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ON THE EQUIVARIANT FORMALITY OF KÄHLER MANIFOLDS WITH FINITE GROUP ACTION

BENJAMIN L. FINE AND GEORGIA TRIANTAFILLOU

ABSTRACT An appropriate definition of equivariant formality for spaces equipped with the action of a finite group G, and for equivariant maps between such spaces, is given Kahler manifolds with holomorphic G-actions, and equivariant holomorphic maps between such Kahler manifolds, are proven to be equivariantly formal, generalizing results of Deligne, Griffiths, Morgan, and Sullivan

Introduction. In this note we study the question of formality for spaces equipped with an action by a finite group G and we prove that G-Kähler manifolds are equivariantly formal. This generalizes a result by Deligne, Griffiths, Morgan and Sullivan in the non-equivariant case [DGMS].

Recall that a space X is formal if its rational homotopy type is determined by its cohomology, *i.e.* there is a cohomology isomorphism

$$\rho\colon \mathcal{M} \longrightarrow H^*(X;\mathbb{Q}),$$

where $\mathcal{M}$ is the minimal model of X as in [S]. There are large classes of spaces which are formal, *e.g.*, Lie groups, classifying spaces [S], $(r-1)$-connected manifolds of dimension $< 4r-1$ [M], 1-connected Kähler manifolds [DGMS]. An algorithmic method to decide when a space is formal is given in [HS].

Now let X be a G-CW complex such that X^H is nonempty and simply-connected for all $H \subseteq G$. The notion of homotopy we consider in this context is equivariant homotopy of G-maps. We recall a result of [B] which characterizes G-homotopy equivalences for G-CW complexes, namely a G-map $f\colon X \longrightarrow Y$ is a G-homotopy equivalence if and only if it induces isomorphisms $(f^H)_*\colon \pi_*(X^H) \longrightarrow \pi_*(Y^H)$ for all subgroups H of G. An appropriate notion of rational G-homotopy type was developed in [T] where all fixed point sets are rationalized at the same time. Also for any space X as above an equivariant minimal model was constructed which determines the equivariant rational homotopy type of X.

We want to investigate the question when the G-homotopy type of a G-space X is determined rationally by the cohomology algebras $H^*(X^H;\mathbb{Q})$ of the fixed point sets of the subgroups $H \subseteq G$. In order to make this precise we recall the following: Let $\mathcal{O}_G$ be the *category of canonical orbits*; its objects are the quotients G/H, $H \subseteq G$, and the morphisms are the equivariant maps. A *system of DGA's* is a contravariant functor from $\mathcal{O}_G$ into the category of graded-commutative differential graded algebras over the rationals.

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Examples of systems of DGA's associated to a G-complex X are the system $\underline{H}^*(X;\mathbb{Q})$ of the cohomology of the fixed point sets and the system $\underline{\mathcal{E}}_X$ of de Rham algebras of the fixed point sets defined by

$$\underline{H}^*(X;\mathbb{Q})(G/H) \equiv H^*(X^H;\mathbb{Q})$$

and

$$\underline{\mathcal{E}}_X(G/H) \equiv \mathcal{E}_{X^H}$$

respectively for $H \subseteq G$. The equivariant minimal model $\mathcal{M}$ of X constructed in [T] is also a system of DGA's and there is a map from $\mathcal{M}$ into $\underline{\mathcal{E}}_X$ which is a cohomology isomorphism. It has the analogous properties in the equivariant category as Sullivan's minimal model in the non-equivariant one.

It seems that the right definition for equivariant formality would be to require that there is a cohomology isomorphism from the equivariant minimal model of a G-space X to the system of cohomology algebras of the fixed point sets $\underline{H}^*(X;\mathbb{Q})$. However, not all systems of DGA's admit an equivariant minimal model. The crucial property needed is *injectivity* in the sense of category theory. A system of DGA's can be considered as an object of a certain abelian category namely the category of functors from O_G into the category of vector spaces (by neglect of structure). A system of DGA's is *injective* if it is an injective object in this abelian category.

As shown in [T] the system $\underline{\mathcal{E}}_X$ of the de Rham algebras of the fixed point sets of X is always injective and equivariant minimal models for G-spaces with nonempty nilpotent fixed point sets can be constructed. However the system $\underline{H}^*(X;\mathbb{Q})$ is almost never injective. Therefore the definition of formality does not generalize in a straightforward way to the equivariant context.

