Textual materiality and abstraction in mathematics

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Abstract

In this paper, we wish to explore the role that textual representations play in the creation of new mathematical objects. We do so by analyzing texts by Joseph-Louis Lagrange (1736–1813) and Évariste Galois (1811–1832), which are seen as central to the historical development of the mathematical concept of groups. In our analysis, we consider how the material features of representations relate to the changes in conceptualization that we see in the texts.

Against this backdrop, we discuss the idea that new mathematical concepts, in general, are increasingly abstract in the sense of being detached from material configurations. Our analysis supports the opposite view. We suggest that changes in the material aspects of textual representations (i.e., the actual graphic inscriptions) play an active and crucial role in conceptual change.

We employ an analytical framework adapted from Bruno Latour's 1999 account of intertwined material and representational practices in the empirical sciences. This approach facilitates a foregrounding of the interconnection between the conceptual development of mathematics, and the construction, (re-) configuration, and manipulation of the materiality of representations. Our analysis suggests that, in mathematical practice, distinctions between the material and structural features of representations are not permanent and absolute. This problematizes the appropriateness of the distinction between concrete inscriptions and abstract relations in understanding the development of mathematical concepts.

Keywords: Galois; Lagrange; Latour; mathematical abstraction; mathematical representations; mathematical textuality; permutation groups

Introduction

One of the many mechanisms involved in creating new mathematical concepts is evolution from established concepts. This idea is, for instance, expressed by Reuben Hersh who claims that "mathematical objects are invented or created by humans. . . . They are created, not arbitrarily, but arise from activity with already existing mathematical objects, and from the needs of science and daily life" (Hersh 1979, 44). The mechanism is often seen as a process of abstracting, which is repeatable and leads to increasingly abstract mathematical objects. A recent example is Robert Thomas (2000), who introduces this idea of iterative abstraction in a commentary on Aristotle: "It is only in a few bottom-level cases that mathematical objects are direct abstractions from physical objects. Most are abstractions from mathematical objects that have been abstracted at a previous stage: the iterative conception of abstraction" (Thomas 2000, 317).

A similar idea of iterative abstraction exists at the core of Anna Sfard's (2008) model of mathematical concept formation. In this model, new mathematical objects are created by the reification of activities with objects established as such, where reification is understood as "discursively turning processes into objects" by "substituting talk about actions with talk about objects" (Sfard 2008, 43–44). For example, according to Sfard's model, natural numbers are seen as

reifications of actual acts of counting physical objects. By repeating this process, starting from mathematical objects (like the newly formed natural numbers) instead of concrete physical objects, further mathematical objects can be created—for example, activities of subtracting natural numbers may eventually be reified as negative numbers.

From such theories and ideas, a consistent narrative emerges: First, some mathematical objects are created as abstractions from actual activities with physical objects—what Sfard calls "tangible, material objects that exist independently of communication" (Sfard 2008, 112). From such "initial" mathematical objects, essentially the same process of abstracting creates other mathematical objects. Each repetition creates an entity that is further away from the actual handling of concrete material objects.

What is missing from this story is the role played by the external representations involved in the process of developing new mathematical concepts.[1] Mathematicians' use of external representations continues to receive a great deal of attention from scholars interested in mathematical practice. Some focus on cognitive and epistemic aspects of representations, such as the cognitive affordances of specific kinds of representations (e.g. Schlimm and Neth 2008; Widom and Schlimm 2012; De Cruz and De Smedt 2013; Carter 2019). Others discuss the epistemic role of visualizations, such as figures and diagrams (e.g. De Toffoli 2017; Giaquinto 2007; Giaquinto 2011; De Toffoli and Giardino 2014). Finally, Sybille Krämer considers mathematical notation as a two-dimensional technology and the text as a special kind of flat medium in which this technology is operated (Krämer 2014).

Another group of questions concerns external representations in relation to historical changes within mathematics. Historically speaking, the material level of mathematics is highly plastic. Mathematicians frequently develop new notations and non-textual representations,[2] and adapt existing ones to their immediate needs.[3] There is now a substantial literature in the history of mathematics supporting the view that such practices are not auxiliary surface phenomena that merely accompany the changes of some "mathematics proper." Rather, the work that historical agents perform with, and on, notation, is integral for understanding the changes mathematics undergoes as historical processes. Elaine Koppelman (1971) does precisely this: in Koppelman's account, English algebra in the first half of the nineteenth century was characterized partly by a new object of study, that is, by shifting from studying entities such as numbers to studying operations defined at the concrete level of notation. By returning to mathematical texts from the seventeenth and eighteenth centuries (e.g. by Gottfried Leibnitz and Joseph-Louis Lagrange), Koppelman traces this shift in focus back to practices where the notational analogy between the power expansion of a sum and the formula for the differential of product is used to manipulate notation. From a philosophical perspective, Emily Grosholz argues that mathematical theories change, and new knowledge emerges, because mathematicians superimpose, juxtapose, and multiply representations, which sometimes creates what Grosholz coins "productive ambiguity" (Grosholz 2007). So, the role of representations is not reducible to a means for mathematicians to

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[1]Sfard challenges the distinction between representational means and mathematical objects by taking representations as the main ontological category in mathematics in the sense that she considers mathematical objects as discursive objects, but she does not delve into the analysis of how representations and objects co-evolve (Sfard 2008, 174).

[2]In this paper, we use the term notation to refer to the actual inscriptions of a textual representation. Marks with a pen on paper, digital renderings on a screen and etchings on clay table are examples of such inscriptions. However, the term also has the connotation of a specific system of such inscriptions: We will sometimes speak about e.g. "capital letter notation" or "table notation" to indicate that a particular inscription belongs to a certain type of notation. However, we do this only when we discuss specific inscriptions that occur in some text directly and these notational types are understood purely relationally, i.e. in opposition to each other. Moreover, a type of notation need not be defined anywhere in the text (or elsewhere) nor to be stable throughout the text.

[3]Albrecht Heeffer has written a very useful and concise primer to the field of the history of mathematical notation as the introductory chapter of Schulte and Heeffer (2015).

express (or otherwise externalize) their thinking. Working with notation produces a non-reducible creative element of mathematics.

Going beyond mathematics, investigating interactions between objects of study and the material representations used is a well-established theme in the context of empirical and experimental sciences. Here, several scholars have systematically challenged the dichotomy between representation and object as suitable for describing scientific knowledge and practices. Ursula Klein, in her work on textual representations used in chemistry (e.g. (2001) and more extensively (2003)), describes how representations can provide a way to manage an "epistemically fluid environment":

> When Berzelius introduced into chemistry the quasi-algebraic notation familiar from mathematics and physics, he used it as a creative sign system that forged the difference between "atoms" in the philosophical tradition and discrete "chemical portions". In their new epistemically fluid environment situated between the traditional vocabulary of philosophical atomic theories and the vocabulary of stoichiometric and volumetric experiments, which nourished chemists' intuition about discrete quantitative units of substances, this new sign system became a medium in which these chemical intuitions were expressed and a tool for constructing and defining a new, more solid chemical concept. (Klein 2001, 29)

Analyzing an episode in the development of biochemistry, Hans Jörg Rheinberger introduces a distinction between "epistemic things" and "technical objects" (Rheinberger 1997). The first of these concepts refers to the entities or processes under investigation, and the second to the instruments, research traditions and laboratory skills that are used in the investigation of the epistemic things. These two aspects of experimental systems are, however, interconnected. On the one hand, the available technical objects condition the investigation of the epistemic objects, while on the other hand, epistemic things can turn into technical objects over time and thus change the conditions for investigation and representation of further epistemic things (Rheinberger 1997, 28–29).

