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A new inner approach for differential subordinations

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In this paper we introduce and examine the differential subordination of the form

$$p(z) + zp'(z)\varphi(p(z), zp'(z)) \prec h(z), \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},$$

where h is a convex univalent function with $0 \in h(\mathbb{D})$. The proof of the main result is based on the original lemma for convex univalent functions and offers a new approach in the theory. In particular, the above differential subordination leads to generalizations of the well-known Briot-Bouquet differential subordination. Appropriate applications among others related to the differential subordination of harmonic mean are demonstrated. Related problems concerning differential equations are indicated.

Keywords: Differential subordination; Briot-Bouquet differential subordination; harmonic mean; convex function

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1. Introduction

Given $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. Let $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. For D being a domain in $\mathbb{C}$ let $\mathcal{H}(D)$ be the family of all analytic functions $f : D \to \mathbb{C}$ and $\mathcal{H} := \mathcal{H}(\mathbb{D})$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ of f normalized by $f(0) = 0 = f'(0) - 1$, and $\mathcal{S}$ be the subclass of $\mathcal{A}$ of univalent functions.

For $r \in (0, 1)$ and $f \in \mathcal{H}$, let $f_r(z) := f(rz)$, $z \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leqslant 1\}$. Clearly, each f_r is analytic in $\overline{\mathbb{D}}$.

A function $f \in \mathcal{H}$ is said to be subordinate to a function $g \in \mathcal{H}$ if there exists $\omega \in \mathcal{H}$ such that $\omega(0) = 0$, $\omega(\mathbb{D}) \subset \mathbb{D}$ and $f = g \circ \omega$ in $\mathbb{D}$. We write then $f \prec g$. If g is univalent, then (e.g., [**3**, Vol. I, p. 85])

$$f \prec g \Leftrightarrow (f(0) = g(0) \wedge f(\mathbb{D}) \subset g(\mathbb{D}))\,. \tag{1.1}$$

Given $\psi : \mathbb{C}^2 \to \mathbb{C}$, let $\mathcal{H}[\psi]$ be the subset of $\mathcal{H}$ of all p such that a function $\mathbb{D} \ni z \mapsto \psi(p(z), zp'(z))$ is well-defined and analytic.

Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ and $h \in \mathcal{H}$ be univalent. We say that a function $p \in \mathcal{H}[\psi]$ satisfies the first-order differential subordination and is called its solution if

$$\psi\left(p(z), zp'(z)\right) \prec h(z), \quad z \in \mathbb{D}. \tag{1.2}$$

If $q \in \mathcal{H}$ is a univalent function such that $p \prec q$ for all p satisfying (1.2), then q is called a dominant of (1.2). A dominant $\widetilde{q}$ is called the best dominant if $\widetilde{q} \prec q$ for all dominants q of (1.2). Finding those q for which the subordination (1.2) yields $p \prec q$, in particular, $p \prec \widetilde{q}$, is the basis in the theory of differential subordinations. Further details and references can be found in the book of Miller and Mocanu [**8**].

The classical example of (1.2) is related to the arithmetic mean and has been studied by many authors (see e.g., [**8**, pp. 120–145]). Given $\varphi \in \mathcal{H}(D)$ and $\alpha \in [0, 1]$, consider

$$p(z) + \alpha z p'(z)\varphi(p(z)) \prec h(z), \quad z \in \mathbb{D}, \tag{1.3}$$

written equivalently as

$$(1-\alpha)p(z) + \alpha(p(z) + zp'(z)\varphi(p(z))) \prec h(z), \quad z \in \mathbb{D},$$

where $p \in \mathcal{H}$ with $p(\mathbb{D}) \subset D$. In particular, if $\alpha = 1$, $\delta, \gamma \in \mathbb{C}$, $\delta \neq 0$, $\varphi(w) := 1/(\delta w + \gamma)$, $w \in D := \mathbb{C} \setminus \{-\gamma/\delta\}$, then (1.3) reduces to

$$p(z) + \frac{zp'(z)}{\delta p(z) + \gamma} \prec h(z), \quad z \in \mathbb{D}, \tag{1.4}$$

which is known as the Briot-Bouquet differential subordination.

Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$, $p \in \mathcal{H}[\varphi]$ and $h \in \mathcal{H}$ be a convex univalent function which means that h maps univalently $\mathbb{D}$ onto a convex domain $h(\mathbb{D})$. In this paper, we propose a study of the differential subordination of the form

$$p(z) + zp'(z)\varphi\left(p(z), zp'(z)\right) \prec h(z), \quad z \in \mathbb{D}. \tag{1.5}$$

The case when $0 \in \partial h(\mathbb{D})$ has been studied in [**5**]. Let us remark that the case where 0 is a boundary point of $h(\mathbb{D})$ requires different methods of proofs than those when the origin is the interior point of $h(\mathbb{D})$. The differential subordination (1.5) is a special case of (1.2), but it offers interesting applications. In particular, it generalizes the Briot-Bouquet differential subordination (1.4). In addition, we prove in a new way some recent results regarding the differential subordination related to the harmonic mean. The problem of the best dominant in the case where h is a linear function is also discussed.

