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On Complex Explicit Formulae Connected with the Möbius Function of an Elliptic Curve

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Abstract. We study analytic properties function $m(z, E)$, which is defined on the upper half-plane as an integral from the shifted L-function of an elliptic curve. We show that $m(z, E)$ analytically continues to a meromorphic function on the whole complex plane and satisfies certain functional equation. Moreover, we give explicit formula for $m(z, E)$ in the strip $|\Im z| < 2\pi$.

1 Introduction

For a complex number z from the upper half-plane let

$$m(z) = \frac{1}{2\pi i} \int_C \frac{e^{sz}}{\zeta(s)}\, ds,$$

where $\zeta(s)$ denotes the classical Riemann zeta function, and the path of integration consists of the half-line $s = -\frac{1}{2} + it, \infty > t \geq 0$, the line segment $[-\frac{1}{2}, \frac{3}{2}]$ and the half-line $s = \frac{3}{2} + it$, $0 \leq t < \infty$. This function was considered in [1] and [5] where the following theorems were proved.

Theorem 1.1 (Bartz [1]) *The function $m(z)$ can be analytically continued to a meromorphic function on the whole complex plane and satisfies the following functional equation*

$$m(z) + \overline{m(\overline{z})} = -2 \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cos\Big(\frac{2\pi}{n} e^{-z}\Big).$$

The only singularities of $m(z)$ are simple poles at the points $z = \log n$, where n is a square-free natural number. The corresponding residues are

$$\mathrm{Res}_{z=\log n}\, m(z) = -\frac{\mu(n)}{2\pi i}.$$

J. Kaczorowski in [5] simplified the proof of this result and gave an explicit formula for $m(z)$ in the strip $|\Im z| < \pi$.

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Theorem 1.2 (Kaczorowski [5]) *For* $|\Im z| < \pi$, $z \neq \log n$, $\mu(n) \neq 0$ *we have*

$$m(z) = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} e\Big(-\frac{1}{ne^z}\Big) - \frac{e^z}{2\pi i} m_0(z) - \frac{1}{2i}\big(m_1(z) + \overline{m_1}(z)\big) + \frac{1}{2i}\big(F_m(z) + \overline{F_m}(z)\big),$$

where

$$m_0(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \frac{1}{z - \log n}$$

is meromorphic on $\mathbb{C}$ *and*

$$m_1(z) = \frac{1}{2\pi i} \int_C \Big(\tan \frac{\pi s}{2} - i\Big) \frac{e^{sz}}{\zeta(z)}\, ds,$$
$$F_m(z) = \frac{1}{2\pi i} \int_1^{1+i\infty} \Big(\tan \frac{\pi s}{2} - i\Big) \frac{e^{sz}}{\zeta(z)}\, ds$$

are holomorphic in the half-plane $\Im z > -\pi$.

In this paper we prove analogous results for the Möbius function of an elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation

$$E/\mathbb{Q} : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q}.$$

Let $L(s, E)$ denote the L-function of E (see for instance [4, pp. 365–366]). For $\sigma = \Re s > 3/2$ we have

$$L(s, E) = \prod_{p|N} (1 - a_p p^{-s})^{-1} \prod_{p\nmid N} (1 - a_p p^{-s} + p^{1-2s})^{-1}, \tag{1.1}$$

where N is the conductor of E. It is well-known that coefficients a_p are real and for $p \nmid N$ one has

$$a_p = p + 1 - \#E(\mathbb{F}_p),$$

where $\#E(\mathbb{F}_p)$ denotes the number of points on E modulo p including the point at infinity, and $a_p \in \{-1, 0, 1\}$, when $p|N$ (for details see [4, p. 365]). The Möbius function of E is defined as the sequence of the Dirichlet coefficients of the inverse of the shifted $L(s, E)$:

$$\frac{1}{L(s + \frac{1}{2}, E)} = \sum_{n=1}^{\infty} \frac{\mu_E(n)}{n^s}, \quad \sigma > 1.$$

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Using (1.1) and the well-known Hasse inequality (see [4, p. 366, (14.32)]) we easily show that μ_E is a multiplicative function satisfying Ramanujan's condition ($\mu_E(n) \ll n^\epsilon$ for every $\epsilon > 0$), and moreover

$$\mu_E(p^k) = \begin{cases} -\frac{a_p}{\sqrt{p}} & \text{if } k = 1, \\ 1 & \text{if } k = 2 \text{ and } p \nmid \Delta, \\ 0 & \text{if } k \geq 3 \text{ or } k = 2 \text{ and } p | \Delta, \end{cases}$$

for every prime p and positive integer k.