We solve this problem by constructing injective envelopes for arbitrary systems of DGA's as follows:

THEOREM 1. *For any system of* DGA*'s $\mathcal{A}$ over a finite group G there is an injective system of* DGA*'s I and an inclusion $i\colon \mathcal{A} \longrightarrow I$ which is a cohomology isomorphism.*

This I satisfies a certain uniqueness property. A construction of injective envelopes for arbitrary systems of vector spaces was given in [T]. The novel feature in the above result is that one has to take into account the algebra structure as well. This makes the construction more involved.

All G-spaces considered in this paper are G-CW-complexes X such that all fixed-point sets X^H are nonempty connected and nilpotent. The first-named author is presently working to extend the theory to the case of non-connected fixed-point sets. For G-spaces which satisfy our assumptions we give the following definition:

DEFINITION 2. A G-space X is said to be *equivariantly formal* if there is a cohomology isomorphism from the equivariant minimal model $\mathcal{M}$ of X into the injective envelope I of $\underline{H}^*(X;\mathbb{Q})$.

This means that the system of the rational cohomology algebras of the fixed point sets of X determines the equivariant rational homotopy type of X. The existence of the injective envelope for systems of DGA's is the crucial step towards attacking the formality problem for various classes of G-spaces. For instance, having the injective envelope one can prove the following:

THEOREM 3. *Let G be a finite group and let X be a Kähler manifold which admits a holomorphic G-action with nonempty connected and simply-connected fixed-point sets. Then X is equivariantly formal. Moreover, equivariant holomorphic maps between G-Kähler manifolds are formal, i.e. they are determined rationally by the maps induced on cohomology.*

We note that all the fixed-point sets of the spaces of the theorem are Kähler manifolds and therefore they are formal by [DGMS]. Formality of the fixed-point sets is not sufficient for equivariant formality, however. The second-named author has constructed in [T3] an infinite set of $\mathbb{Z}_p \times \mathbb{Z}_q$-spaces which belong to distinct rational equivariant homotopy types but which have formal fixed-point sets and the same system of cohomology algebras of fixed point sets.

REMARK. In the dual category of systems of differential graded Lie algebras one can construct equivariant minimal models for arbitrary—not necessarily projective—systems. Therefore, the definition of equivariant formality is straightforward in this dual sense. This is the context in which [RT] proved that the equivariant classifying space $BO(\alpha)$ is equivariantly formal when G is abelian. The space $BO(\alpha)$ classifies certain G-vector bundles modeled by a given representation $\alpha: G \longrightarrow O(n)$. Theorem 3 above is applied in [RT2] to prove that the equivariant classifying space $BU(\alpha)$ is equivariantly formal for an *arbitrary* finite group G, where $BU(\alpha)$ classifies complex vector bundles modeled by a representation $\alpha: G \longrightarrow U(n)$.

Several other attempts have been made to determine the G-homotopy type rationally from the cohomology of the fixed point sets. In [T3] it is shown that H-spaces with compatible $\mathbb{Z}_{p^k}$-action are equivariantly formal (in fact they split rationally into products of equivariant Eilenberg-Mac Lane spaces) and that this is not in general the case for more complicated groups. An alternative definition of equivariant formality is given in [L] for the case of $\mathbb{Z}_{p^k}$-actions. It is also shown in [L] that a $\mathbb{Z}_p$-space X is equivariantly formal if and only if the inclusion map $X^{\mathbb{Z}_p} \longrightarrow X$ is formal. The latter result can not be generalized to arbitrary G-actions as the above mentioned examples of $\mathbb{Z}_p \times \mathbb{Z}_q$-actions of [T3] show. Although all inclusion maps of the fixed point sets are formal the spaces themselves are not equivariantly formal.

As in the non-equivariant case we show first that G-Kähler manifolds are equivariantly formal over the complex numbers $\mathbb{C}$ and then we apply the following result.

THEOREM 4. *A system of* DGA*'s $\mathcal{A}$ over $\mathbb{Q}$ is formal iff $\mathcal{A} \otimes \mathbb{C}$ is formal over $\mathbb{C}$.*

Proofs. In order to prove Theorem 1 we need the following constructions.