Moritz Epple has adapted Rheinberger's theoretical framework to the history of mathematics (Epple 2004). Epple shows how epistemic objects (the objects of investigation) can turn into epistemic techniques (the tools and techniques used in the investigation) and vice versa. What is used as a tool or technique for an investigation in one research situation can become the object under investigation in another. Epple has used the framework to analyze the influence of the local research situations on the historical development of mathematics by analyzing these tool-object shifts as driven by the local research situation, and at the same time themselves a dynamic of the local research situation.[4,5]

In light of these insights, one problem with a simple story about chains of iterative abstraction is that it suggests that the distance or detachment from material configurations increases as the abstraction chain grows. However, this is a somewhat reductionist reconstruction of mathematical practice. It rather seems as if material representations continue to multiply and re-configure as the abstraction chain lengthens. Roy Wagner points out further problems; in particular, he notes the tendency to consider some kinds of representations as inherently more abstract than others, mentioning formal and axiomatic definitions (Wagner 2019, 2). We will discuss an example of this in our analysis.

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[4]For further applications of Epple's approach, see Kjeldsen (2008; 2009). For a more internalist application, see Johansen and Misfeldt (2015) and Steensen and Johansen (2016).

[5]Krömer (2007) could also be added here. However, it should be noted that Krömer discusses tools and objects in mathematics from a different theoretical vantage point (pragmatism and Peircean semiotics). Krömer is interested in situations where a tool one has been using is later scrutinized (treated as an object) to understand how it works and determine whether it can be trusted (e.g. Krömer 2007, 33) but he – in contrast to the other authors we have mentioned – does not consider the role of materiality in play in tool-object relationships.

In this paper, we will add to the understanding of the use of representations in mathematics by performing a semiotic analysis that explores how a chain of iterative abstraction unfolds. We take Bruno Latour's 1999 essay "Circulating Reference" as our point of departure. In it, Latour describes the intertwined material and representational practices of a group of scientists making systematic observations (soil sampling, etc.) on a field trip in the Amazonian rainforest and turning them into a published scientific paper. Although Latour's own focus is empirical science, his approach is useful for understanding chains of iterative abstraction in mathematics as well, since it allows us to overcome inherent distinctions between mathematical inscriptions on the one hand, and mathematical objects and concepts on the other.

We will describe the emergence of a new mathematical object—the permutation group—as it appears in two mathematical texts, Lagrange's "Réflexions sur la résolution algébrique des équations" (1869) and Évariste Galois' *Mémoire sur les conditions de résolubilité des équations par radicaux*, (Galois and Neumann 2011), which, in retrospect, were central elements in an early episode of creating the mathematical concept of groups. However, the type of analysis we propose here is not historical, but semiotic: we do not aim to reconstruct the intentions and interpretations of historical actors (either the author or their typical readers), but rather the life that the text as a concrete set of graphic marks can have for today's readers—including historians of mathematics. From the semiotic perspective, a reading is an interpretation, and semiotics is the "science of interpretation." Here, we follow Julia Kristeva, who writes that "the fundamental gesture of semiotics: a formalization or production of models . . . that is, of formal systems whose structure is isomorphic or analogous to the structure of another system (the system under study). . . . In each particular case of semiotic research, a theoretical reflection isolates the signifying function being axiomatized, which is then represented in a formal manner" (Kristeva 1986, 76–77). Such a semiotic approach provides a useful analytical perspective that complements (rather than replaces) historical, philosophical, and cognitive analysis, as demonstrated by Wagner and Alain Herreman (Wagner 2009a; 2009b; Herreman 2000). In line with these authors, we do not consider this kind of analysis as providing an absolute, trans-historical account of the text. Rather, it provides interpretations that are grounded in a conceptual system which is chosen with respect to the specific research question, and positioned in the present historical situation. Semiotic analysis thus seeks to find contemporarily relevant interpretations of the relations and impacts of signs. In our case, the "formal" aspect referred to simply describes the signifying functions under study, which are mathematical representations, in terms of the conceptual framework, and not (or only secondarily) using any presupposed content or context of the representations. Simply put, we ask what abstract concepts of a group a reader may form from these texts today.

There is no consensus about how to define abstraction with respect to mathematics. One approach is to view abstraction as essentially a thought process. From this position, Benis Sinaceur develops a typology of thought processes through which we regulate relationships, such as that of the particular to the general (Sinaceur 2014, 94–98). Abstraction is associated with the "logical technique" of subsuming or subtracting certain features or information, and is seen as the process by which we form concepts (Sinaceur 2014, 84). Another approach, as explored by Karine Chemla, is to focus on abstraction as epistemic value, and to study how agents shape their mathematical practice to promote such values and how such values in turn influence the mathematics they develop (Chemla 2003).[6] Wagner proposes a definition of abstraction as a mathematical practice based on the idea of "abstraction as a collection of different translations between available

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[6]"With respect to the theoretical dimensions of mathematical activity in ancient China, this thesis holds that generality was a value that ancient China's practitioners of mathematics placed above abstraction. In other words, what mattered most was the generality of the algorithms, and not the abstract character of the framework within which they were presented" (Chemla 2003, 416). A philosophical point to take from this quotation is that the tendency to see an intimate relation (sometimes even a necessary relation) between generality and abstraction is not a cognitive universal, but is relative to historical and local conditions.

mathematical representations. The innovation is that I do away with the suggestion that such translations generate a set of invariants that serve as the goal or content of abstraction" (Wagner 2019, 3). Following the same line of reasoning, we will say that something is an abstraction (or is being abstracted) if the text (in our interpretation) indicates a detachment from a material configuration. Furthermore, if the process of detachment is followed by the appearance of a new material configuration, we shall refer to such new materiality and to its manipulation, construction, re-configuration, etc., as reification.[7] We stress that we do not propose this definition as a general definition of abstraction. Our definition of abstraction is not intended to disqualify other definitions, but to serve as an operational description meant to enable us to analyze the texts with respect to the problem that interests us here: the relationship between textual materiality and abstraction. Our analysis starts with Lagrange's application of permutations (in the non-technical sense of interchanging the places of things) to study algebraic equations. We will see that groups of such permutations do not emerge suddenly, with the introduction of a new representation, but in a series of gradual representational changes involving both abstraction and reification.

Theoretical approach

Latour's 1999 essay seeks to replace a traditional picture of reference—a picture where references reach across the metaphysical gap that separates representations from objects. Instead of the constellation representation-gap-object, Latour proposes that we have chains of elements of representation (Latour 1999, 70). In these elements, the representational and material intertwine. More precisely, what is distinguishable as the material and the representational aspect of representations in the traditional picture of reference are not distinguishable in a Latourian element of representation. By creating and linking such elements of representation, scientists produce chains of representations that carry reference in the sense that they allow, for example, a scientific paper to speak about the site of data collection. With these chains of representation, Latour models the production of scientific knowledge from fieldwork to publication as an iterative process that, in each step, combines abstraction and reification. Scientific knowledge production is thus not about establishing connections between metaphysically different domains—one of states of affairs, another of representations; it is about extending and maintaining chains of reference. The philosophical problem is, therefore, not how practitioners bridge the "rupture between things and signs" but how they develop the next link in "an unbroken series of well-nested elements [of representation]" (Latour 1999, 56).

   To illustrate Latour's model, we can look at a chain of reference that a botanist may create. (One of the scientists that Latour followed is a botanist and the following example is loosely based on Latour's account of her work.) In Latour (1999), the botanist is collecting leaves from three species of trees at different locations in a forest. Back at her laboratory, she arranges the leaves in a cabinet composed of several columns of shelves (Latour 1999, 35). For instance, we could imagine the leaves arranged according to color. The botanist can then quantify the color variations and write a sequence of numbers to represent the collection of leaves, which, in turn, can be compared with sequences produced from other geographical, temporal, or meteorological parameters. Thus, by treating "average temperature" or the "decade after 1900" as parameters, the sequences make it possible to address questions such as how climate change affects this tree species.