Furthermore, C. Breuil, B. Conrad, F. Diamond and R. Taylor, using the method pioneered by A. Wiles, proved in [3] that every L−function of an elliptic curve analytically continues to an entire function and satisfies the following functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s, E) = \eta \left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s) L(2-s, E), \tag{1.2}$$

where $\eta = \pm 1$ is called the root number.

In analogy to $m(z)$ we define $m(z, E)$ by

$$m(z, E) = \frac{1}{2\pi i} \int_C \frac{1}{L(s + \frac{1}{2}, E)} e^{sz}\, ds,$$

where the path of integration consists of the half-line $s = -\frac{1}{4} + it$, $\infty > t \geq 0$, the simple and smooth curve l (which is parametrized by $\tau\colon [0, 1] \to \mathbb{C}$ such that $\tau(0) = -\frac{1}{4}$, $\tau(1) = \frac{3}{2}$, $\Im\tau(t) > 0$ for $t \in (0, 1)$ and $F(s)$ has no zeros on l and between l and the real axis), and the half-line $s = \frac{3}{2} + it$, $0 \leq t < \infty$.

Using (1.2) and Stirling's formula (see [4, p. 151, (5.112)]) it is easy to see that $m(z, E)$ is holomorphic on the upper half-plane.

Our main goal in this paper is to prove the following results, which are extensions of Theorems 1.1 and 1.2.

Theorem 1.3 *The function $m(z, E)$ can be continued analytically to a meromorphic function on the whole complex plane and satisfies the following functional equation*

$$m(z, E) + \overline{m}(z, E) = -\frac{2\pi}{\eta\sqrt{N}} \sum_{n=1}^{\infty} \frac{\mu_E(n)}{n} J_1\left(\frac{4\pi}{\sqrt{Nn}} e^{-\frac{z}{2}}\right) - R(z),$$

where $R(z) = \sum \operatorname{Res}_{s=\beta} \frac{e^{sz}}{L(s+\frac{1}{2}, E)}$ (summation is over real zeros of $L(s + \frac{1}{2}, E)$ in $(0, 1)$, if there are any) and $J_1(z)$ denotes the Bessel function of the first kind:

$$J_1(z) = \sum_{k=1}^{\infty} \frac{(-1)^k (z/2)^{2k+1}}{k!\Gamma(k+2)}.$$

The only singularities of $m(z, E)$ are simple poles at the points $z = \log n$, $\mu_E(n) \neq 0$ with the corresponding residues

$$\operatorname{Res}_{z=\log n} m(z, E) = -\frac{\mu_E(n)}{2\pi i}.$$

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Let $Y_1(z)$ be the Bessel function of the second kind and let

$$H_1^{(2)}(z) = J_1(z) - iY_1(z)$$

denote the classical Hankel function (see [2, p. 4]). Moreover, let

$$R^*(z) = \operatorname{Res}_{s=\frac{1}{2}}\Big(\tan \pi s \frac{e^{sz}}{L(s+\frac{1}{2},E)}\Big) + \sum \operatorname{Res}\Big(\tan \pi s \frac{e^{sz}}{L(s+\frac{1}{2},E)}\Big),$$

summation is over real zeros of $L(s+\frac{1}{2},E)$ in $(0,1)\setminus\{\frac{1}{2}\}$ (if there are any),

$$m_0(z,E) = \sum_{n=1}^{\infty} \frac{\mu_E(n)}{n^{\frac{3}{2}}} \frac{1}{z-\log n},$$

$$m_1(z,E) = \frac{1}{2\pi i}\int_C (\tan \pi s - i)\frac{e^{sz}}{L(s+\frac{1}{2},E)}\,ds,$$

$$H(z,E) = \frac{1}{2\pi i}\int_{\frac{3}{2}}^{\frac{3}{2}+i\infty} (\tan \pi s - i)\frac{e^{sz}}{L(s+\frac{1}{2},E)}\,ds.$$

It is easy to see that $R(z)$ and $R^*(z)$ are entire functions, $m_0(z,E)$ is meromorphic on the whole plane, whereas $m_1(z,E)$ and $H(z,E)$ are holomorphic for $\Im z > -2\pi$. With this notation we have the following result.