Let G be a finite group and let O_G be the category of canonical orbits of G defined above. We fix a field $\mathbb{F}$ of characteristic 0 and we define a (contravariant) *coefficient system for* G to be a contravariant functor from O_G to the category of vector spaces over $\mathbb{F}$. Let Vec_G denote the category of such coefficient systems. A functor from O_G to the category of DGA's is called a *system of* DGA*'s*.

The injective objects of the abelian category Vec_G play a crucial role in the construction of the equivariant minimal model of a G-space in [T]. A description of the injectives of Vec_G is given in [T] as follows: An injective system S can be written as

$$S = \bigoplus_{(H)} \underline{W}_H,$$

where the direct sum is over the congugacy classes of the subgroups of G. Here each $\underline{W}_H$ is determined by an $\mathbb{F}\big(N(H)/H\big)$-module W_H by the formula

$$\underline{W}_H(G/K) = \operatorname*{Hom}_{\mathbb{F}(N(H)/H)}\Big(\mathbb{F}\big((G/H)^K\big), W_H\Big),$$

for $K \subseteq G$.

DEFINITION 5. Let $\mathcal{A}_H$ be a DGA over $\mathbb{F}$ such that N(H)/H acts on it by DGA automorphisms. The *associated system* $\underline{\mathcal{A}}_H$ of $\mathcal{A}_H$ is a system of DGA's defined as follows: Let V_H be a copy of $\mathcal{A}_H$ considered as a graded $\mathbb{F}\big(N(H)/H\big)$-module by neglect of structure and let $\underline{V}_H$ be the induced injective system of vector spaces as above. Let $s\underline{V}_H$ be a copy of $\underline{V}_H$ with a shift of degree by +1. We denote by Λ_H the system of acyclic DGA's generated by $\underline{V}_H \oplus s\underline{V}_H$, where $d(\underline{V}_H) = s\underline{V}_H$. Now we define the associated system $\underline{\mathcal{A}}_H$ by

$$\underline{\mathcal{A}}_H(G/K) \equiv \Lambda_H(G/K),$$

for $(K) = (H)$ and

$$\underline{\mathcal{A}}_H(G/K) \equiv \operatorname*{Hom}_{\mathbb{F}(N(H)/H)}\big((G/H)^K, \mathcal{A}_H\big),$$

for $(K) \neq (H)$, where (H) is the conjugacy class of H in G. The value of this functor on morphisms is the obvious one.

PROPOSITION 6. *Let $\mathcal{B}$ be an injective system of* DGA*'s and let f be an $N(H)/H$-equivariant algebra map from $\mathcal{A}_H$ as above into the subalgebra $\bigcap_{H' \underset{\neq}{\supset} H} \ker \beta_{H,H'}$ of $\mathcal{B}(G/H)$, where $\beta_{H,H'}\colon \mathcal{B}(G/H) \longrightarrow \mathcal{B}(G/H')$ is the morphism induced by the projection $G/H \longrightarrow G/H'$. Then f extends to a map of systems of* DGA*'s from $\underline{\mathcal{A}}_H$ to $\mathcal{B}$.*

PROOF. Because each DGA $\Lambda_H(G/K)$ is free and acyclic for any subgroup K of G not conjugate to H, it suffices to extend the map f to a map of systems of vector spaces from $\underline{V}_H$ into $\mathcal{B}$. Since $\mathcal{B}$ is injective it can be written as a direct sum $\mathcal{B} = \oplus_{(H)}\underline{W}_H$ as we mentioned above. By assumption the map f takes values in $W_H \equiv \underline{W}_H(G/H)$. Now a map f between two $\mathbb{F}\big(N(H)/H\big)$-modules V_H and W_H extends uniquely to a map between the induced injective systems $\underline{V}_H$ and $\underline{W}_H$ This is an immediate consequence of the definition of the induced injective systems of vector spaces This completes the proof

