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OPTIMAL PRESENTATIONS FOR SOLVABLE 2-KNOT GROUPS

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We find presentations for the groups of Cappell-Shaneson 2-knots and other solvable 2-knot groups which are optimal in terms of deficiency and number of generators.

Let π be a virtually torsion free solvable 2-knot group. If π has deficiency 1 then $\pi \cong Z$ or $\Phi = Z*_2$ (the group with presentation $\langle a, t \mid tat^{-1} = a^2 \rangle$), by [**4**, Theorems III.1 and 2]. Otherwise π has deficiency $\leqslant 0$, and either is torsion free polycyclic of Hirsch length 4 or has finite commutator subgroup, by [**4**, Theorems VI.13 and 14]. Using more recent work it can be shown that these are the only coherent, elementary amenable 2-knot groups [**5**]. In this note we shall find presentations for the torsion free polycyclic 2-knot groups which are optimal both in terms of deficiency and numbers of generators. We shall also show that 2-knot groups with nontrivial finite commutator subgroup have 2-generator presentations of deficiency -1 or 0.

A choice of normal generator for π determines an isomorphism $\mathbb{Z}[\pi/\pi'] \cong \Lambda = \mathbb{Z}[Z] = \mathbb{Z}[t, t^{-1}]$. If X is a space with $\pi_1(X) \cong \pi$ and X' is its infinite cyclic covering space we shall let $H_*(X; \Lambda)$ denote $H_*(X'; \mathbb{Z})$ with the natural Λ-module structure.

LEMMA. *Let π be a finitely presentable group such that $\pi/\pi' \cong Z$, and let $R = \Lambda$ or $\Lambda/p\Lambda$ for some prime $p \geqslant 2$. Then*

(i) *if π can be generated by 2 elements $H_1(\pi; R)$ is cyclic as an R-module;*
(ii) *if $\operatorname{def}(\pi) = 0$ then $H_2(\pi; R)$ is cyclic as an R-module.*

PROOF: If π is generated by two elements t and x, say, we may assume that the image of t generates π/π' and that $x \in \pi'$. Then π' is generated by the conjugates of x under powers of t, and so $H_1(\pi; R) = R \otimes_\Lambda (\pi'/\pi'')$ is generated by the image of x.

If X is the finite 2-complex determined by a deficiency 0 presentation for π then $H_0(X; R) = R/(t-1)$ and $H_1(X; R)$ are R-torsion modules, and $H_2(X; R)$ is a submodule of a finitely generated free R-module. Hence $H_2(X; R) \cong R$, as it has rank 1 and R is an UFD. Therefore $H_2(\pi; R)$ is cyclic as an R-module, since it is a quotient of $H_2(X; R)$, by Hopf's theorem. ∎

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If π is torsion free polycyclic and $\pi' \neq 1$ then π' is either virtually Abelian or virtually nilpotent, of Hirsch length 3, and the knot complement is determined up to homeomorphism by π and the conjugacy class of a normal generator. (See [**4**, Chapter VIII]. If $\pi' \cong Z^3$ then $\pi \cong Z^3 \times_C Z$, where $C \in SL(3,\mathbb{Z})$ and $\det(C-I) = \pm 1$. (The corresponding knots are the Cappell-Shaneson 2-knots [**2**].) Every such matrix is conjugate to one with first row $(0,0,1)$. (See [**1**, Theorem A.3].) Hence π has a deficiency -1 presentation

$$\langle t,x,y,z \mid xz = zx,\ yz = zy,\ txt^{-1} = y^m z^n,\ tyt^{-1} = y^p z^q,\ tzt^{-1} = xy^r z^s \rangle.$$

We may easily obtain a 3-generator 4-relator presentation by using the final relation to eliminate x. If $n = p = 0$ then π has a 2-generator 2-relator presentation

$$\langle t,x \mid txt^{-1}x = xtxt^{-1},\ t^3xt^{-3} = xtx^r tx^s t^{-2} \rangle.$$

THEOREM. *Let $\pi = Z^3 \times_C Z$ be the group of a Cappell-Shaneson 2-knot, and let $\Delta(t) = \det(tI - C)$. Then the following are equivalent*

(i) *π has a 2-generator 2-relator presentation;*
(ii) *π is generated by 2 elements;*
(iii) *$\operatorname{def}(\pi) = 0$;*
(iv) *π' is cyclic as a Λ-module;*
(v) *the ideal generated by $(t-p, m+pr)$ in the domain $\Lambda/(\Delta(t))$ is principal.*

(Here we assume that the presentation is as given above.)