Theorem 1.4 *For $z = x+iy$, $|y| < 2\pi$, $x \in \mathbb{R}$, $z \neq \log n$, and $\mu_E(n) \neq 0$, we have*

$$(1.3)\quad m(z,E) = \frac{-\pi}{\eta\sqrt{N}} \sum_{n=1}^{\infty} \frac{\mu_E(n)}{n}\Big(H_1^{(2)}\Big(\frac{4\pi}{\sqrt{Nn}}e^{-\frac{z}{2}}\Big) - \frac{2}{\pi} i \Big(\frac{4\pi}{\sqrt{Nn}}e^{-\frac{z}{2}}\Big)^{-1}\Big)$$
$$- \frac{1}{2}\big(R(z) - iR^*(z)\big) + \frac{1}{2i}\big(H(z,E) + \overline{H}(z,E)\big) - \frac{e^{\frac{3}{2}z}}{2\pi i} m_0(z,E)$$
$$- \frac{1}{2i}\big(m_1(z,E) + \overline{m_1}(z,E)\big).$$

2 An Auxiliary Lemma

We need the following technical lemma.

Lemma 2.1 *Let $z = x+iy$, $y > 0$, $s = Re^{i\theta}$, $R\sin\theta \geq 1$, $\frac{\pi}{2} \leq \theta \leq \pi$. Then for $R \geq R(x,y)$ we have*

$$\Big|\frac{e^{sz}}{L(s+\frac{1}{2},E)}\Big| \leq e^{-y\frac{R}{2}}.$$

Proof Using (1.2), the Stirling's formula and estimate

$$\log L(\sigma+it,E) \ll \log(|t|+2), \quad |\sigma| \geq \frac{3}{2}, \quad |t| \geq 1$$

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(see [7, p. 304]) we obtain

(2.1)
$$\log\Big|\frac{e^{sz}}{L(s+\frac{1}{2},E)}\Big| = \Re\log\frac{e^{sz}}{L(s+\frac{1}{2},E)} = 2R\log R\cos\theta + Rf(\theta,x,y) + O(\log R),$$

where $f(\theta,x,y) = \Big(x + 2\log\frac{\sqrt{N}}{2\pi} - 2\Big)\cos\theta - (y + 2\theta - \pi)\sin\theta$.

For $\frac{\pi}{2} \le \theta \le \frac{\pi}{2} + \frac{1}{\sqrt{\log R}}$ we have

$$f(\theta,x,y) = -(y + 2\theta - \pi) + O\Big(\frac{1}{\sqrt{\log R}}\Big)$$

and hence

$$\log\Big|\frac{e^{sz}}{L(s+\frac{1}{2},E)}\Big| \le -\frac{yR}{2}.$$

For $\frac{\pi}{2} + \frac{1}{\sqrt{\log R}} \le \theta \le \pi$ we have

$$|\cos\theta| \gg \frac{1}{\sqrt{\log R}}$$

and consequently

$$\log\Big|\frac{e^{sz}}{L(s+\frac{1}{2},E)}\Big| = -2|\cos\theta|R\log R + O(R) \le -yR \le -\frac{yR}{2}$$

for sufficently large R, and the lemma easily follows. ■

3 Proof of Theorem 1.3

We shall first prove that $m(z,E)$ has meromorphic continuation to the whole complex plane.

Let us write

$$\begin{aligned} 2\pi i m(z,E) &= \int_{-\frac{1}{4}+i\infty}^{-\frac{1}{4}} \frac{e^{sz}}{L(s+\frac{1}{2},E)}\,ds + \int_{l} \frac{e^{sz}}{L(s+\frac{1}{2},E)}\,ds + \int_{\frac{3}{2}}^{\frac{3}{2}+i\infty} \frac{e^{sz}}{L(s+\frac{1}{2},E)}\,ds \\ &= n_1(z) + n_2(z) + n_3(z), \end{aligned}$$

say.

Notice that $n_2(z)$ is an entire function.

We compute $n_3(z)$ explicitly. Term by term integration gives

$$n_3(z) = -e^{\frac{3}{2}z}\sum_{n=1}^{\infty}\frac{\mu_E(n)}{n^{\frac{3}{2}}(z-\log n)}.$$

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This shows that $n_3(z)$ is meromorphic on the whole complex plane and has simple poles at the points $z = \log n$, $\mu_E(n) \neq 0$, with residues

$$\mathrm{Res}_{z=\log n}\, n_3(z) = -\mu_E(n).$$

Let us now consider $n_1(z)$. Let C_1 consist of the half-line $s = \sigma + i$, $-\infty < \sigma \leq -\frac{1}{4}$ and the line segment $[-\frac{1}{4} + i, -\frac{1}{4}]$. Using Lemma 2.1, we can write

$$n_1(z) = \int_{C_1} \frac{e^{sz}}{L(s + \frac{1}{2}, E)}\, ds.$$

Putting $s = \sigma + i$, $\sigma \leq 0$ in (2.1) we obtain

$$\Big| \frac{e^{(\sigma+i)z}}{L(\frac{1}{2} + \sigma + i, E)} \Big| \ll e^{-c_0|\sigma| \log(|\sigma|+2)},$$

hence $n_1(z)$ is an entire function.