   We could view the whole process (from collecting leaves to writing a scientific paper analyzing the numerical sequences) as the stepwise extraction of a form—a structural property—by gradually getting rid of material (first, concrete leaf material, then, individual number sequences, etc.) and, thus, moving away from matter. In this view, the written conclusion (reached by comparing the numerical sequences) is less material than a pile of leaves, and as such, it is more

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[7]Note that we give "reification" a meaning that differs from the one used by Sfard (2008).

abstract. However, Latour's point (and ours) is that such a view is too insensitive to the material configurations that the botanist made in each step. First, there are a stack of leaves and the empty shelves. The stack of leaves represents the forest precisely by consisting of the concrete material leaves picked out by the botanist. When the botanist fills the shelves of the cabinet with leaves according to their color, she re-configures the given material—the stack of leaves—and uses the cabinet to give form to a new material configuration, a cabinet full of color-coded leaves. This new configuration represents the stack of leaves precisely by being a concrete cabinet filled with the same concrete leaves that made up the stack. At the same time, filling the cabinet draws out a structural property that the stack does not have: placing a leaf on a shelf emphasizes color, while other material properties lose significance. The next transformation assigns numerical values to colors and produces a new material configuration, the numerical sequence. Again, the sequence represents the cabinet full of leaves, precisely by being an actual sequence of those concrete number marks that the botanist assigns to the colors on the shelves. And again, the actual sequence of concrete number marks is the color-coding in a new form, this time a discrete quality that enables comparisons with other numerical sequences in the next transformation.

The example illustrates the prominence that Latour ascribes to materiality in the practices of the empirical sciences. Here, we adapt the theoretical framework to apply a similar analysis to mathematics. Specifically, we analyze how mathematical objects may emerge through series of transformations of representations that are similar in structure to the sequence of transformations from the forest to the scientific paper.

In the following section, we apply this approach to the case of the permutation group to explore whether the interconnection between the materiality of representation and the formation of concepts in practice can be understood in this way. For Latour, it is essential that each element in the chain belongs to matter by its origin and to form by its destination (Latour 1999, 56–57). Similarly, based on the analysis, we argue that mathematical objects emerging by iterative abstraction do not become increasingly independent of the material features of representations. We believe that our analysis shows how the lack of permanent distinction between material features and form in representation makes it possible for the mathematician to use material configurations to produce new structural relations. The appropriateness of making an essential distinction between matter and form when seeking to understand the development of mathematical objects and concepts thus becomes hard to maintain.

Analysis: The emergence of groups

The case we consider is the work conducted by Joseph-Louis Lagrange (1736–1813) and Évariste Galois (1811–1832) on solving polynomial equations—recognized as focal contributions to the development of the concept of mathematical groups. We do not aim to account for this development from a mathematical perspective, as such an account can be found, for instance, in Edwards (1984) and Kiernan (1971). Instead, we are interested in the representations that Lagrange and Galois used in connection with this development and, in particular, how the representations change. Our approach involves analyzing the changing representations as a chain of elements of representation in the adapted Latourian sense, addressing, specifically, the interplay between gradual abstraction and the transformations that concrete inscriptions undergo.

Consequently, we will suppress much of what is often considered to be the mathematical content of representations. The points we wish to make do not require knowledge of this mathematical content In the following section, we provide the mathematical–historical background knowledge useful for following the transformations of representations. We base this summary mainly on Edwards and Kiernan (Edwards 1984, 2, 19–22 and 32–35; Kiernan 1971, 40–56 and 79–90).

For the history of the development of the concept of mathematical groups, we refer to Hans Wussing's work. In his account, Wussing includes the two texts that we consider here, but does

not take the notational changes into consideration as an active part of the conceptual development (Wussing [2007], 71–118).[8] For an account of Galois' contribution with more focus on his notation, see Ehrhardt ([2016]).

Background and context regarding the emergence of groups

Our primary sources are Lagrange's 1770 paper "Réflexions sur la résolution algébrique des équations" and Galois' *Mémoire sur les conditions de résolubilité des équations par radicaux*, known as his "first memoir" (hereafter, *PM*), a manuscript submitted in 1831 but first published posthumously in 1846.[9] It has been established that Galois knew some of Lagrange's writings (Galois and Neumann [2011], 5). As such, there is a basis for claiming some intellectual continuity between the two authors.

The mathematical terminology of *PM* differs from that of "Réflexions." For the sake of brevity, we will not go into the differences here (see Galois and Neumann [2011], 18–27; Kiernan [1971], 40–45). Instead, we simply introduce the following conventions: a "rational expression" in $x_1, x_2, x_3, \ldots, x_n$ is an expression only in terms of $x_1, x_2, x_3, \ldots, x_n$, the four basic arithmetical operations and certain quantities. Precisely which quantities that are allowed depends on the coefficients of the polynomial. To begin with, we may think of these quantities as any number that can be obtained from the coefficients by applying the four arithmetical operations. For example, $3x_1 x_2 + 47 x_3$ is a rational expression in x_1, x_2 and x_3. We call the expression in $x_1, x_2, x_3, \ldots, x_n$ "algebraic" if it is an expression in terms of $x_1, x_2, x_3, \ldots, x_n$, the four basic arithmetical operations, the quantities that can be obtained by applying the four arithmetical operations to the coefficients, as well as a finite number of radicals.[10] The expression $x_1 x_2 + \sqrt{2} x_3$ is an example. The radicals are said to be (successively) adjoined to the equation.

In "Réflexions," Lagrange explores the possibility of expressing the roots of general polynomial equations as algebraic expressions in the coefficients of the polynomial. A polynomial equation for which this is possible is said to be solvable by the adjunction of radicals (solvable, for short). Lagrange was interested in permutations of the roots because he saw a connection between patterns relating to how the roots behave under permutations and the possibility of expressing them algebraically, (i.e., of solving the polynomial). Galois can be said to have continued this line of work. Since permutations will be central in what follows, we briefly outline their relevance in the solvability of polynomials.

In short, in 1797, Carl Friedrich Gauss proved that there are n roots for the general polynomial of degree n (with possible repetitions) but only the general polynomials of degrees 2, 3, and 4 were known to be solvable in the above sense (see, e.g., Kiernan [1971], 44). Therefore, Lagrange and his contemporaries were interested in figuring out the kinds of roots of different types of polynomials. One way of addressing this question was to search for criteria that guaranteed the existence of solutions of specific kinds of polynomials without having to determine the actual algebraic expressions.

Lagrange's strategy was to search for a common feature among the techniques for solving the general polynomials of degrees 2, 3, and 4 and then to use this for solving polynomials of a higher

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[8]Wussing's approach is to use today's concept of automorphism to search in the texts for "implicit group theoretic modes of thought" and among these "spotting those paths of development of implicit group theory that have made a causal contribution to the rise of explicit group theory. ... We are not concerned with the historical manifestation of some logical development but with the logical manifestation of the historical development" (Wussing [2007], 17). Moreover, Wussing bases his interpretation of Galois to a large extent on Galois' methodological reflections in other texts (Wussing [2007], 102–118). Both points highlight differences between our approaches.

[9]We use the bilingual version (French and English) of the first memoir from Peter Neumann's 2011 annotated translation of the collected writings of Évariste Galois. We use page numbers from Galois and Neumann ([2011]). In quoting Lagrange, we indicate our own English translation and include the text of the French original in a footnote.

[10]Generally speaking, a radical is an expression of the form $\sqrt[k]{u}$.

degree. One common feature seems to be the introduction of new polynomials constructed in such a way that their roots can be used to find the roots of the original polynomial. The crux is that the roots of these new polynomials are simpler to find because they possess certain symmetry properties with respect to the roots of the original polynomial. This is where permutations enter the investigation of solvability: the roots of the new polynomials must be constructed such that they relate to each other by permutations of the roots of the original polynomial.