The proof of the main result is based on the original lemma 2.1 on convex univalent functions. Therefore, the proof of Theorem 2.4 is strictly analytical in nature, while until now in the proofs of analogous propositions, analytical arguments have been used in conjunction with geometric considerations (cf. [**8**]). By applying lemma 2.1, a series of theorems from the monographs [**8**]) underlying the theory of the differential subordinations can be proved again by using a purely analytical argumentation.

2. Main result

2.1.

A function $h \in \mathcal{H}$ is said to be convex if it is univalent and $h(\mathbb{D})$ is a convex domain. Study [**12**] (e.g., [**9**, p. 44]) has shown that a function $h \in \mathcal{H}$ with $h'(0) \neq 0$ is convex if, and only if,

$$\operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

If h is a convex function, then $h(\mathbb{D}_r)$ for every $r \in (0, 1)$, is a convex domain (e.g., [**2**, p. 42], [**4**, p. 14]), so every h_r, $r \in (0, 1)$, is convex function also. Let $\mathcal{S}^c$ be the class of all convex functions normalized by $h(0) = 0$. For $h \in \mathcal{S}^c$ the following inequality due to Sheil-Small [**10**] and Suffridge [**13**] (see also [**9**, p. 44]) holds

$$\operatorname{Re}\left\{\frac{2\zeta h'(\zeta)}{h(\zeta) - h(z)} - \frac{\zeta + z}{\zeta - z}\right\} \geqslant 0, \quad z, \zeta \in \mathbb{D}. \tag{2.1}$$

The inequality (2.1) with $z = 0$ reduces to the inequality

$$\operatorname{Re}\frac{\zeta h'(\zeta)}{h(\zeta)} > \frac{1}{2}, \quad \zeta \in \mathbb{D}, \tag{2.2}$$

due to Marx [**6**] and Storhhäcker [**11**] (see also [**9**, p. 45]), which means that h is a starlike function of order $1/2$ (cf. [**3**, p. 138]).

Let $\mathcal{Q}$ be the subclass of $\mathcal{S}^c$ of all convex functions analytic on $\overline{\mathbb{D}}$ with $h'(\zeta) \neq 0$ at every $\zeta \in \mathbb{T}$.

We will now prove the lemma that will be used in the proof of the main theorem. This results is geometrically obvious.

LEMMA 2.1. *If $h \in \mathcal{Q}$, then*

$$\operatorname{Re}\frac{h(z) - h(\zeta)}{\zeta h'(\zeta)} < 0, \quad z \in \mathbb{D},\ \zeta \in \mathbb{T}. \tag{2.3}$$

Proof. Since h_r for every $r \in (0, 1)$, is analytic on $\overline{\mathbb{D}}$ and convex in $\mathbb{D}$, from (2.1) it follows that for $z \in \mathbb{D}$ and $\zeta \in \mathbb{T}$,

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{2\zeta h_r'(\zeta)}{h_r(\zeta) - h_r(z)} - \frac{\zeta + z}{\zeta - z}\right\} \\
&\quad = \operatorname{Re}\left\{\frac{2r\zeta h'(r\zeta)}{h(r\zeta) - h(rz)} - \frac{r\zeta + rz}{r\zeta - rz}\right\} \\
&\quad = \operatorname{Re}\left\{\frac{2uh'(u)}{h(u) - h(v)} - \frac{u + v}{u - v}\right\} \geqslant 0,
\end{aligned}$$

where $u := r\zeta \in \mathbb{D}$ and $v := rz \in \mathbb{D}$. Hence and by the fact that $h_r(\zeta) \to h(\zeta)$ and $h_r'(\zeta) \to h'(\zeta)$ as $r \to 1^-$, we deduce that

$$\operatorname{Re}\frac{2\zeta h'(\zeta)}{h(\zeta) - h(z)} \geqslant \operatorname{Re}\frac{\zeta + z}{\zeta - z} = \frac{1 - |z|^2}{|\zeta - z|^2} > 0, \quad z \in \mathbb{D},\ \zeta \in \mathbb{T},$$

which shows (2.3). □

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We need also the following lemma which is a special case of lemma 2.2d [8, p. 22].

LEMMA 2.2. *Let h be an analytic univalent function in $\overline{\mathbb{D}}$, $p \in \mathcal{H}$ be a nonconstant function with $p(0) = h(0)$. If p is not subordinate to h, then there exist $z_0 \in \mathbb{D} \setminus \{0\}$ and $\zeta_0 \in \mathbb{T}$ such that $p(\mathbb{D}_{|z_0|}) \subset h(\mathbb{D})$,*

$$p(z_0) = h(\zeta_0) \tag{2.4}$$

and

$$z_0 p'(z_0) = m\zeta_0 h'(\zeta_0) \tag{2.5}$$

for some $m \geqslant 1$.