DEFINITION 7 Let $\mathcal{A}$ be a system of DGA's and let $\mathcal{A}_H$ be the subalgebra of $\mathcal{A}(G/H)$ which is equal to $\bigcap_{H' \supsetneq H} \ker \alpha_{H H'}$, where $\alpha_{H H'}$ is the morphism induced by the projection $G/H \longrightarrow G/H'$ Let $\underline{\mathcal{A}}_H$ be the associated system to $\mathcal{A}_H$ The *enlargement* of $\mathcal{A}$ at H is the system of DGA's $I_H(\mathcal{A})$ defined by

$$I_H(\mathcal{A})(G/K) \equiv \mathcal{A}(G/K) \otimes \underline{\mathcal{A}}_H(G/K),$$

for $K < H$, and

$$I_H(\mathcal{A})(G/K) \equiv \mathcal{A}(G/K)$$

otherwise, where $K < H$ means that K is a proper subgroup of a conjugate of H The value of the functor $I_H(\mathcal{A})$ on morphisms is the obvious one, namely, they are equal to the old morphisms when restricted to the subsystems $\mathcal{A}$ and $\underline{\mathcal{A}}_H$ respectively

PROPOSITION 8 *Let μ $\mathcal{A} \longrightarrow \mathcal{B}$ be a map of systems of* DGA*'s, where $\mathcal{B}$ is injective Then μ can be extended to the enlargement $I_H(\mathcal{A})$ of $\mathcal{A}$ at H*

PROOF This is an easy consequence of Proposition 6

PROOF OF THEOREM 1 The construction of I is done inductively by taking successive enlargements at the subgroups of G, where we consider one subgroup for each conjugacy class of subgroups Consider first $\mathcal{A}(G/G)$, its associated system $\underline{\mathcal{A}}(G/G)$, and the enlargement $I_G(\mathcal{A})$ of $\mathcal{A}$ at G Note that by construction there is an inclusion of $\mathcal{A}$ into its enlargement at any H and this map is a cohomology isomorphism Next we choose a maximal subgroup K of G and we construct the enlargement of $I_G(\mathcal{A})$ at K We pick only one subgroup of G in a given conjugacy class in this process of successive enlargements Now let (H) be an arbitrary conjugacy class of subgroups of G and assume inductively that the system of DGA's $I_{>H}(\mathcal{A})$ has been constructed by successive enlargements at the subgroups H' of G which properly contain H or a conjugate of it We construct the enlargement of $I_{>H}(\mathcal{A})$ at H and we continue this process until H is the trivial subgroup We denote the resulting system I It remains to be shown that I is injective, *i e* that it can be written additively as a direct sum of $\underline{V}_H$'s Again the argument goes inductively over the subgroups of G The first enlargement $I_G(\mathcal{A})$ contains by construction the injective system of vector spaces $\underline{V}_G$ and therefore it splits

$$I_G(\mathcal{A}) = \underline{V}_G \oplus \mathcal{R}_G,$$

where $\mathcal{R}_G(G/G) = 0$ Assume inductively that

$$I_{>H} = \bigoplus_{(H') H' > H} \underline{V}_{H'} \oplus \mathcal{R}_H,$$

where $\mathcal{R}_H(G/H') = 0$ for all subgroups $H' > H$ The next enlargement at H contains the injective subsystem of vector spaces $\underline{V}_H$, where V_H is the intersection of kernels of

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maps emanating from $I_{>H}(\mathcal{A})(G/H)$. This intersection is exactly $\mathcal{R}_H(G/H)$. Moreover, $\underline{V}_H$ does not intersect any of the $\underline{V}_{H'}$. Therefore, there is a splitting

$$I_{H'\geq H}(\mathcal{A}) = \bigoplus_{H'>H} \underline{V}_{H'} \oplus \underline{V}_H \oplus \tilde{\mathcal{R}},$$

where $\tilde{\mathcal{R}}(G/H') = 0$ for all subgroups $H' \geq H$. This completes the inductive step and the proof of Theorem 1.

Following [S], we shall prove that a minimal system is formal over $\mathbb{C}$ iff it is formal over $\mathbb{Q}$. Henceforward, if $\mathcal{A}$ is a system of DGA's we denote by $\tilde{\mathcal{A}}$ its injective envelope in the sense of Theorem 1.

DEFINITION 9. Let $\underline{H}$ be the cohomology system associated to a system of DGA's $\mathcal{A}$ defined over a field $\mathbb{F}$ of characteristic 0. Given $\alpha \in \mathbb{F}$, $\alpha \neq 0$, we define the *grading automorphism* ϕ_α of $\underline{H}$ by $\phi_\alpha(x) = \alpha^{\deg x} \cdot x$ on homogeneous elements.