PROOF: Condition (i) implies (ii) and (iii), since $\operatorname{def}(\pi) \leqslant 0$, as observed above, while (ii) implies (iv), by the Lemma. Conversely if π' is generated as a Λ-module by x then it is easy to see that π has a presentation of the form $\langle t,x \mid [x,\ txt^{-1}] = 1,\ t^3xt^{-3} = t^2x^a t^{-2} tx^b t^{-1} x^c \rangle$, and so (i) holds. Conditions (iv) and (v) are equivalent since the isomorphism class of π' is that of its Steinitz-Fox-Smythe row invariant, which is the class of the ideal $(t-p, m+pr)$. (See [**3**, Theorem III.12].) Suppose finally that $\operatorname{def}(\pi) = 0$. Then $H_2(\pi;\Lambda)$ is cyclic as a Λ-module, by the Lemma. Since $\pi' = H_1(\pi;\Lambda) \cong \overline{H^3(\pi;\Lambda)} \cong \overline{Ext^1_\Lambda(H_2(\pi;\Lambda),\Lambda)}$, by Poincaré duality and the Universal Coefficient spectral sequence, it is also cyclic and so (iv) holds. ∎

See the tables in [**1**] for some computations of the ideal class groups for such domains $\Lambda/(\Delta)$, with Δ a cubic knot polynomial. In particular, their concluding example gives rise to the group with (optimal) presentation

$$\langle t,y,z \mid tzt^{-1}z = ztzt^{-1},\ yz = zy,\ tz^7tzt^{-2} = y^{-5}z^{-8},\ tyt^{-1} = y^2z^3 \rangle,$$

whose commutator subgroup is not cyclic as a Λ-module.

If π' is virtually Abelian but not Z^3 then it is the fundamental group of the flat orientable 3-manifold with noncyclic holonomy. There are two such groups $\pi \cong G(\pm)$, with presentations

$$\langle t, x, y \mid xy^2x^{-1} = y^{-2},\ txt^{-1} = (xy)^{\mp 1},\ tyt^{-1} = x^{\pm 1}\rangle.$$

Using the final relation to eliminate the generator y gives 2-generator presentations of deficiency 0. The group $G(+)$ is the group of the 3-twist spin of the figure eight knot. On the other hand $G(-)$ is not the group of any twist spin.

If π' is a torsion free nonabelian nilpotent group then it is isomorphic to the group Γ_q with presentation $\langle x, y, z \mid xz = zx,\ yz = zy,\ [x, y] = z^q\rangle$, for some odd $q \geqslant 1$. There are three groups with $\pi' \cong \Gamma_1$, with presentations

$$\langle t, x, y \mid xyxy^{-1} = yxy^{-1}x,\ txt^{-1} = xy,\ tyt^{-1} = w\rangle,$$

where $w = x^{-1}$, xy^2 or x. The groups with $\pi' \cong \Gamma_q$ for $q > 1$ (and odd) have presentations

$$\langle t, u, z \mid [u, tut^{-1}] = z^q,\ z = t^{-1}ututu^{-1}t^{-1},\ tzt^{-1} = z^{-1}\rangle.$$

In all cases we may use one of the relations to eliminate one of the generators, giving 2-generator presentations of deficiency 0. The group of the 6-twist spin of the trefoil has commutator subgroup Γ_1 (corresponding to $w = x^{-1}$ above). None of the other groups with infinite nilpotent commutator subgroup are realised by twist spins.

The other polycyclic 2-knot groups are the groups $\pi(b, \varepsilon)$ of the 2-twist spins of the Montesinos knots $K(0|b; (3,1), (3,1), (3,\varepsilon))$, where b is even and $\varepsilon = \pm 1$. These groups have presentations

$$\langle t, x, y \mid x^3 = y^3 = \left(x^{1-3b}y\right)^{-3\varepsilon},\ txt^{-1} = x^{-1},\ tyt^{-1} = xy^{-1}x^{-1}\rangle.$$

(Note that there is an error near the end of [**4**, Theorem VI.11]. The outer automorphism classes containing meridianal automorphisms should be jc, jck and jck^2.) In all cases $\pi'/\pi'' \cong (\Lambda/(3, t+1))^2$, and so $H_1(\pi; R) \cong H_2(\pi; R) \cong (R/(t+1))^2$, where $R = \Lambda/3\Lambda$. Thus these 3-generator deficiency -1 presentations are optimal, by the Lemma.

If the commutator subgroup π' of a 2-knot group π is finite then $\pi' \cong P \times (Z/nZ)$ where $P = 1$, $Q(8)$ (the quaternion group of order 8), $I^* = SL(2, \mathbb{F}_5)$ (the binary icosahedral group), or $T_k^* = Q(8) \widetilde{\times} (Z/3^kZ)$ (a central extension of the binary tetrahedral group T_1^*), and $(n, 2|P|) = 1$, and the ("meridianal") action of $\pi/\pi' \cong Z$ on π' is essentially unique. (See [**4**, Theorem IV.3].) Excepting only the cases with

$\pi' \cong Q(8) \times (Z/nZ)$ with n odd and > 1 each of these groups is realised by a twist spin of a classical knot [**6**]. In particular, if $\pi' \cong Z/nZ$ then π is the group of the 2-twist spin of a 2-bridge knot, and has a 2-generator deficency 0 presentation $\langle a, t \mid tat^{-1} = a^{-1},\ a^n = 1\rangle$. If $\pi' \cong Q(8)$, T_1^* or I^* then π is the group of the 3-, 4- or 5-twist spin of the trefoil knot (respectively), and so again has a 2-generator presentation of deficiency 0, of the form $\langle a, t \mid tat^{-1} = at^2at^{-2},\ t^r a = at^r\rangle$, for $r = 3$, 4 or 5. These presentations are clearly optimal.