The symmetry property of the new polynomials' roots is a formal relationship between the coefficients and the roots of any polynomial, and it is a key point in Lagrange's strategy (and later in Galois' approach, cf. Edwards 1984, 8–9). When some rational expression in the letters $a, b, c, d, \ldots$ is symmetric, in the sense that the expression always remains the same when the letters are permuted among each other, then this symmetric rational expression can be rewritten as a rational expression in the coefficients of the various powers of x that arise from expanding the product $(x - a)(x - b)(x - c)(x - d) \cdots$. In essence, every symmetric rational expression in the roots of a polynomial can be expressed rationally by its coefficients.[11]

To illustrate the use of permutations in the solution technique, let p be the general polynomial of degree n, and let f be a rational expression in the roots $a, b, c, d, \ldots$ of p. If we permute the roots, the rational expression f will be changed. Now, let $f_1, f_2, \ldots$ be the expressions that all possible permutations of the roots produce from f in this way. Since p has n roots (which we assume to be distinct for simplicity), there are $n!$ permutations of the roots and, thus, $n!$ such permutation variants of f. At least one of them will be equal to f, since one of the permutations leaves every root fixed, but other permutations may also produce expressions equal to f.[12] Such relations of symmetry among the f_i are used for the construction of the first of the new polynomials mentioned above as follows: we form the polynomial $F(x) = (x - f_1)(x - f_2) \cdots (x - f_{n!})$; the coefficients of F (i.e., of the various powers of x) consist in sums and products of the f_i and are, thus, rational expressions in the roots of p. Moreover, the coefficients of F are symmetric in the roots of p, i.e., no permutation of the roots of p changes any of the coefficients of F, since any permutation of the roots of p simply changes the order in which $f_1, f_2, \ldots, f_{n!}$ occur in the expression $F(x)$ above, which does not change F. Therefore, the coefficients of F are rational in the coefficients of p by the symmetry property.

Lagrange's insight is that the roots of F can be used to determine the roots of p. Obviously, the roots of F are precisely the expressions $f_1, f_2, \ldots$, but we do not immediately know if they can be expressed algebraically in the coefficients of p. At first glance, this seems like a more challenging problem than that of solving p because the degree of F may be higher than the degree of p. However, the difficulty depends on the relations of symmetry among the f_i. If f was chosen wisely, then the f_i are not distinct expressions. In this case, it may be possible to rewrite F as a polynomial G of lower degree (with coefficients that are algebraic (if not rational) in the coefficients of p), thus simplifying the problem. For example, if p is the general quartic (cf. Edwards 1984, §17, 20-21) with roots a, b, c, d, taking $f = (a + b)(c + d)$ gives three different permutation variants $f_1 = (a + b)(c + d)$, $f_2 = (a + c)(b + d)$ and $f_3 = (a + d)(c + b)$, each of which arises eight times when the roots a, b, c, d are permuted. This choice of f thus makes it possible to rewrite F in the form G^8, where G is a polynomial in x^3 (to see this, imagine what happens if we expand $F(x) = (x - f_1)^8 (x - f_2)^8 (x - f_3)^8$). This essentially reduces the problem of finding the roots of F and, therefore, the solution of p to solving a cubic and extracting an 8th root as follows. Since the coefficients of F are rational in the coefficients of p (by the symmetry property), and the

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[11] For example, if we consider the quadratic $(x - a)(x - b) = x^2 - (a + b)x + ab$, the expression $-ab(a + b)$ is symmetric in a and b and is equal to the product of the constant term and the coefficient of x.

[12] That is, the expressions may be equal up to operations according to the rules of the arithmetical operations. For example, if we permute a and b in the expression $c(a + b)$, we get $c(b + a)$, which is regarded as the same expression because addition is commutative.

coefficients of G are algebraic in those of F,[13] the coefficients of G are algebraic in those of p. Because G as a cubic is solvable, we known that the roots $f_1 = (a + b)(c + d), f_2 = (a + c)(b + d)$ and $f_3 = (a + d)(c + b)$ are algebraic in the coefficients of G and, hence, algebraic in the coefficients of p. With some algebraic consideration and manipulation, this eventually leads to the fact that a, b, c, and d are algebraic in the coefficients of p. Thus, with the choice of f, solving the general quartic is transformed into solving a cubic and taking an eighth root. In the general case, we can repeat this technique, if necessary, now rewriting G instead of F, and so on until we reach a polynomial that we can solve. This is why Lagrange and Galois were interested in permutations of the roots of p: which kinds of rational expressions in the roots are changed by some permutations of the roots but not by others? In other words, what are the symmetry properties of rational expressions in the roots of p? With this short introduction, we now turn to the analysis.

First transformation: Describing permutations as movements

In "Réflexions," Lagrange simply begins by verbally describing the specific movements of roots, for example, "If we suppose that the roots $x', x'', \ldots, x^{(\omega)}$ are exchanged respectively into $x^{(\omega+1)}, x^{(\omega+2)} \ldots, x^{(2\omega)}$, or into $x^{(2\omega+1)}, x^{(2\omega+2)}, \ldots x^{(3\omega)}$, or, etc., it will result in the same changes in the preceding equations as if one exchanged ζ' into ζ'', or into ζ''', or, etc." (Lagrange 1869, 323).[14] The "preceding equations" that Lagrange mentions are rational expressions in the roots x', x'', etc. (what ζ' etc. stands for is not important). Importantly, in this quotation, Lagrange explains the permutations by describing how letters denoting roots move with respect to each other (are "exchanged"). The quotation also illustrates that such a description enabled Lagrange to pass smoothly to a description of how the movements of roots affect "the preceding equations", (i.e., certain given rational expressions in the roots). However, Lagrange is interested in saying something more general about such effects than what can be said by choosing particular explicit expressions in the roots, say $x'x'' + x^{(\omega)}$, and exchanging its letters. His next step should be seen in light of this interest.

Lagrange introduces a notation for the effects of permutations on expressions as follows: if f is a rational expression in x and y, then f is denoted by $f\big[(x)(y)\big]$. If f remains the same expression when x and y are interchanged, then f is denoted by $f\big[(x, y)\big]$ instead of $f\big[(x)(y)\big]$. For example, we can denote both expressions xy and $2x + y$ by $f\big[(x)(y)\big]$, but only xy can be denoted by $f\big[(x, y)\big]$. We will call this new notation "the bracket notation" and its expressions, e.g., $f\big[(x)(y)\big]$ and $f\big[(x, y)\big]$, "the bracket expressions."

Not only does a bracket expression immediately state how permutations affect f, it also makes it easy to compare f to other expressions in the same roots based on how permutations of the roots affect the expressions. Lagrange introduces the convention of using the same bracket expression to denote all rational expressions that are affected in the same way by permutations, and to call such rational expressions *similar*:

> Finally, if we have several functions of the same quantities, we will call those functions *similar* which vary at the same time or remain the same when we make the same permutations between the quantities of which they are composed, so that they can be designated in an analogue way. Thus, taking the characteristics f and ϕ to designate different functions, the

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[13]When we reduce the degree of F by passing to G in this way, we pass from a rational to an algebraic relationship between their coefficients, i.e., we need to allow the expressions of the coefficients of G in terms of those of F to contain radicals because we extract an 8th root.

[14]Que si l'on suppose qu'on échange respectivement les racines $x', x'', \ldots, x^{(\omega)}$ en $x^{(\omega+1)}, \ldots, x^{(2\omega)}$, ou $x^{(2\omega+1)}, x^{(2\omega+2)}, \ldots x^{(3\omega)}$, ou, etc., il en résultera dans les équations précédentes les mêmes changements que si l'on échangeait ζ' en ζ'', ou en ζ''', ou, etc. (Lagrange 1869, 323).

functions $f\big[(x)(y)\big]$ and $\phi\big[(x)(y)\big]$ will be similar, as well as the functions $f\big[(x,y)\big]$ and $\phi\big[(x,y)\big]$, and so on (Lagrange 1869, 358–359).[15]

This convention makes the content of the bracket a more important part of a bracket expression than the letter referring to the function (the left-most letter in the bracket expression, such as f and ϕ in the quote above). Lagrange thus uses the bracket notation to organize expressions in the roots according to how permutations of the roots affect them.