The theorem below follows directly from the Lindelöf Principle (e.g., [3, Vol. I, p. 86]). However, it will be useful in proving the main theorem.

THEOREM 2.3. *Let f, $h \in \mathcal{H}$, h be univalent and $f(0) = h(0)$. Then*

$$f \prec h \tag{2.6}$$

if and only if for every $r \in (0, 1)$,

$$f_r \prec h_r. \tag{2.7}$$

Proof. Suppose that (2.6) holds. Then by the Lindelöf Principle (e.g., [3, Vol. I, p. 86]) for every $r \in (0, 1)$,

$$f_r(\mathbb{D}) = f(\mathbb{D}_r) \subset h(\mathbb{D}_r) = h_r(\mathbb{D}). \tag{2.8}$$

Since $f_r(0) = f(0) = h(0) = h_r(0)$ and every h_r is univalent, from (1.1) and (2.8) it follows (2.7).

Suppose now that (2.7) holds for every $r \in (0, 1)$. Then by (1.1) the inclusion (2.8) holds for every $r \in (0, 1)$, and therefore

$$f(\mathbb{D}) = \bigcup_{r\in(0,1)} f(\mathbb{D}_r) \subset \bigcup_{r\in(0,1)} h(\mathbb{D}_r) = h(\mathbb{D}).$$

Hence and from (1.1) we obtain (2.6). □

2.2.

We now prove the main theorem of this paper. In the proof we apply lemma 2.1 and Theorem 2.3. Therefore the argumentation is purely analytical without using a geometrical property based on the behaviour of the normal vector to the boundary curve $\partial h(\mathbb{D})$, standardly used in the theory (cf. [8]). In further discussion we present new type of the differential subordination generalizing the well known Briot-Bouquet differential subordination. The significance of Theorem 2.4 is emphasized also in the presented applications.

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Theorem 2.4. *Let h be a convex function and $\varphi : \mathbb{C}^2 \to \mathbb{C}$ be such that for each $m \geqslant 1$ a function*

$$\mathbb{D} \ni z \mapsto \varphi(h(z), mzh'(z)) \tag{2.9}$$

is well-defined and analytic satisfying the condition

$$\operatorname{Re}\varphi(h(z), mzh'(z)) \geqslant 0, \quad z \in \mathbb{D}. \tag{2.10}$$

If $p \in \mathcal{H}[\varphi]$ with $p(0) = h(0)$, and

$$p(z) + zp'(z)\varphi(p(z), zp'(z)) \prec h(z), \quad z \in \mathbb{D}, \tag{2.11}$$

then

$$p \prec h. \tag{2.12}$$

Proof. Note first that if $\operatorname{Re}\varphi(h(z_0),\, mz_0h'(z_0)) = 0$ for a certain $z_0 \in \mathbb{D}$, then by the minimum principle for harmonic function $\operatorname{Re}\varphi(h(z),\, mzh'(z)) = 0$ for all $z \in \mathbb{D}$ and hence $\varphi(h(z),\, mzh'(z)) = ia$ for some $a \in \mathbb{R}$ and all $z \in \mathbb{D}$.

Let $p \in \mathcal{H}[\varphi]$ with $p(0) = h(0)$. Define $\psi : \mathbb{D} \to \mathbb{C}$ as follows

$$\psi(z) := p(z) + zp'(z)\varphi(p(z), zp'(z)), \quad z \in \mathbb{D}. \tag{2.13}$$

Since $\psi(0) = p(0) = h(0)$, by Theorem 2.3 the condition (2.11) is equivalent to

$$\psi_r \prec h_r, \quad r \in (0, 1). \tag{2.14}$$

On the contrary, suppose that p is not subordinate to h. By Theorem 2.3, there exists $r_0 \in (0,\, 1)$ such that p_{r_0} is not subordinate to h_{r_0}. Since h_{r_0} is analytic in $\overline{\mathbb{D}}$, by lemma 2.2 there exist $z_0 \in \mathbb{D} \setminus \{0\}$, $\zeta_0 \in \mathbb{T}$ and $m \geqslant 1$ such that (2.4) and (2.5) hold with $p := p_{r_0}$ and $h := h_{r_0}$, i.e.,

$$p_{r_0}(z_0) = h_{r_0}(\zeta_0) \tag{2.15}$$

and

$$z_0p'_{r_0}(z_0) = m\zeta_0h'_{r_0}(\zeta_0). \tag{2.16}$$

Hence

$$\begin{aligned} \psi_{r_0}(z_0) &= \psi(r_0z_0) \\ &= p(r_0z_0) + r_0z_0p'(r_0z_0)\varphi(p(r_0z_0), r_0z_0p'(r_0z_0)) \\ &= p_{r_0}(z_0) + z_0p'_{r_0}(z_0)\varphi(p_{r_0}(z_0), z_0p'_{r_0}(z_0)) \\ &= h_{r_0}(\zeta_0) + m\zeta_0h'_{r_0}(\zeta_0)\varphi(h_{r_0}(\zeta_0), m\zeta_0h'_{r_0}(\zeta_0)). \end{aligned} \tag{2.17}$$