Then for $z \in \mathbb{C}$, $z \neq \log n$, and $\mu_E(n) \neq 0$, we have

$$m(z, E) + \overline{m}(z, E) = -\frac{1}{2\pi i} \int_{\overline{C_1} \cup (-C_1)} \frac{e^{sz}}{L(s + \frac{1}{2}, E)}\, ds - R(z)$$

where minus before a contour denotes the opposite direction.

Using the equality (1.2), we get

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\overline{C_1} \cup (-C_1)} \frac{e^{sz}}{L(s + \frac{1}{2}, E)}\, ds \\ &\qquad = \frac{\pi}{\eta\sqrt{N}} \sum_{n=1}^{\infty} \frac{\mu_E(n)}{n} \cdot \frac{1}{2\pi i} \int_{\overline{C_1} \cup (-C_1)} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{3}{2} - s)} \Big(\frac{Nne^z}{(2\pi)^2} \Big)^s ds. \end{aligned}$$

The last integrand has simple poles at $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$. Computing residues we obtain

$$\frac{1}{2\pi i} \int_{\overline{C_1} \cup (-C_1)} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(\frac{3}{2} - s)} \Big(\frac{Nne^z}{(2\pi)^2} \Big)^s ds = J_1\Big(\frac{4\pi}{\sqrt{Nn}} e^{-\frac{z}{2}} \Big).$$

4 Proof of Theorem 1.4

Let us now consider the function

$$m^*(z, E) = \frac{1}{2\pi i} \int_C \tan(\pi s) \frac{e^{sz}}{L(s + \frac{1}{2}, E)}\, ds.$$

Using Lemma 2.1 we can write

$$m^*(z, E) = \frac{1}{2\pi i} \Big(\int_{C_1 \cup l} + \int_{\frac{3}{2}}^{\frac{3}{2}+i\infty} \Big) \tan \pi s \frac{e^{sz}}{L(s + \frac{1}{2}, E)}\, ds = m^*_a(z, E) + m^*_b(z, E).$$

Using again estimation

$$\Big|\frac{e^{(\sigma+i)z}}{L(\frac{1}{2}+\sigma+i,E)}\Big| \ll e^{-c_0|\sigma|\log(|\sigma|+2)}, \quad \sigma \le 0$$

and $\tan\big(\pi(\sigma+i)\big) \ll 1$ it is easy to see that $m_a^*(z,E)$ is an entire function.

Moreover

$$m_b^*(z,E) = H(z,E) - \frac{e^{\frac{3}{2}z}}{2\pi} m_0(z,E).$$

This gives the meromorphic continuation of $m^*(z,E)$ to the half-plane $\Im z > -2\pi$ and $m^*(z,E)$ has poles at the points $\log n$, $n = 1,2,3,\dots,\mu_E(n) \neq 0$, with residues

$$\operatorname{Res}_{s=\log n} m^*(z,E) = -\frac{\mu_E(n)}{2\pi}.$$

Now we consider the function $\overline{m^*}(z,E)$. Changing s to $\bar{s}$ we get

$$\overline{m^*}(z,E) = \frac{1}{2\pi i}\int_{-\overline{C}} \tan\pi s \frac{e^{sz}}{L(s+\frac{1}{2},E)}\,ds, \quad \Im z < 2\pi.$$

Further we have

$$\overline{m^*}(z,E) = \frac{1}{2\pi i}\int_{-(\overline{C_1}\cup\bar{l})} \tan\pi s \frac{e^{sz}}{L(s+\frac{1}{2},E)}\,ds + \overline{H}(z,E) - \frac{e^{\frac{3}{2}z}}{2\pi} m_0(z,E).$$