To describe permutations, Lagrange also begins to use bracket notation instead of always explicating the specific letter movements. For example, rather than writing that f remains the same when x and y are interchanged, he now sometimes writes that $f\big[(x,y)\big]$ is the form of f: "it is necessary, by the hypothesis, that this function also remains the same by changing both x' and x'' into x''' and x^{IV}; therefore, since by these permutations the two quantities y' and y'' change into each other, the function $f\big[(y')(y'')\big]$ must be of the form $f\big[(y',y'')\big]$ (Lagrange 1869, 359)."[16] As a new representation, the bracket notation thus enables Lagrange to describe the effect of a permutation on a rational expression in the roots, without giving a verbal description of how the permutation actually moves the roots.

In addition, notice that there is a subtle difference between $f\big[(x)(y)\big]$ and $f\big[(x,y)\big]$, both materially and referentially. The first bracket expression $f\big[(x)(y)\big]$, which has no comma in it, and with x and y each in their own set of parentheses, simply denotes an unspecified but specifically "chosen" expression f that is rational in x and y—and this may be all we know about f. In contrast, the second bracket expression $f\big[(x,y)\big]$ is sometimes used by Lagrange to say that the unspecified but specifically "chosen" f has a form that is not changed by interchanging x and y; at other times, this same bracket expression $f\big[(x,y)\big]$ is used to designate this form of symmetry itself. The following passage illustrates the difference: "To find the general form of the function in question, I take ... another arbitrary function, designated by $\phi\big[(x')(x'')(x''')(x^{IV})\big]$, and I first reduce it to the form $\phi\big[(x',x'')(x''',x^{IV})\big]$, so that it remains the same ... changing x' to x'' or x''' to x^{IV}" (Lagrange 1869, 395).[17] Thus, in our view, Lagrange's conceptualization of symmetry as a relation of similarity between functions co-evolves with, and is inseparable from, his ability to capture the idea in a new material configuration, that of the bracket notation. As a new representation, the bracket notation is at once an abstraction and a reification.

The bracket notation states what effect a movement of the roots has on an expression in the roots, even if we do not know what the expression actually looks like (i.e., without knowing specifically how the expression is made up of roots, arithmetical operations, and quantities). This abstraction of the effects of permutations as forms of symmetry makes them independent of one material configuration, namely, a written-out arrangement of the permutable things before and after permutation. But the abstraction comes with a reification, in the form of bracket notation.

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[15]"Enfin, si l'on a plusieurs fonctions des mêmes quantités, on appellera fonctions semblables celles qui varient en même temps ou demeurent les mêmes lorsqu'on y fait les mêmes permutations entre les quantités dont elles sont composées, de manière qu'elles puissent être désignées d'une manière analogue. Ainsi prenant les caractéristiques f et ϕ pour désigner des fonctions différentes, les fonctions $f\big[(x)(y)\big]$ et $\phi\big[(x)(y)\big]$ seront semblables, ainsi que les fonctions $f\big[(x,y)\big]$ et $\phi\big[(x,y)\big]$ et ainsi des autres" (Lagrange 1869, 358–359).

[16]"mais il faut, par l'hypothèse, que cette fonction demeure aussi la même en y changeant à la fois x' et x'' en x''' et x^{IV}; donc, puisque par ces permutations les deux quantités y' et y'' se changent l'une dans l'autre, il faudra que la fonction $f\big[(y')(y'')\big]$ soit de la forme $f\big[(y',y')\big]$" (Lagrange 1869, 359).

[17]"Pour trouver la forme générale de la fonction dont il s'agit, je prends ... une autre fonction quelconque, désignée par $\phi\big[(x')(x'')(x''')(x^{IV})\big]$, et je la réduis d'abord à la forme $\phi\big[(x',x'')(x''',x^{IV})\big]$, pour qu'elle demeure la même ... changeant x' en x'' ou x''' en x^{IV}."

Table 1. List of bracket expressions (Lagrange 1869, 394).

$f\left[(x')(x'')(x''')(x^{IV})\right],$	$f\left[(x'')(x')(x''')(x^{IV})\right],$
$f\left[(x''')(x'')(x')(x^{IV})\right],$	$f\left[(x'')(x''')(x')(x^{IV})\right],$
$f\left[(x')(x''')(x'')(x^{IV})\right],$	$f\left[(x''')(x')(x'')(x^{IV})\right],$
$f\left[(x^{IV})(x'')(x''')(x')\right],$	$f\left[(x'')(x^{IV})(x''')(x')\right],$
$f\left[(x''')(x'')(x^{IV})(x')\right],$	$f\left[(x'')(x''')(x^{IV})(x')\right],$
$f\left[(x^{IV})(x''')(x'')(x')\right],$	$f\left[(x'')(x^{IV})(x'')(x')\right],$
$f\left[(x')(x^{IV})(x''')(x'')\right],$	$f\left[(x^{IV})(x')(x''')(x'')\right],$
$f\left[(x''')(x^{IV})(x')(x'')\right],$	$f\left[(x^{IV})(x''')(x')(x'')\right],$
$f\left[(x')(x''')(x^{IV})(x'')\right],$	$f\left[(x''')(x')(x^{IV})(x'')\right],$
$f\left[(x')(x'')(x^{IV})(x''')\right],$	$f\left[(x'')(x')(x^{IV})(x''')\right],$
$f\left[(x^{IV})(x'')(x')(x''')\right],$	$f\left[(x'')(x^{IV})(x')(x''')\right],$
$f\left[(x')(x^{IV})(x'')(x''')\right],$	$f\left[(x^{IV})(x')(x'')(x''')\right],$

Although the move may be seen as an abstraction in a traditional sense, the dependence on material configurations does not decrease.

The bracket notation is a form (in the Latourian sense) that takes rational expressions in the roots of a polynomial as matter, similar to the way in which the botanist's cabinet takes leaves as matter. Just as sorting the leaves in the cabinet creates new possibilities for further work, so does the novel material configuration offered by the bracket notation. In both cases, the organization draws out certain properties (the color of the leaf and the symmetries of the expression) and suppresses others. That the organization works like this is, however, hardly surprising, but once it is complete (the cabinet is full, effects of permutations are expressed in the bracket notation), we also see in the mathematical case that, in the new representation, the forms that the effects of permutations take, and the matter for a subsequent organization, are not clearly distinct.

Second transformation: Tables

The bracket notation is also a central part of the second transformation. As we have seen, Lagrange knew that for a polynomial of degree n, the possibility of obtaining algebraic expressions in the polynomial's coefficients for its roots depended on the effects of permutations of the roots on some chosen rational expression also found in the roots. As we have also seen, the bracket notation makes it possible to work with the effects of permutations on expressions without committing to a specific explicit expression. The next transformation concerns this use of the bracket notation.

In paragraph 107 of "Réflexions," Lagrange considers the general polynomial of degree 4, the quartic, after having dealt with the cubic in paragraphs 105–6 (Lagrange 1869, 394–397). He denotes the roots x', x'', x''', x^{IV} and considers an arbitrary rational expression in the roots without further specifying it. He uses the new bracket notation to identify the expression. Thus, in the bracket notation, the solvability of the quartic now depends on what Lagrange calls the form of f.

Lagrange then lists the $4! = 24$ bracket expressions, which can be obtained by moving x', x'', x''', x^{IV} around in f. The list appears in two columns in the original text, as reproduced in Table 1. From here, Lagrange continues as outlined in the background and context section: take a rational expression f in the roots of the general quartic and permute its roots x', x'', x''', x^{IV} in all possible ways. This produces $4! = 24$ expressions $f_1, f_2, \ldots, f_{24}$ (one of them is of course f itself). In the bracket notation, f is f, and the 24 bracket expressions listed above denote the 24 permutation variants f_i. Lagrange then states:

It will therefore be necessary to try to lower this equation to a lesser degree than the fourth, that is to say to the second or to the third degree, and it will be advisable to choose the latter, as being the highest that one can admit in this research. For this, it will therefore be necessary to ensure that the twenty-four roots that we have just found are equal eight to eight, and we will achieve this by comparing these roots with each other in all possible ways, until we find a combination which gives precisely eight equal roots, because then the other sixteen will also be equal eight to eight (97). (Lagrange 1869, 394)[18]

Recall that the possibility of solving the quartic depends on reducing the number of f_i, which can be done by choosing f appropriately: we can choose f to be an expression that remains unchanged by some of the twenty-four permutations.[19] In Lagrange's terminology, the f_i in the same subset have the same form, (i.e., they are similar). Lagrange's next move is to *assume* that any bracket expression in Table 1 is of one of only three forms, namely, f, f or f, which is equivalent to assuming that f has been chosen such that the permutations of the roots x', x'', x''', x^{IV} will only produce three different permutation variants of f, in effect reducing the solution of the quartic to the solution of the cubic. In terms of Table 1, this choice corresponds to a subdivision of the twenty-four bracket expressions into three groups of eight.