Note that ϕ_α extends to $\underline{\tilde{H}}$, and the extension denoted again ϕ_α is diagonal with eigenvalues equal to powers of α. This follows from the definition of enlargement and Proposition 6.

Let $\mathcal{M}$ be the minimal model of $\tilde{\mathcal{A}}$.

PROPOSITION 10. *The following are equivalent:*

(1) $\mathcal{A}$ is formal;

(2) All automorphisms of $\underline{H}$ lift to automorphisms of $\mathcal{M}$; and

(3) All grading automorphisms of $\underline{H}$ lift to $\mathcal{M}$.

Before we proceed with the proof we will prove the following lemma and its corollary.

LEMMA 11. *If there is a lift $f\colon \mathcal{M} \longrightarrow \mathcal{M}$ of ϕ_α then there is a diagonalizable lift. Moreover the eigenvalues of any lift are equal to natural powers of α.*

PROOF OF LEMMA. We recall from [T2] that $\mathrm{Aut}(\mathcal{M})$ is a linear algebraic group and specifically it is an algebraic subgroup of $\mathrm{GL}\big(\bigoplus_{H\subseteq G} \mathcal{M}(G/H)\big)$, where we consider $\mathcal{M}(G/H)$ as a vector space by neglect of structure. We recall that a linear algebraic group over $\mathbb{F}$ is a subgroup of some $\mathrm{GL}(V)$ defined by polynomial equations on the entries of its elements considered as matrices. Here V is a vector space over $\mathbb{F}$. Also it is shown in [T2] that the map $A\colon \mathrm{Aut}(\mathcal{M}) \longrightarrow \mathrm{Aut}(\underline{H})$ is a map of algebraic groups.

By a basic fact in the theory of algebraic groups any element of an algebraic group over an algebraically closed field of characteristic 0 can be written uniquely as a product of its semisimple and its unipotent part. Moreover this splitting is canonical with respect to algebraic homomorphisms (*cf.* [H] p. 99). This is the Jordan-Chevalley multiplicative decomposition. By the exact same argument this result holds for any field $\mathbb{F}$ of characteristic 0 provided that all eigenvalues of the given element lie in $\mathbb{F}$. We will show below that the eigenvalues in our case satisfy this property.

Granted the latter fact, any automorphism of $\mathcal{M}$ which is a lift of ϕ_α can be written as a composition of a diagonalizable automorphism (diagonalizable at every G/H in a compatible way) and a unipotent automorphism Furthermore its unipotent part induces the identity on cohomology because the identity is the unipotent part of ϕ_α Hence the diagonalizable part of any lift of ϕ_α is also a lift

It remains to show that the eigenvalues of any lift of ϕ_α are natural powers of α There are two approaches in showing this We can either work with an arbitrary lift f itself over the field $\mathbb{F}$ or we can go to the algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$ and consider the diagonalizable part of $f \otimes \bar{\mathbb{F}}$ There is no essential difference in the inductive procedure in either case and we choose the second approach

So from now on let Φ_α be a diagonal lift of ϕ_α We proceed by induction on the natural filtration

$$\subseteq \mathcal{M}(n-1) \subseteq \mathcal{M}(n) \quad \subseteq \mathcal{M}$$

of $\mathcal{M}$, where $\mathcal{M}(n)$ is characterized by the properties that it is a minimal subsystem of $\mathcal{M}$ and the inclusion induces an isomorphism on cohomology in degrees $\leq n$ and an injection on degree $n+1$ Assume inductively that the eigenvalues of Φ_α when restricted to $\mathcal{M}(n-1)$ are natural powers of α We know from [T] that $\mathcal{M}(n)$ is constructed from $\mathcal{M}(n-1)$ by adjoining W_n and W'_n and their minimal injective resolutions Here

$$W_n \cong \operatorname{coker}\Big(\underline{H}^n\big(\mathcal{M}(n-1)\big) \longrightarrow \underline{H}^n(\mathcal{M})\Big)$$

and

$$0 \longrightarrow W_n \longrightarrow W_{n\,0} \longrightarrow W_{n\,1} \longrightarrow \quad \longrightarrow W_{n\,k} \longrightarrow 0$$

is a minimal injective resolution of W_n Similarly $\{W'_{n_J}\}$ is a minimal injective resolution of

$$W'_n \cong \ker\Big(\underline{H}^{n+1}\big(\mathcal{M}(n-1)\big) \longrightarrow \underline{H}^{n+1}(\mathcal{M})\Big)$$