Moreover by (2.10),

$$\begin{aligned} \operatorname{Re}\varphi(h_{r_0}(\zeta_0), m\zeta_0h'_{r_0}(\zeta_0)) &= \operatorname{Re}\varphi(h(r_0\zeta_0), mr_0\zeta_0h'(r_0\zeta_0)) \\ &= \operatorname{Re}\varphi(h(u_0), mu_0h'(u_0)) \geqslant 0, \end{aligned} \tag{2.18}$$

where $u_0 := r_0\zeta_0 \in \mathbb{D}$. In view of (2.14), $\psi_{r_0} \prec h_{r_0}$, so $\psi_{r_0}(\mathbb{D}) \subset h_{r_0}(\mathbb{D})$. Thus $\psi_{r_0}(z_0) \in h_{r_0}(\mathbb{D})$ and therefore $\psi_{r_0}(z_0) = h_{r_0}(z_1)$ for some $z_1 \in h_{r_0}(\mathbb{D})$. Hence from

(2.17) and (2.18) we get

$$\operatorname{Re}\frac{h_{r_0}(z_1)-h_{r_0}(\zeta_0)}{\zeta_0 h'_{r_0}(\zeta_0)} = \operatorname{Re}\frac{\psi_{r_0}(z_0)-h_{r_0}(\zeta_0)}{\zeta_0 h'_{r_0}(\zeta_0)}$$
$$= m\operatorname{Re}\varphi(h_{r_0}(\zeta_0), m\zeta_0 h'_{r_0}(\zeta_0)) \geqslant 0.$$

Since $h_{r_0} \in \mathcal{Q}$, it follows that the above inequality contradicts (2.3) with $h := h_{r_0}$, $z := z_1$ and $\zeta := \zeta_0$. Thus we conclude that ψ_{r_0} is not subordinate to h_{r_0}, which contradicts (2.14) and completes the proof. □

In Theorem 2.4 instead of φ we can put a function $\phi : D \to \mathbb{C}$ such that a function $\mathbb{D} \ni z \mapsto \phi(h(z))$ is well-defined and analytic satisfying the condition $\operatorname{Re}\phi(h(z)) \geqslant 0$ for $z \in \mathbb{D}$. Then we obtain the result due to Miller and Mocanu [**7**] (see also [**8**, Theorem 3.4a, p. 120]).

Corollary 2.5 [**7**]. *Let h be a convex function, $\phi \in \mathcal{H}(D)$ be such that $h(\mathbb{D}) \subset D$ and*

$$\operatorname{Re}\phi(h(z)) \geqslant 0, \quad z \in \mathbb{D}.$$

If $p \in \mathcal{H}$, $p(0) = h(0)$, $p(\mathbb{D}) \subset D$ and

$$p(z) + zp'(z)\phi(p(z)) \prec h(z), \quad z \in \mathbb{D},$$

then

$$p \prec h.$$

In the following theorem the assumption (2.19) is based on the idea of [**1**] (see also [**8**, pp. 124–125]), where in the proof of the main result Löwner chains were used. Our argumentation is analogous to that in the proof of Theorem 2.4.

Theorem 2.6. *Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and $p \in \mathcal{H}[\varphi]$ be such that*

$$\operatorname{Re}\varphi(p(z), zp'(z)) \geqslant 0, \quad z \in \mathbb{D}. \tag{2.19}$$

If h is a convex function with $h(0) = p(0)$ and

$$p(z) + zp'(z)\varphi(p(z), zp'(z)) \prec h(z), \quad z \in \mathbb{D}, \tag{2.20}$$

then

$$p \prec h. \tag{2.21}$$

Proof. Let ψ be defined as in (2.11). By Theorem 2.3 the condition (2.20) is equivalent to (2.14). On the contrary, suppose that p is not subordinate to h. As in the proof of Theorem 2.4, there exist $z_0 \in \mathbb{D} \setminus \{0\}$, $\zeta_0 \in \mathbb{T}$ and $m \geqslant 1$ such that (2.15) and (2.16) hold. Thus