The novelty in this second transformation is the table. In Table 1, the difference between two bracket expressions shows the movement required to transform one into the other. Notice especially that Table 1 is organized according to the following principle:[20] the six bracket expressions in the top three-by-two block have the same letter x^{IV} in the fourth position, and each of the remaining three three-by-two blocks has a different letter in the fourth position. This means that in order to ensure all twenty-four possibilities get into the table, Lagrange can limit his attention to the first three positions: in each block, he just has to permute the remaining letters (occupying the first three positions) in every possible way. To do so, he permutes the "positions" and, in all the blocks, he does it according to the same pattern, as if the positions were empty. From Table 1's organization emerges a conceptualization of a permutation that does not require a specific object to move (e.g., a letter) but only positions between which to move. This detachment of the movement itself changes the concept of permutation toward that of an operator that is conceivable without things (e.g., letters) on which to operate.[21] Thus, there seems to be an abstraction of the process of moving letters for the sake of bookkeeping, which detaches the process from the concrete letters. However, this does not move the concept of the permutation further away from materiality since it happens with the material configuration of the table.

The two transformations show that matter and form are in rapid flux. The first transformation organized rational expressions as matter in the form of the bracket notation, and now, in the second transformation, the bracket expressions serve as matter formed by Table 1. Then the table becomes matter with Lagrange's assumption that it can be divided into three groups of eight similar expressions. Through these transformations, we juxtapose several representations but, in Wagner's terms (2019), moving between them does not generate a common invariant independent of the material features of the representations. To return to Latour's example,

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[18]"Il faudra donc tâcher d'abaisser cette équation à un degré moindre que le quatrième, c'est-à-dire au second ou au troisième degré, et il conviendra de choisir ce dernier, comme étant le plus haut qu'on puisse admettre dans cette recherche. Pour cela, il faudra donc faire en sorte que les vingt-quatre racines que nous venons de trouver soient égales huit à huit, et l'on y parviendra en comparant ces racines les unes avec les autres de toutes les manières possibles, jusqu'à ce qu'on trouve une combinaison qui donne justement huit racines égales, car alors les seize autres seront aussi égales huit à huit (97)" (Lagrange 1869, 394).

[19]In fact, such a choice of f implies that the set of 24 f_i *always* splits into subsets of equal size, such that the f_i in each set are equal to each other as expressions and only to each other. Lagrange already dealt with this (Lagrange 1869, 370–371). The reference "(97)" in the quote refers precisely to the point in his text.

[20]We stress that this is our interpretation; Lagrange does not say anything about the organization of the table.

[21]Lagrange does not seem to make further use of such conceptualization to investigate the roots of the quartic.

when the botanist uses a leaf to represent a location in the forest and uses a particular material aspect of the leaf (color) to place it in the cabinet, she constructs and transforms representations by using the mutability of form and matter, rather than stabilizing a separation of form from matter. In the case of a bracket expression, the feature that makes it represent all rational expressions obtained by certain permutations of the roots is the distribution of parentheses and letters between the brackets. So, when Lagrange compiled the table, he organized the twenty-four bracket expressions according to this same aspect, the distribution of the letters x', x'', x''', x^{IV}. Like color, the distribution of the letters does not belong to either form or matter.

Third transformation: Permutations and substitutions

We now pass from Lagrange to Galois. As mentioned earlier, we can assume that Galois draws on some of Lagrange's work on the use of the systematic interchange of letters to investigate the solvability of equations.[22] Like Lagrange, Galois introduces various new representations. However, before we consider them, we need to address an important point at which Galois' terminology diverges from that of Lagrange. Recall that Lagrange uses "permutation" to refer to a re-ordering of the letters that denote the roots. In contrast, Galois uses "permutation" to refer to a string of either single letters, or terms composed of several letters, which denote the roots of some polynomial. The following two examples (only the first is string-like; the second is more row-like, with wide spaces) are what Galois calls permutations (Galois and Neumann 2011, 125 and 117, respectively).

$$acdb$$

$$\phi V \quad \phi_1 V \quad \phi_2 V \ldots \ldots \phi_{m-1} V$$

In addition, Galois uses the term "substitution," which Lagrange did not. While Galois does not directly explain what he meant by a "permutation," he explicitly describes substitutions as the "passage from one permutation to another" (Galois and Neumann 2011, 115). Thus, in Galois' use of the terms, "permutation" assumes the sense of a concrete, static inscription. Meanwhile, "substitution" has the sense of the active re-ordering of the individual letters or terms of a permutation, changing it into another permutation (cf. Galois and Neumann 2011, 20–21). In the following paragraphs, "permutation" and "substitution" refer to Galois' terminology unless we explicitly state otherwise.

The transformation we now consider involves the relationship between permutations and substitutions. Like Lagrange's notion of permutation, Galois' notion of substitution refers to the act of re-ordering letters. But while Lagrange considered these acts relative to expressions in the roots of an equation, Galois, with the distinction between permutations and substitutions, introduces new objects—concrete strings—which provide a new material configuration that allows him to write down substitutions as pairs of permutations. We begin with a passage in which Galois directly addresses this relationship:

> The permutation from which one starts in order to indicate substitutions is completely arbitrary, as far as functions are concerned; for there is no reason at all why a letter should occupy one place rather than another in a function of several letters.

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[22]Precisely what Galois had read is an open question. Neumann doubts that Galois had read "Réflexions," whereas Kiernan mentions that Galois refers to a proof by Lagrange from "Réflexions" (Galois and Neumann 2011, 5; Kiernan 1971, 81). In any case, for our analysis, we need not assume that Galois had actually seen such representations as the bracket notation or the table of bracket expressions. The continuity between Galois and Lagrange that we assume is the association of solvability with the effects of exchanging certain letters in certain rational expressions. This continuity is well-documented, for example in the two secondary sources just mentioned (Galois and Neumann 2011; Kiernan 1971).

> Nevertheless, since it is impossible to grasp the idea of a substitution without that of a permutation, we will make frequent use of permutations in the language, and we shall not consider substitutions other than as the passage from one permutation to another.

> When we wish to group some substitutions we make them all begin from one and the same permutation.

> As the concern is always with questions where the original disposition of the letters has no influence, in the groups that we will consider one must have the same substitutions whichever permutation it is from which one starts. Therefore, if in such a group one has substitutions S and T, one is sure to have the substitution ST. (Galois and Neumann 2011, 115)[23]

In the first paragraph, Galois asserts that substitutions do not depend on any particular permutation: as long as we know how a substitution operates in terms of the chosen letters, it can be represented by taking an arbitrary permutation (i.e., a string of these letters) and re-ordering the letters accordingly to produce another permutation. This pair of permutations then "indicates" the substitution—but so does any other pair of permutations obtained in the same way. For example, the permutations abc, bac together describe the substitution that interchanges a and b and fixes c. This substitution is also described by the pair of permutations acb, bca.[24]

The first paragraph of the quotation thus states an abstraction. Substitutions are abstract in the sense that different pairs of permutations can represent the same substitution. Yet, as the second paragraph makes clear, this abstraction depends on the concrete materiality of permutations. It is impossible to "grasp the idea of a substitution" without having permutations "in the language" (Galois and Neumann 2011, 115). From our theoretical viewpoint, substitutions are thus introduced as reified in the material configurations of permutations. In Latourian terms, (pairs of) permutations are the matter that displays substitutions as particular forms. Once again, abstraction comes with reification, and form is closely interwoven with matter.