In degree n we have

$$\mathcal{M}^n = \mathcal{M}(n-1)^n \oplus W_{n\,0} \oplus W'_{n\,0}$$

The splitting can be chosen so that all summands are invariant under the map Φ_α This can be seen from the fact that any injective system V splits naturally $V = \oplus_{(H)} \underline{V}_H$, where $V_H = \bigcap_{H \underset{\neq}{\supset} H} \ker\big(V(G/H) \longrightarrow V(G/H')\big)$ In our case each $\mathcal{M}^n_H$ which is certainly invariant under Φ_α splits further into $\mathcal{M}(n-1)^n_H \oplus W_H \oplus W'_H$, where the differential is equal to 0 on W_H and it is decomposable on W'_H mapping into nontrivial cocycles of $\mathcal{M}(n-1)(G/H)$ This splitting can obviously be done so as to be compatible with Φ_α and equivariant with respect to the action of $N(H)/H$ Taking sums we have $W_{n\,o} = \oplus_{(H)} \underline{W}_H$, $W'_{n\,o} = \oplus_{(H)} \underline{W}'_H$ and so on This gives the desired splitting of $\mathcal{M}^n$ which is compatible with Φ_α

In order to complete the inductive step it suffices to show that the eigenvalues of the restrictions of Φ_α to W_H and W'_H are natural powers of α since these restrictions uniquely determine the map on the entire minimal injective resolutions of W_n and W'_n respectively and this map has the same eigenvalues By the properties of the differential when restricted to W_H and W'_H these spaces inject into the cohomology of the indecomposables $H^n\big(Q\mathcal{M}(G/H)\big)$, where the latter is the dual homotopy of the differential algebra $\mathcal{M}(G/H)$. Moreover W_H injects into the spherical homotopy whereas W'_H injects into a complement of it. By the nonequivariant argument Φ_α acts on the spherical part of dual homotopy by multiplication by α^n and on the complement by multiplication by powers α^p where $p > n$. Therefore:

CLAIM. The restriction of Φ_α on $\{W_{n,\imath}\}$ is given by multiplication by α^n, and its restriction on $\{W'_{n,\jmath}\}$ is given by multiplication by powers α^p for $p > n$.

This completes the proof of the lemma.

COROLLARY 12. *The kernel of the map* A: $\mathrm{Aut}(\mathcal{M}) \longrightarrow \mathrm{Aut}\big(\underline{H}(\mathcal{M})\big)$ *is a unipotent subgroup.*

PROOF. As before any lift of the identity on cohomology can be written as a product of a diagonalizable and a unipotent element. Applying the argument of Lemma 11 for $\alpha = 1$ yields that the diagonalizable part is the identity.

PROOF OF PROPOSITION 10. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear. To show (3) $\Rightarrow$ (1), we use Lemma 11 to construct a map ρ: $\mathcal{M} \longrightarrow \underline{\tilde{H}}$ which induces an isomorphism on cohomology. As in Lemma 11 we proceed by induction on the natural filtration $\cdots \subseteq \mathcal{M}(n-1) \subseteq \mathcal{M}(n) \cdots \subseteq \mathcal{M}$ of $\mathcal{M}$.

Assume there is a map ρ: $\mathcal{M}(n-1) \longrightarrow \underline{\tilde{H}}$ such that $\rho \circ \phi_\alpha = \Phi_\alpha \circ \rho$, *i.e.* ρ is compatible with the splitting into eigenspaces of powers of α. We also assume that ρ induces the same map on cohomology as the inclusion $\mathcal{M}(n-1) \longrightarrow \mathcal{M}$.

We will extend ρ to $\mathcal{M}(n)$. We know that $\mathcal{M}(n)$ is constructed from $\mathcal{M}(n-1)$ by adjoining $W_n \oplus W'_n$ and their minimal injective resolutions. We only need to define ρ on W_H and W'_H because then the map is automatically defined on W_n and W'_n and their resolutions and by product building on the entire $\mathcal{M}(n)$. Here we need the injectivity of the range $\underline{\tilde{H}}$ of the map in order to be able to extend it from W_n and W'_n to their injective resolutions respectively. We define ρ on the elements of W_H to be their cohomology classes and on the elements of W'_H to be 0. By the claim in the proof of Lemma 11 this map is compatible with multiplication with powers of α. Moreover it has the right cohomological properties as can be shown by looking at each G/H separately and employing the non-equivariant case.

