $r(t)^{\frac{1}{2k-1}} \tau(t) \geqslant 1$ for $t \in [t_1, t_2]$, which contradicts the definition of t_1. Thus for all $t \in [0, 2\pi]$, one has $\tau(t) \geqslant r(t)^{-\frac{1}{2k-1}}$. The proof of the claim is completed.

Now we prove that every solution of system (5.2) with the initial point $(r(0),\, \tau(0)) \in D$ and $r(0) \geqslant r^*$ is unbounded.

From the claim, if the initial point $(r(0),\, \tau(0)) \in D$ and $r(0) \geqslant r^*$, then for any $t \in [0,\, 2\pi]$, one has $r(t)^{-\frac{1}{2k-1}} \leqslant \tau(t) \leqslant r(t)^{-\frac{1}{3k}} \leqslant 2(r^*)^{-\frac{1}{3k}}$. Thus, from the second equation of (5.2), for any $t \in [0,\, 2\pi]$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{1}{2\pi} m\rho r(t)^{\frac{1}{2}} \Phi_f'(\tau(t)+\theta^*) - \partial_\tau P_1(\tau(t)+\theta^*, t) - \partial_\tau P_2(r(t), \tau(t)+\theta^*, t) \\
&= \frac{1}{2\pi} m\rho r(t)^{\frac{1}{2}} \Phi_f'(\tau(t)+\theta^*) + O(1) \\
&\geqslant \frac{1}{2\pi} m\rho r(t)^{\frac{1}{2}} \frac{c_1}{(k-1)!} \tau(t)^{k-1} \Phi_f^{(k)}(\theta^*) + O(1) \\
&\geqslant \frac{1}{2\pi} m\rho r(t)^{\frac{1}{2}} \frac{c_1}{(k-1)!} r(t)^{-\frac{k-1}{2k-1}} \Phi_f^{(k)}(\theta^*) + O(1) \\
&= \frac{1}{2\pi} \frac{c_1}{(k-1)!} m\rho r(t)^{\frac{1}{2(2k-1)}} \Phi_f^{(k)}(\theta^*) + O(1) \\
&\geqslant \frac{1}{3\pi} \frac{c_1}{(k-1)!} m\rho r(t)^{\frac{1}{2(2k-1)}} \Phi_f^{(k)}(\theta^*).
\end{aligned}$$

Choose r^* sufficiently large such that $\frac{1}{3\pi}\frac{c_1}{(k-1)!} m\rho (\frac{r^*}{2})^{\frac{1}{2(2k-1)}} \Phi_f^{(k)}(\theta^*) \geqslant 1$. Then $r(2\pi) \geqslant r(0) + 2\pi > r^*$.

In a word, if $(r(0),\, \tau(0)) \in D$ and $r(0) \geqslant r^*$, then $(r(2\pi),\, \tau(2\pi)) \in D$ and $r(2\pi) \geqslant r(0) + 2\pi > r^*$.

Using the above argument repeatedly, if $(r(0),\, \tau(0))$ satisfies the above initial conditions, then $r(2\pi i) \geqslant r(0) + 2\pi i$ for any $i \in \mathbb{N}$, which means that the solution $(r(t),\, \tau(t))$ is unbounded. Up to now theorem 1.1 is proved.

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References

1 J. Alonso and R. Ortega. Roots of unity and unbounded motions of an asymmetric oscillator. *J. Differ. Equ.* **143** (1998), 201–220.

2 E. Dancer. On the Dirichlet problem for weakly nonlinear partial differential equations. *Proc. R. Soc. Edinb., Sect. A, Math.* **76** (1977), 283–300.

3 S. Fuçik. *Solvability of Nonlinear Equations and Boundary Value Problems* (Dordrecht: Reidel, 1980).

4 T. Gallouet and O. Kavian. Resonance for jumping nonlinearities. *Commun. Partial Differ. Equ.* **7** (1982), 325–342.

5 P. Habets, M. Ramos and L. Sanchez. Jumping nonlinearities for Neumann boundary value problems with positive forcing. *Nonlinear Anal.* **20** (1993), 533–549.

6 S. Hu, B. Liu and R. Liu. *Invariant curves for quasi-periodic area-preserving mappings and its application*, preprint.

7 A. Lazer and P. McKenna. A semi-Fredholm principle for periodically forced systems with homogeneous nonlinearities. *Proc. Am. Math. Soc.* **106** (1989), 119–125.

8 A. Lazer and P. McKenna. Large-amplitude periodic oscillations in suspensions bridges: some new connections with nonlinear analysis. *SIAM Rev.* **32** (1990), 537–578.

9 M. Li and X. Li. *Unbounded solutions for asymmetric oscillations in the degenerate resonant case*, preprint.

10 M. Li and X. Li. Boundedness in asymmetric oscillations under the non-resonant case. *J. Differ. Equ.* **274** (2021), 828–856.

11 B. Liu. Boundedness in asymmetric oscillations. *J. Math. Anal. Appl.* **231** (1999), 355–373.

12 R. Ortega. Asymmetric oscillators and twist mappings. *J. London Math. Soc.* **53** (1996), 325–342.

13 R. Ortega. On Littlewood's problem for the asymmetric oscillator. *Milan J. Math.* **68** (1998), 153–164.

14 R. Ortega. Invariant curves of mappings with averaged small twist. *Adv. Nonlinear Stud.* **1** (2001), 14–39.

15 X. Xing, J. Wang and Y. Wang. Boundedness of semilinear Duffing equations at resonance in a critical situation. *J. Differ. Equ.* **266** (2019), 2294–2326.

16 M. Zhang. Nonresonance conditions for asymptotically positively homogeneous differential systems: the Fuçik spectrum and its generalization. *J. Differ. Equ.* **145** (1998), 332–366.