Although Galois seems to attribute a certain primality to permutations in the representation of substitutions, he also includes several other representations of substitutions in *PM*. One of these is introduced in the latter lines of the quoted passage, with two substitutions denoted by the single capital letters S and T, and the composition of the two substitutions denoted by the concatenation ST.[25] Thus, Galois introduces a new notation, which he uses to describe the set of substitutions as closed in the sense that when the set contains substitutions S and T, then it contains the substitution ST.[26]

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[23]Although Galois uses the term "group" in connection with collections of substitutions, "group" does not have an established technical meaning in mathematics when Galois writes *PM*. According to Neumann, Galois first uses the term in its ordinary meaning of "set" or "collection", its technical meaning emerging from his repeated use of it rather than from explicit definition (Galois and Neumann 2011, 22). Since the very point in question here is the emergence of an abstract notion of group, we will not consider Galois' "group" as a group in today's sense but simply as a collection of substitutions that may or may not have additional relations between them. In order to prevent today's notion of a group from confusing our description of Galois' text, we will call the property that Galois describes in the quoted passage a "closure" (shorthand for "closed under composition of substitutions").

[24]We stress that Galois represents substitutions as substitutions of the letters themselves, not as substitutions between positions on strings. Thus, the substitution described by the permutations abc, bac should not be understood as interchanging the first and second positions and fixing the third.

[25]The composition of substitutions S and T is again a substitution, the substitution obtained by first applying S to some arbitrary permutation and then applying T to the resulting permutation (following the modern convention of applying substitutions in reading order). The substitution ST is the direct passage from the initial to the third and final permutation.

[26]Equivalently, for the group of permutations, take any two pairs of permutations, and consider the substitution obtained by combining the two substitutions determined by the two pairs; there will always be somewhere in the group of permutations a pair of permutations that corresponds precisely to this substitution.

With this capital letter notation, we can designate individual substitutions and, without involving permutations, describe them as compositions of other substitutions. This advances the gradual detachment of substitutions from the material configurations of permutations. However, the chain of abstraction does not take off into pure abstraction since this step of detachment only happens with the reification in the capital letters. At the textual level, a process (that of substituting letters arranged in concrete strings) is turned into a concrete capital letter.[27] With each *tick* of abstraction follows a *tock* of reification.

There is a rich commentary on Galois' use of capital letter notation. A common point of view is that this kind of notation signals an increased level of abstraction compared with a notation that specifies terms, such as the string of letters (or the tables of strings that we consider in the next transformation). This idea is expressed by Ehrhardt when she considers Galois' use of capital letters G and H to denote groups.

> The single-letter notation led to another way to work on groups. ... [T]he group could be seen as <u>one</u> mathematical object, and not as a kind of cluster. ... [W]ith this notation, there was no need to precise what were exactly the elements contained in the group and to explain how they "moved" from one place to another within the group. Moreover, even if these elements were always substitutions in Galois's research, ... a group could eventually be *not* attached an equation. (Ehrhardt 2016, 104, emphases in original)

Here, Ehrhardt suggests that the capital letter notation functions in such a way that its occurrence in *PM* is central to the development of the mathematical concept of a group. In the context of natural science, this kind of representation has been described as similar to "an object." Klein, for example, writes, "the fact that a letter is a visible, discrete, and indivisible thing (unlike a written name) constitutes a minimal isomorphy with the postulated object it stands for, namely, the indivisible unit or portion of chemical elements" (Klein 2003, 26). In the context of mathematical representations, Grosholz (2007) adds (referring to Klein) that, "typographical isolation, for example, is iconic in intent. ... I came to similar conclusions [as Klein] about the iconic dimensions of symbols and the symbolic dimensions of icons in mathematics" (Grosholz 2007, 28). We do not question the validity of these comments, nor the importance of the capital letter notation for the historical development of today's conceptualization of a group. However, we emphasize that the notation of strings of letters is much more prevalent in *PM*. In fact, the capital letter notation for substitutions occurs only once in *PM* (in the passage already quoted). More importantly, however, as we will see in the following transformation, Galois actually uses the strings of letters to prove results about the solvability of polynomials.

Fourth transformation: Groups and subgroups

In the discussion of substitutions, Galois alludes to a practical implication of the independence of substitutions from permutations (Galois and Neumann 2011, 115). Since a substitution does not depend on any specific pair of permutations, we can produce a representation of a given set of substitutions simply by applying each of the substitutions to a single arbitrary permutation. The list of all the permutations thus obtained (including the initial permutation) describes the group of these substitutions. The next transformation concerns how the permutations that arise from "grouping" substitutions in this way are written. Galois organizes these permutations in rectangular arrangements, as shown in the following passage:

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[27]In Sfard's (2008) terms, we substitute talk about processes (of moving letters) with talk about objects (*S* and *T*).

Adjoining this square root to the equation of the 4th degree the group of the equation, which contains 24 substitutions in all, is decomposed into two which contain only 12 of them. Denoting the roots by a, b, c, d, here is one of these groups:

abcd,	*acdb*,	*adbc*,
badc,	*cabd*,	*dacb*,
cdab,	*dbac*,	*bcad*,
dcba,	*bdca*,	*cbda*.

(Galois and Neumann 2011, 125)

In the following, we refer to this arrangement as Table 2.[28] Each string of letters is a permutation, and each (ordered) pair of strings from anywhere in this table describes a substitution as the re-ordering of the letters a, b, c, d, required to pass from one string to the other.

By choosing any one of these permutations as the initial permutation and passing from this permutation to each of the others, we obtain twelve substitutions (including the passage from the initial permutation to itself, i.e., the substitution that fixes all the letters). The point is that no matter which permutation we start from, we will get the same twelve substitutions. Equivalently, any substitution determined by a pair of permutations in the table is one of these twelve substitutions.

At this point, the great difference between Table 2 and Table 1 becomes clear. Table 2 is not used for bookkeeping. The twelve permutations are organized in three columns of four, such that each column independently describes a closed set of substitutions. Galois states: *"these groups will enjoy the remarkable property that one will pass from one to another by operating on all the permutations of the first with one and the same substitution of letters"* (Galois and Neumann 2011, 119, emphases in original). For example, the substitution that fixes a and moves b to where d is, c to where b is, and d to where c is takes any entry in the first column to the corresponding entry in the second column. Moreover, *"the substitutions of letters are the same in each group"* (Galois and Neumann 2011, 121, emphases in original).[29] Galois does not give this property a name, but for simplicity, we will say that the group "splits."[30] The tabular representation thus makes it possible to express splitting as a feature of the organization of the table. Clearly, splitting is more specific than closure: we can easily distort Table 2 such that this feature is lost without changing the (closed) set of substitutions that Table 2 determines.