THEOREM 4. *A system* $\mathcal{A}$ *over* $\mathbb{Q}$ *is formal iff* $\mathcal{A} \otimes \mathbb{C}$ *is formal over* $\mathbb{C}$.

PROOF. The map A: $\mathrm{Aut}\big(\mathcal{M}(\mathcal{A})\big) \longrightarrow \mathrm{Aut}\big(\underline{\tilde{H}}(\mathcal{A})\big)$ is, by [T2], a map of $\mathbb{Q}$-algebraic groups. By Corollary 12 the kernel of this map is unipotent. By a result in [Se], any $\mathbb{Q}$-point in $\mathrm{Aut}\big(\underline{\tilde{H}}(\mathcal{A})\big)$ that is the image of a $\mathbb{C}$-point in $\mathrm{Aut}\big(\mathcal{M}(\mathcal{A})\big)$ is already the image of a $\mathbb{Q}$-point. Thus, if all the $\mathbb{C}$-grading automorphisms lift to the $\mathbb{C}$-minimal model, all the $\mathbb{Q}$-grading automorphisms lift to the $\mathbb{Q}$-minimal model. This proves Theorem 4.

Before we prove Theorem 3 we recall the definition of a Kähler manifold.

DEFINITION 13. A Kähler manifold is a complex manifold M admitting a positive definite Hermitian metric

$$H(x,y) = S(x,y) + \sqrt{-1}A(x,y), \quad \text{for } x, y \in T(M),$$

which satisfies the following property: The imaginary part A of the metric, which is a (1,1)-form, is a closed form, *i.e.* $dA = 0$.

We can immediately observe the following: Let X be a compact Kähler manifold, and suppose a finite group G acts holomorphically on X. Then if H is a subgroup of G, X^H is a compact Kähler manifold.

We establish the following notation. For X any complex manifold, let $\mathcal{A}(X)$ denote the PL de Rham algebra on X, $\mathcal{E}(X)$ the complex de Rham algebra on X, $\mathcal{E}^c(X)$ the subalgebra of $\mathcal{E}(X)$ consisting of d^c-closed forms, where $d^c \equiv i(\partial - \bar{\partial})$, $H^c(X)$ the cohomology of $\mathcal{E}^c(X)$ with respect to the coboundary map d^c, and $\mathcal{E}^{ps}(X)$ the piecewise smooth forms on X with complex coefficients. Now define the following systems of DGA's on X: let

$$\begin{aligned} \underline{H}^c_X(G/H) &\equiv H^c(X^H; \mathbb{C}), \\ \mathcal{E}^c_X(G/H) &\equiv \mathcal{E}^c(X^H), \\ \mathcal{E}_X(G/H) &\equiv \mathcal{E}(X^H), \\ \mathcal{E}^{ps}_X(G/H) &\equiv \mathcal{E}^{ps}(X^H), \text{ and} \\ \mathcal{A}_X(G/H) &\equiv \mathcal{A}(X^H). \end{aligned}$$

If $\mathcal{A}$ is a system of DGA's we denote by $\tilde{\mathcal{A}}$ its injective envelope in the sense of Theorem 1.

PROOF OF THEOREM 3. We have the following commutative diagram:

$$\begin{array}{ccccccccc} \tilde{\mathcal{A}}_X \otimes \mathbb{C} & \longrightarrow & \tilde{\mathcal{E}}^{ps}_X & \longleftarrow & \tilde{\mathcal{E}}_X & \longleftarrow & (\tilde{\mathcal{E}}^c_X, d) & \longrightarrow & (\underline{\tilde{H}}^c_X, d) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{A}_X \otimes \mathbb{C} & \longrightarrow & \mathcal{E}^{ps}_X & \longleftarrow & \mathcal{E}_X & \longleftarrow & (\mathcal{E}^c_X, d) & \longrightarrow & (\underline{H}^c_X, d). \end{array}$$