$$\begin{aligned}\psi_{r_0}(z_0) &= \psi(r_0 z_0)\\ &= p(r_0 z_0) + r_0 z_0 p'(r_0 z_0)\varphi(p(r_0 z_0), r_0 z_0 p'(r_0 z_0))\\ &= p_{r_0}(z_0) + z_0 p'_{r_0}(z_0)\varphi(p(r_0 z_0), r_0 z_0 p'(r_0 z_0))\\ &= h_{r_0}(\zeta_0) + m\zeta_0 h'_{r_0}(\zeta_0)\varphi(p(r_0 z_0), r_0 z_0 p'(r_0 z_0))\\ &= h_{r_0}(\zeta_0) + m\zeta_0 h'_{r_0}(\zeta_0)\varphi(p(u_0), u_0 p'(u_0)),\end{aligned} \tag{2.22}$$

where $u_0 := r_0 z_0 \in \mathbb{D}$. In view of (2.14), $\psi_{r_0} \prec h_{r_0}$, so $\psi_{r_0}(\mathbb{D}) \subset h_{r_0}(\mathbb{D})$. Thus $\psi_{r_0}(z_0) \in h_{r_0}(\mathbb{D})$ and therefore $\psi_{r_0}(z_0) = h_{r_0}(z_1)$ for some $z_1 \in \mathbb{D}$. Hence from (2.22) and (2.19) it follows that

$$\begin{aligned}\operatorname{Re}\frac{h_{r_0}(z_1) - h_{r_0}(\zeta_0)}{\zeta_0 h'_{r_0}(\zeta_0)} &= \operatorname{Re}\frac{\psi_{r_0}(z_0) - h_{r_0}(\zeta_0)}{\zeta_0 h'_{r_0}(\zeta_0)}\\ &= m\operatorname{Re}\varphi(p(u_0), u_0 p'(u_0)) \geqslant 0.\end{aligned}$$

Since $h_{r_0} \in \mathcal{Q}$, it follows that the above inequality contradicts (2.3) with $h := h_{r_0}$, $z := z_1$ and $\zeta := \zeta_0$. Thus we conclude that ψ_{r_0} is not subordinate to h_{r_0}, which contradicts (2.14), so (2.20) and completes the proof. □

3. Special cases

3.1.

We now discuss special cases of Theorem 2.4.

Corollary 3.1. *Let $\beta \geqslant 0$, $h \in \mathcal{S}^c$, $\phi \in \mathcal{H}(D)$ be such that $h(\mathbb{D}) \subset D$ and*

$$\operatorname{Re}\phi(h(z)) \geqslant 0, \quad z \in \mathbb{D}. \tag{3.1}$$

If $p \in \mathcal{H}$, $p(0) = 0$, $p(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$, $p(\mathbb{D}) \subset D$ and

$$p(z) + zp'(z)\phi(p(z)) + \beta p(z)\left(\frac{zp'(z)}{p(z)}\right)^2 \prec h(z), \quad z \in \mathbb{D}, \tag{3.2}$$

then

$$p \prec h. \tag{3.3}$$

Proof. Let $\beta \geqslant 0$, $h \in \mathcal{S}^c$ and $\phi \in \mathcal{H}(D)$ be such that $h(\mathbb{D}) \subset D$. Define $\varphi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ as

$$\varphi(u, v) := \phi(u) + \beta\frac{v}{u}, \quad (u, v) \in (D \setminus \{0\}) \times \mathbb{C}.$$

Since $h(0) = 0$, $h(z) \neq 0$ for $z \neq 0$ and $h'(0) \neq 0$, it follows that

$$\lim_{z\to 0}\frac{zh'(z)}{h(z)} = 1. \tag{3.4}$$

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Thus the function

$$\mathbb{D} \setminus \{0\} \ni z \mapsto \varphi(h(z), mzh'(z)) = \phi(h(z)) + \beta \frac{mzh'(z)}{h(z)}$$

has an analytic extension on $\mathbb{D}$ by setting $\phi(0) + \beta m$ at zero. Moreover by (2.2) and (3.1) for every $m \geqslant 1$,

$$\begin{aligned} &\operatorname{Re} \varphi(h(z), mzh'(z)) \\ &= \operatorname{Re} \phi(h(z)) + \beta m \operatorname{Re} \frac{zh'(z)}{h(z)} \geqslant \frac{\beta}{2} \geqslant 0, \quad z \in \mathbb{D}. \end{aligned} \tag{3.5}$$

Let $p \in \mathcal{H}$, $p(0) = 0$, $p(z) \neq 0$ for $z \neq 0$ and $p(\mathbb{D}) \subset D$. Because $p(0) = 0$, there exists a positive integer k such that $p(z) = z^k q(z)$, $z \in \mathbb{D}$, where $q \in \mathcal{H}$ and $q(0) \neq 0$. Hence and by the fact that $p(z) \neq 0$ for $z \neq 0$, it follows that $q(z) \neq 0$ for $z \in \mathbb{D}$. Thus the function

$$\mathbb{D} \ni z \mapsto \varphi(p(z), zp'(z)) = \phi(p(z)) + \beta \frac{zp'(z)}{p(z)}$$

has an analytic extension on $\mathbb{D}$ by setting $\phi(0) + \beta k$ at zero, i.e., $p \in \mathcal{H}[\varphi]$. At the end note that

$$\begin{aligned} &p(z) + zp'(z)\varphi\left(p(z), zp'(z)\right) \\ &= p(z) + zp'(z)\phi(p(z)) + \beta p(z) \left(\frac{zp'(z)}{p(z)}\right)^2, \quad z \in \mathbb{D}. \end{aligned}$$