If $P = Q(8)$ then π has a 2-generator deficiency 0 presentation

$$\langle t, u \mid tu^2t^{-2} = u^{-2},\ t^2u^nt^{-2} = u^ntu^nt^{-1}\rangle.$$

(Let $x = u^n$, $y = tu^nt^{-1}$ and $z = u^4$. Then these relations imply $y^2 = x^{-2}$, $x^2 = (xy)^2$, $xz = zx$ and $tzt^{-1} = z^{-1}$, and so $yz = zy$ and $x = yxy$. Hence $x^3 = yxy^{-1}$ and $x^6 = yx^2y^{-1} = x^2$, and so $x^4 = 1$ and $z^n = 1$. Thus we obtain an equivalent presentation $\langle t, x, y, z \mid x^2 = (xy)^2 = y^2,\ xz = zx,\ yz = zy,\ z^n = 1,\ txt^{-1} = y,\ tyt^{-1} = xy,\ tzt^{-1} = z^{-1}\rangle$, from which it is easy to see that $\pi' \cong Q(8) \times (Z/nZ)$ and that the conjugation by t is as in [**4**, Theorem IV.3]. When $n = 1$ we may relate this to the presentation derived from the 3-twist spin of the trefoil knot, if we set $x = a$, $y = tat^{-1}$ and replace t by $t_1 = ta$.)

If $P = T_k^*$ then π has a presentation $\langle s, x, y, z \mid x^2 = (xy)^2 = y^2,\ zxz^{-1} = y,\ zyz^{-1} = xy,\ z^\alpha = 1,\ sxs^{-1} = y^{-1},\ sys^{-1} = x^{-1},\ szs^{-1} = z^{-1}\rangle$, where $\alpha = 3^kn$. (When $n = k = 1$ we may relate this to the presentation derived from the 4-twist spin of the trefoil knot, if we set $x = tat^{-1}a$, $y = atat^{-1}$, $z = ax^2$ and $s = xat$.) This is equivalent to the presentation $\langle s, x, y, z \mid z^\alpha = 1,\ zxz^{-1} = y,\ zyz^{-1} = xy,\ sxs^{-1} = y^{-1},\ szs^{-1} = z^{-1}\rangle$. (For conjugating $y = zxz^{-1}$ by s gives $sys^{-1} = x^{-1}$, while conjugating $zyz^{-1} = xy$ by s gives $x = yxy$, so $x^2 = y^2$, and conjugating this by z gives $y^2 = (xy)^2$.) Let $t = sxz$. Then $tx = xt$ and $tzt^{-1} = sxzx^{-1}s^{-1} = szxyx^{-1}s^{-1} = z^{-1}x$. Hence we obtain the presentation

$$\langle t, x, y, z \mid z^\alpha = 1,\ x = ztzt^{-1},\ y = z^2tzt^{-1}z^{-1},\ zyz^{-1} = xy,\ tx = xt\rangle.$$

We may use the second and third relations to eliminate the generators x and y, to obtain a 2-generator presentation of deficiency -1.

If $P = I^*$ then $(n, 30) = 1$ and π has a presentation $\langle s, x, y, z \mid x^2 = (xy)^3 = y^5,\ xz = zx,\ yz = zy,\ z^n = 1,\ sx = xs,\ sy = ys,\ szs^{-1} = z^{-1}\rangle$, and y represents an element of order 10. (Hence $y^5 = y^{-5}$.) Let $t = sx^{-1}y^{-1}$, $a = yxy$ and $b = y^{-1}$. Then the presentation is equivalent to $\langle t, a, b, z \mid a = btat^{-1},\ b = t^{-1}at,\ az = za,\ z^n = 1,\ t^5a = at^5,\ tzt^{-1} = z^{-1}\rangle$, and a represents an element of order 10. Let p and q satisfy $np \equiv 1$ modulo (10) and $10q \equiv 1$ modulo (n), and let $w = a^pz^q$. Then $w^n = a$ and $w^{10} = z$. Hence we obtain the equivalent 2-generator presentation

$$\langle t, w \mid tw^nt^{-1} = w^nt^2w^nt^{-2},\ t^5w^n = w^nt^5,\ tw^{10}t^{-1} = w^{-10}\rangle.$$

Since $w^{10n} = 1$ this simplifies to the deficiency 0 presentation $\langle t, w \mid twt^{-1} = wt^2wt^{-2},\ t^5w = wt^5\rangle$ when $n = 1$.

We do not know whether any of the groups with $\pi' \cong T_k^* \times (Z/nZ)$ and $nk > 1$ or $I^* \times (Z/nZ)$ and $n > 1$ have deficiency 0.

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