The lower horizontal maps are induced from the nonequivariant ones given in [DGMS] which are functorial. They are all cohomology isomorphisms. Moreover, the differential induced on $\underline{H}^c_X$ by the ordinary differential d is identically 0, so $\underline{H}^c_X \cong \underline{H}_X$. These maps lift to the injective envelopes by Theorem 1 and they are cohomology isomorphisms as well. Applying the construction of equivariant minimal models for the injective systems of the diagram we show that the system of de Rham algebras of the fixed-point sets of X and the system of the cohomology algebras with complex coefficients have isomorphic minimal models. This means that the space is equivariantly formal over the complex numbers. By Theorem 4, formality over the complex numbers is equivalent to formality over the rationals. This proves the first part of Theorem 3.

In order to prove the second part about the formality of equivariant holomorphic maps we first give the following definition.

DEFINITION 14. Let X and Y be compact formal G-spaces with non-empty connected and simply connected fixed-point sets, and let $X \xrightarrow{f} Y$ be an equivariant map. We say that f is *formal over* $\mathbb{F}$ if the map between the equivariant minimal models over $\mathbb{F}$ induced by f is homotopic to the map between the models induced by $f^*\colon \underline{H}(Y;\mathbb{F}) \longrightarrow \underline{H}(X;\mathbb{F})$.

PROPOSITION 15. *Suppose X and Y are compact G-Kähler manifolds with non-empty connected and simply connected fixed-point sets, and let $X \xrightarrow{f} Y$ be an equivariant holomorphic map. Then f is formal over $\mathbb{C}$.*

PROOF. As above, we have the following commutative diagram:

$$\begin{array}{ccccc} \tilde{\mathcal{E}}_X & \longleftarrow & \tilde{\mathcal{E}}^c_X & \longrightarrow & \underline{\tilde{H}}(X;\mathbb{C}) \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{\mathcal{E}}_Y & \longleftarrow & \tilde{\mathcal{E}}^c_Y & \longrightarrow & \underline{\tilde{H}}(Y;\mathbb{C}). \end{array}$$

Consider now the map $\bar{f}$ induced on the minimal models by $\widetilde{\mathcal{E}^c_Y} \xrightarrow{f^*} \widetilde{\mathcal{E}^c_X}$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \mathcal{M}_X \otimes \mathbb{C} & \longrightarrow & \widetilde{\mathcal{E}^c_X} \\ \uparrow & & \uparrow \\ \mathcal{M}_Y \otimes \mathbb{C} & \longrightarrow & \widetilde{\mathcal{E}^c_Y}. \end{array}$$

Composing this diagram with the previous one proves the proposition.

PROPOSITION 16. *A map is formal over $\mathbb{C}$ iff it is formal over $\mathbb{Q}$.*

PROOF. Formality over $\mathbb{Q}$ clearly implies formality over $\mathbb{C}$. Conversely, $\mathcal{M}_Y = \bigcup_{n=0}^{\infty} \mathcal{M}_Y(n)$, where $\mathcal{M}_Y(n)$ is an elementary extension of $\mathcal{M}_Y(n-1)$ by some V_n. By [T], two maps between minimal systems are homotopic iff a well-defined sequence of obstructions to extending the homotopy from $\mathcal{M}_Y(n-1) \otimes \mathbb{C}$ to $\mathcal{M}_Y(n) \otimes \mathbb{C}$ vanishes for each n, namely, that a certain morphism $V_n \otimes \mathbb{C} \xrightarrow{\sigma_n \otimes \mathbb{C}} \underline{Z}^{n+1}(\mathcal{M}_X) \otimes \mathbb{C}$ be a coboundary. But if σ_n is a coboundary after tensoring with $\mathbb{C}$, it must already have been a coboundary over $\mathbb{Q}$. Thus, formality of the map over $\mathbb{Q}$ follows.

Hence we have the following corollary, which establishes the second part of Theorem 3:

COROLLARY 17. *Let $X \xrightarrow{f} Y$ be an equivariant holomorphic map between compact G-Kähler manifolds with non-empty connected and simply connected fixed-point sets. Then f is formal.*

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Department of Mathematics
Indiana University
Bloomington, Indiana 47405
U S A

Department of Mathematics
University of Chicago
Chicago, Illinois 60637
U S A