Thus, the assumptions of Theorem 2.4 are satisfied, which ends the proof of the corollary. □

For $\beta = 0$ the above theorem reduces to Corollary 2.5, with the additional assumption that $h(0) = 0$. In fact, this assumption is not required in Corollary 2.5.

For $\mathcal{H}(\mathbb{C}) \ni \phi \equiv \alpha$, where $\operatorname{Re} \alpha \geqslant 0$, Corollary 3.1 takes the following form.

Corollary 3.2. *Let* $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha \geqslant 0$, $\beta \geqslant 0$ *and* $h \in \mathcal{S}^c$. *If* $p \in \mathcal{H}$, $p(0) = 0$, $p(z) \neq 0$ *for* $z \in \mathbb{D} \setminus \{0\}$, *and*

$$p(z) + \alpha zp'(z) + \beta p(z) \left(\frac{zp'(z)}{p(z)}\right)^2 \prec h(z), \quad z \in \mathbb{D}, \tag{3.6}$$

then

$$p \prec h.$$

For $\alpha = 0$ from Corollary 3.2 we deduce

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Corollary 3.3. *Let* $\beta \geqslant 0$ *and* $h \in \mathcal{S}^c$. *If* $p \in \mathcal{H}$, $p(0) = 0$, $p(z) \neq 0$ *for* $z \in \mathbb{D} \setminus \{0\}$, *and*

$$p(z) + \beta p(z) \left(\frac{zp'(z)}{p(z)} \right)^2 \prec h(z), \quad z \in \mathbb{D},$$

then

$$p \prec h.$$

The differential subordination (3.9), which is a special case of (3.2), is an interesting generalization of the Briot-Bouqet subordination. Briot-Bouquet differential subordination plays a fundamental role in the theory of the differential subordinations. We get it from (3.9) for $\beta = 0$ (e.g. [**8**, pp. 80–105]). The following corollary follows from Corollary 3.1.

Corollary 3.4. *Let* $\beta \geqslant 0$, δ, $\gamma \in \mathbb{C}$, $\delta \neq 0$, *and* $h \in \mathcal{S}^c$ *be such that*

$$h(z) \neq -\gamma/\delta, \quad z \in \mathbb{D}, \tag{3.7}$$

and

$$\operatorname{Re}(\delta h(z) + \gamma) > 0, \quad z \in \mathbb{D}. \tag{3.8}$$

If $p \in \mathcal{H}$, $p(0) = 0$, $p(z) \neq 0$ *for* $z \in \mathbb{D} \setminus \{0\}$, $p(z) \neq -\gamma/\delta$ *for* $z \in \mathbb{D}$, *and*

$$p(z) + \frac{zp'(z)}{\delta p(z) + \gamma} + \beta p(z) \left(\frac{zp'(z)}{p(z)} \right)^2 \prec h(z), \quad z \in \mathbb{D}, \tag{3.9}$$

then

$$p \prec h.$$

Proof. Take

$$\phi(w) := \frac{1}{\delta w + \gamma}, \quad w \in D := \mathbb{C} \setminus \{-\gamma/\delta\}.$$

Then by (3.7) the function $\phi \circ h$ is analytic in $\mathbb{D}$, and by (3.8),

$$\operatorname{Re} \phi(h(z)) = \operatorname{Re} \frac{1}{\delta h(z) + \gamma} > 0, \quad z \in \mathbb{D},$$

and we apply Corollary 3.1. □

In the same way as Corollary 3.4 the following result follows.

Corollary 3.5. *Let* $\beta \geqslant 0$, $\delta, \gamma \in \mathbb{C}$, $\delta \neq 0$, *and* $h \in \mathcal{S}^c$ *be such that*

$$\operatorname{Re}(\delta h(z) + \gamma) > 0, \quad z \in \mathbb{D}.$$

If $p \in \mathcal{H}$, $p(0) = 0$, $p(z) \neq 0$ *for* $z \in \mathbb{D} \setminus \{0\}$, *and*

$$p(z) + zp'(z)(\delta p(z) + \gamma) + \beta p(z) \left(\frac{zp'(z)}{p(z)} \right)^2 \prec h(z), \quad z \in \mathbb{D},$$

then

$$p \prec h.$$

3.2.