The splitting of Table 2 is essential when Galois demonstrates that the quartic is solvable.[31] In his demonstration, Galois presents a series of successive transformations of Table 2: first reducing it to its left-most column, then dividing and rearranging this column in a two-by-two table, and finally reducing this resulting two-column table to its left-most column:

Now this group [Table 2] is itself partitioned into three groups, as is indicated by Theorems II and III. Thus, by the extraction of a single radical of the 3rd degree, there remains simply the group

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$$abcd,$$

$$badc,$$

$$cdab,$$

$$dcab.$$

This group, in turn, is partitioned into two groups:

$$abcd, \; cdab,$$

$$badc, \; dcab$$

Thus, after a simple extraction of a square root, there will remain

$$abcd,$$

$$badc;$$

which finally is solved by a simple extraction of a square root. (Galois and Neumann 2011, 125–127)

The operations "extraction of a single radical of the 3$^{\text{rd}}$ degree" and "extraction of a square root," with which Galois explains the transformations, refer to adjunctions of a third degree radical and a square root (see the background and context section). The second "extraction of a square root," after the table has been reduced to the last single column table containing *abcd* and *badc*, will reduce this two-permutation table to a single permutation. This successive reduction proves that the general quartic is solvable: "I shall observe to begin with that, to solve an equation, it is necessary to reduce its group successively to the point where it does not contain more than a single permutation. For, when an equation is solved, an arbitrary function of its roots is known, even when it is not invariant under any permutation" (Galois and Neumann 2011, 121). Galois later adds that this condition is also sufficient (Galois and Neumann 2011, 125). More importantly, from our point of view, the relationship of the two columns of the middle two-by-two table to each other, and to the whole of this table, is the same as in Table 2, i.e., the group of substitutions of the middle two-by-two table splits just like that of Table 2. Moreover, just like in the case of Table 2, splitting is expressed as a feature of the concrete organization of the table.[32] The tabular representation makes it possible to show that the general quartic is solvable by successively manipulating a table. Such manipulation occurs on the condition that the table is organized in a way that does not just make splitting a structural property of each of the (smaller and smaller) sets of substitutions, but also a material feature of the organization of the (smaller and smaller) tables.

Harold Edwards has also commented on the structural properties of Galois' tables. Using the term "presentation" for those tables of permutations that we have called "closed," Edwards writes, "Galois singled out those subgroups with the property that the various presentations of the subgroup differ from one another by the application of a single substitution" (Edwards 1984, 50). Galois does this without abstracting the splitting property from the tables. For such an abstraction to happen—in our sense of the term—Galois would have to insert yet another representation, such that the move between them would render splitting independent of the tables. Admittedly, Galois does represent property using a different representation, namely the verbal description of

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[32]Galois could have explicated the final step, the second "extraction of a square root," by rearranging the last two-permutation column into a single row. This row, when viewed as a two-column table with a single entry in each column, would then also split.

how a table that splits looks like, but that description refers too directly to the tables to render splitting an abstract property.[33,34]

In our interpretation, the material design of Galois' tabular representation is crucial, as the fundamental property of splitting is reified in the physical layout of the inscription on the paper. Materiality is central, and even a sophisticated structural property—such as splitting—of sets of substitutions is not an abstraction away from material configurations in this case. Rather, splitting emerges as a special organization and reorganization of material configurations. It is a concrete procedure connected to the redistribution of actual inscriptions. However, we cannot say that splitting is strictly a property of the table. This claim can be seen as instantiating Latour's point that, in dealing with representations, we do not encounter a clear separation between form and matter; the matter constituted by the permutations intertwines with the form constituted by the group. Thus, we see that Galois produces a chain of elements of representation from which an abstract notion of group emerges, yet never ceases to be material.[35]

To recapitulate, we have observed that both Lagrange and Galois investigate the solvability problem using a range of representations. Following the chain of representations from Lagrange's verbal description of permutations of the roots of polynomials to Galois' groupings, we did not see a process of even greater detachment from materiality culminating in the abstract concept of mathematical groups. Rather, we saw a series of gradual transformations of representations, with each step involving both an abstraction and a reification in, for instance, the spatial design or typography of a new representation.

Discussion

Through the above case, we have asked what role representations play in the generation of new mathematical objects by discussing several representations used in two texts that played a central role in the development of the notion of mathematical groups. We have analyzed these inscriptions as a series of representational transformations of, and made in connection with, gradual conceptual changes. The analysis has shown that the representations cannot be seen only as designators of some abstract content. Rather, the materiality of the representations—the concrete inscriptions—and the affordances offered for (re-)configuring these inscriptions were crucially linked to the conceptual development. Theoretical advances and representational advances seem to go hand in hand.

The case thus shows that the more traditional idea of iterative abstraction—which sees mathematical objects as created through an iterative process that leads further and further away from materiality—is not satisfactory when it comes to accounting for the material aspects of mathematical innovation. We did not see the inscriptions gradually lose importance through the

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[33]Following Chemla (2003), we could see this as generality without abstraction since the series of tables together with the written language solves the general quartic (and later in *PM*, the general quintic). Recall that Chemla is concerned with how different mathematical cultures value abstraction and generality. In contrast, when we suggest that the use of the table notation in *PM* may be understood as general rather than as abstract, we are not making a claim about Galois' values but merely reflecting on the relationship between Chemla's definition of abstraction and ours. The question of the relationship between generality and abstraction in the mathematical practices of Galois and Lagrange would be an interesting question for further study.

[34]It is also interesting to compare with Wussing's (2007) interpretation. As mentioned, in the analysis section of our paper, in Wussing's reconstruction of Galois' work, the representations serve as denotations and as means for expressing thoughts. One limitation of such an approach is that it sets up a dichotomy between text and thoughts (such as group theoretical thinking) and tends to trivialize the relationship between these two terms. This is especially interesting since Wussing sees it as central to Galois' achievements that "[i]n essence, he posed the question how mathematical thinking should be externalized" (2007, 103). Wussing interprets Galois as proving by "group-theoretic means" without a criterion in terms of concrete text for how to identify such means (Wussing 2007, 113).

[35]Not *the* abstract notion of group, since Galois' group remains a group of substitutions and not, for example, an axiomatically given structure.

series of transformations; rather they remain central in an ongoing interaction between steps of abstraction and reification. The process seems to be better understood as a juxtaposition on the micro level of many conceptual changes (that may only partially conform to each other), creating a fluid interplay rather than a stable common core. At the level of external representation, the mathematical group appears to emerge as part of this fluid interplay. The emergence of the group as a mathematical object can thus be understood in terms of a chain of tightly nested elements of representation, and the emergence seems to happen as a successful interplay between these elements (the bracket notation, the various tables etc.). From a Latourian perspective, what makes it possible today to speak mathematically about groups in Galois' text is the maintaining of this fluid interplay, not a common invariant of multiple representations that mathematicians use and have used to work with groups.

In more general terms, since mathematical practice involves representational changes of various kinds—some of which may very well involve material changes—we question whether the creation of new mathematical objects and notions such as abstraction can be addressed in a meaningful way outside of specific contexts, especially material ones. They must be addressed with analytical categories phrased relative to the intra- and intertextual references and referential processes. As our analysis has shown, a particular type of representation—such as Lagrange's bracket notation—may in one context be said to function as a tool for representing certain categories of expressions, while in another context, it may be an object, itself being structured in a table. Thus, to speak about mathematical tools without their context is to overlook the ways in which representations, practices, and ontology co-evolve within mathematics.

In our interpretation, we emphasize the interconnection between representational and conceptual development. Surely, it would be possible do away with the added complexity by following the slow and gradual development of the representations used by the mathematicians in the case discussed; the theoretical advances of Lagrange and Galois can be reconstructed with present-day concepts and notations. For example, Edwards (1984) not only translates Galois into the language of algebraic number fields and field extensions but often also explains how he did so. Such translations may make it possible to follow a conceptual development independently of the specific representations that were used and developed by Lagrange and Galois. Employing a Latourian approach, we have tried to describe the limitations of this way of telling the story. Namely, the omission of the interconnections between the development of mathematical theory and specific—sometimes contingent—aspects of the materiality of the representations in use.[36]

Developing new kinds of representations is not a trivial task. Nevertheless, mathematicians are apparently willing to spend the time and effort needed to undertake it. A history of mathematics that detaches the theoretical development of mathematical concepts from the development of new representational forms, and even from the extension of mathematical practices to new kinds of materiality (such as the migration of mathematics onto paper or to digital technologies), makes it difficult to understand these ongoing processes. With our interpretation of the development of the mathematical concept of groups, we offer another view of the relationship between theoretical and representational development. Although we do not claim that a similar analysis can be applied universally, we hope that this reading can also inform our understanding of current mathematical practices.

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[36]This is not to say that the scheme that we have applied is complete or that it can describe every creation of a new mathematical object; on the contrary, mathematical objects may be created and their creation understood in many different ways.

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