Then for $|\Im(z)| < 2\pi$ we have

$$m^*(z,E) + \overline{m^*}(z,E) = -J(z,E) - \frac{e^{\frac{3}{2}z}}{\pi} m_0(z,E) + H(z,E) + \overline{H}(z,E) - R^*(z),$$

where

$$J(z,E) = \frac{1}{2\pi i}\int_{\overline{C_1}\cup(-C_1)} \tan\pi s \frac{e^{sz}}{L(s+\frac{1}{2},E)}\,ds.$$

Using functional equation (1.2), we get

$$\begin{aligned} J(z,E) &= \frac{1}{2\pi i}\int_{\overline{C_1}\cup(-C_1)} \tan(\pi s)\frac{e^{sz}\Gamma(s+\frac{1}{2})}{\eta\Gamma(\frac{3}{2}-s)L(\frac{3}{2}-s)}\Big(\frac{\sqrt{N}}{2\pi}\Big)^{2s-1} ds \\ &= \frac{-2\pi}{\eta\sqrt{N}}\sum_{n=1}^{\infty}\frac{\mu_E(n)}{n}\Big(\frac{1}{2\pi i}\int_{\overline{C_1}\cup(-C_1)} \frac{\Gamma(s+\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)\Gamma(1-s)}\Big(\frac{e^z N n}{4\pi^2}\Big)^s ds\Big). \end{aligned}$$

We can compute the last integral using inverse Mellin transform (see [6, p. 407])

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\overline{C_1}\cup(-C_1)} \frac{\Gamma(s+\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)\Gamma(1-s)}\Big(\frac{e^z N n}{4\pi^2}\Big)^s ds \\ &\qquad = -Y_1\Big(\frac{4\pi}{\sqrt{Nn}}e^{-\frac{z}{2}}\Big) - \frac{2}{\pi}\Big(\frac{4\pi}{\sqrt{Nn}}e^{-\frac{z}{2}}\Big)^{-1}. \end{aligned}$$

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Therefore

$$J(z,E)=\frac{2\pi}{\eta\sqrt{N}}\sum_{n=1}^{\infty}\frac{\mu_E(n)}{n}\Big(-Y_1\Big(\frac{4\pi}{\sqrt{Nn}}e^{-\frac{z}{2}}\Big)-\frac{2}{\pi}\Big(\frac{4\pi}{\sqrt{Nn}}e^{-\frac{z}{2}}\Big)^{-1}\Big).$$

For $x\in\mathbb{R}$, $x\neq\log n$ we have

$$\Re\big(m^*(x,E)\big)=\frac{\pi}{\eta\sqrt{N}}\sum_{n=1}^{\infty}\frac{\mu_E(n)}{n}\Big(-Y_1\Big(\frac{4\pi}{\sqrt{Nn}}e^{-\frac{x}{2}}\Big)-\frac{2}{\pi}\Big(\frac{4\pi}{\sqrt{Nn}}e^{-\frac{x}{2}}\Big)^{-1}\Big)$$
$$-\frac{e^{\frac{3}{2}x}}{2\pi}m_0(x,E)+\frac{1}{2}\big(H(x,E)+\overline{H}(x,E)\big)-\frac{1}{2}R^*(x).$$

Obviously

$$m^*(z,E)=im(z,E)+m_1(z,E),$$

therefore we get

$$\Im\big(m(x,E)\big)=-\frac{\pi}{\eta\sqrt{N}}\sum_{n=1}^{\infty}\frac{\mu_E(n)}{n}\Big(-Y_1\Big(\frac{4\pi}{\sqrt{Nn}}e^{-\frac{x}{2}}\Big)-\frac{1}{\pi}\frac{e^{\frac{x}{2}}\sqrt{Nn}}{2\pi}\Big)$$
$$+\frac{e^{(\frac{3}{2}x}}{2\pi}m_0(x,E)-\frac{1}{2}\big(H(x,E)+\overline{H}(x,E)\big)$$
$$+\frac{1}{2}\big(m_1(x,E)+\overline{m_1}(x,E)\big)+\frac{1}{2}R^*(x). \tag{4.1}$$

On the other hand

$$\Re\big(m(x,E)\big)=-\frac{\pi}{\eta\sqrt{N}}\sum_{n=1}^{\infty}\frac{\mu_E(n)}{n}J_1\Big(\frac{4\pi}{\sqrt{Nn}}e^{-\frac{x}{2}}\Big)-\frac{1}{2}R(x). \tag{4.2}$$

The equations (4.1) and (4.2) imply the formula for $z\in\mathbb{R}$, $z\neq\log n$, and $\mu_E(n)\neq 0$, and by the analytic continuation, formula (1.3) is valid in the strip $|\Im z|<2\pi$.

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