By selecting $h \in \mathcal{S}^c$ we can get a number of new results. It is natural to take into account the following convex functions keeping the origin fixed: for $M > 0$,

$$h_1(z) = Mz, \quad h_2(z) = \frac{2Mz}{1 - z}, \quad z \in \mathbb{D},$$
$$h_3(z) = \frac{M}{\pi} \log \frac{1 + z}{1 - z}, \quad \log 1 := 0, \ z \in \mathbb{D}.$$

Then Corollary 3.1 takes respectively the form

Corollary 3.6. *Let* $\beta \geqslant 0$, $\phi \in \mathcal{H}(D)$ *be such that* $\mathbb{D}_M \subset D$ *and*

$$\operatorname{Re} \phi(Mz) \geqslant 0, \quad z \in \mathbb{D}.$$

If $p \in \mathcal{H}$, $p(0) = 0$, $p(z) \neq 0$ *for* $z \in \mathbb{D} \setminus \{0\}$, $p(\mathbb{D}) \subset D$ *and*

$$\left| p(z) + zp'(z)\phi(p(z)) + \beta p(z) \left(\frac{zp'(z)}{p(z)} \right)^2 \right| < M, \quad z \in \mathbb{D},$$

then

$$|p(z)| < M, \quad z \in \mathbb{D}.$$

Corollary 3.7. *Let* $\beta \geqslant 0$, $\phi \in \mathcal{H}(D)$ *be such that* $A := \{w \in \mathbb{C} : \operatorname{Re} w > -M\} \subset D$ *and*

$$\operatorname{Re} \phi(w) \geqslant 0, \quad w \in A.$$

If $p \in \mathcal{H}$, $p(0) = 0$, $p(z) \neq 0$ *for* $z \in \mathbb{D} \setminus \{0\}$, $p(\mathbb{D}) \subset D$ *and*

$$\operatorname{Re} \left\{ p(z) + zp'(z)\phi(p(z)) + \beta p(z) \left(\frac{zp'(z)}{p(z)} \right)^2 \right\} > -M, \quad z \in \mathbb{D},$$

then

$$\operatorname{Re} p(z) > -M, \quad z \in \mathbb{D}.$$

Corollary 3.8. *Let $\beta \geqslant 0$, $\phi \in \mathcal{H}(D)$ be such that $A := \{w \in \mathbb{C} : |\operatorname{Im} w| < M\} \subset D$ and*

$$\operatorname{Re} \phi(w) \geqslant 0, \quad w \in A.$$

If $p \in \mathcal{H}$, $p(0) = 0$, $p(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$, $p(\mathbb{D}) \subset D$ and

$$\left| \operatorname{Im} \left\{ p(z) + zp'(z)\phi(p(z)) + \beta p(z) \left(\frac{zp'(z)}{p(z)} \right)^2 \right\} \right| < M, \quad z \in \mathbb{D},$$

then

$$|\operatorname{Im} p(z)| < M, \quad z \in \mathbb{D}.$$

4. The best dominant

To find the best dominant of (2.12) is an interesting problem to study related to the theory of the differential equations. By applying Theorem 2.3e of [8] we can expect that the best dominant $\widetilde{q}$ of (1.5) should be a univalent solution of the differential equation

$$p(z) + zp'(z)\varphi(p(z), zp'(z)) = h(z), \quad z \in \mathbb{D},$$

if such a solution exists. Here we restrict our interest to the differential subordination (3.6) with $h(z) := Mz$, $z \in \mathbb{D}$, where $M > 0$. For this purpose, we will find a univalent solution of the differential equation

$$q(z) + \alpha z q'(z) + \beta q(z) \left(\frac{zq'(z)}{q(z)} \right)^2 = Mz, \quad z \in \mathbb{D}. \tag{4.1}$$

The following theorem provides a solution to this problem.

Theorem 4.1. *Let $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha \geqslant 0$, $\beta \geqslant 0$ and $M > 0$. If $p \in \mathcal{H}$, $p(0) = 0$, $p(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$, and*

$$p(z) + \alpha z p'(z) + \beta p(z) \left(\frac{zp'(z)}{p(z)} \right)^2 \prec Mz, \quad z \in \mathbb{D}, \tag{4.2}$$

then

$$p(z) \prec \frac{M}{1 + \alpha + \beta} z =: \widetilde{q}(z), \quad z \in \mathbb{D},$$

and $\widetilde{q}$ is the best dominant of (4.2).

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Proof. We apply the technique of power series to find a univalent solution of (4.1) of the form

$$q(z)=\sum_{n=1}^{\infty}a_nz^n,\quad z\in\mathbb{D}. \tag{4.3}$$

Since q is assumed to be univalent, we see that

$$a_1=q'(0)\neq 0. \tag{4.4}$$

From (4.1) we equivalently have

$$q^2(z)+\alpha zq(z)q'(z)+\beta z^2(q'(z))^2=Mzq(z),\quad z\in\mathbb{D}.$$

Hence using (4.3) we have

$$\begin{aligned}
&a_1^2z^2+2a_1a_2z^3+\left(2a_1a_3+a_2^2\right)z^4+\left(2a_1a_4+2a_2a_3\right)z^5+\dots\\
&\qquad+\alpha\left[a_1z^2+3a_1a_2z^3+\left(4a_1a_3+2a_2^2\right)z^4+\left(5a_1a_4+5a_2a_3\right)z^5+\dots\right]\\
&\qquad+\beta\left[a_1^2z^2+4a_1a_2z^3+\left(6a_1a_3+4a_2^2\right)z^4+\left(8a_1a_4+12a_2a_3\right)z^5+\dots\right]\\
&\quad=M\left(a_1z^2+a_2z^3+a_3z^4+c_4z^5+\dots\right),\quad z\in\mathbb{D}.
\end{aligned}$$

Comparing coefficients we obtain

$$\begin{aligned}
&a_1^2(1+\alpha+\beta)=Ma_1,\\
&a_1a_2(2+3\alpha+4\beta)=Ma_2,\\
&a_1a_3(2+4\alpha+6\beta)+a_2^2(1+2\alpha+4\beta)=Ma_3,
\end{aligned} \tag{4.5}$$

and in general, for $n=2k-1,\ \ k\geqslant 2$,

$$\begin{aligned}
&a_1a_{2k-2}\left[2+(2k-1)\alpha+2(2k-2)\beta\right]\\
&\quad+a_2a_{2k-3}\left[2+(2k-1)\alpha+2\cdot 2(2k-3)\beta\right]+\dots\\
&\quad+a_{k-1}a_k\left[2+(2k-1)\alpha+2(k-1)k\beta\right]=Ma_{2k-2},
\end{aligned} \tag{4.6}$$

and for $n=2k,\ \ k\geqslant 2$,

$$\begin{aligned}
&a_1a_{2k-1}\left[2+2k\alpha+2(2k-1)\beta\right]\\
&\quad+a_2a_{2k-2}\left[2+2k\alpha+2\cdot 2(2k-2)a_2a_{2k-2}\right]+\dots\\
&\quad+a_{k-1}a_{k+1}\left[2+2k\alpha+2(k-1)(k+1)\beta\right]\\
&\quad+a_k^2\left[1+k\alpha+k^2\beta\right]=Ma_{2k-1}.
\end{aligned} \tag{4.7}$$

Taking into account (4.4) from the first equation in (4.5) it follows that

$$a_1=\frac{M}{1+\alpha+\beta}. \tag{4.8}$$

This and the second equation in (4.5) yield $a_2=0$. Substituting $a_2=0$ into the third equation in (4.5) in view of (4.8) we see that $a_3=0$. In this way, by using mathematical induction we can prove that

$$a_2 = a_3 = \cdots = a_{2k-3} = 0 \tag{4.9}$$

and that the formula (4.6) reduces to

$$a_1 a_{2k-2}\left[2 + (2k-1)\alpha + 2(2k-2)\beta\right] = M a_{2k-2},$$

which in view of (4.8) yield $a_{2k-2} = 0$. Hence by using (4.9) the equation (4.7) reduces to

$$a_1 a_{2k-1}\left[2 + 2k\alpha + 2(2k-1)\beta\right] = M a_{2k-1},$$

which in view of (4.8) yield $a_{2k-1} = 0$. Thus we proved that $a_n = 0$ for all $n \geqslant 2$. In this way by (4.3) and (4.8) we see that

$$q(z) = \frac{M}{1+\alpha+\beta} z =: \widetilde{q}(z), \quad z \in \mathbb{D},$$

is a unique univalent solution of (4.1). From Theorem 2.3e of [**8**] it follows that $\widetilde{q}$ is the best dominant of (4.2) which completes the proof of the lemma. □

For $\alpha = 1$, $\beta = 1$ and $M = 1$, the above result reduces to the well known special case of the first order Euler differential subordination (see [**8**, pp. 334–340]).

Corollary 4.2. *If* $p \in \mathcal{H}$, $p(0) = 0$ *and*

$$p(z) + zp'(z) \prec z, \quad z \in \mathbb{D},$$

then

$$p(z) \prec \frac{1}{2} z =: \widetilde{q}(z), \quad z \in \mathbb{D},$$

and $\widetilde{q}$ *is the best dominant.*

References

1 Y. Chunfang. Univalence of analytic solutions to some first-order differential equations. *Acta Math. Sinica* **35** (1992), 483–491.

2 P. L. Duren. *Univalent Functions* (Springer Verlag, New York, 1983).

3 A. W. Goodman. *Univalent Functions* (Mariner Publishing Company Inc, Tampa, Florida, 1983).

4 D. J. Hallenbeck and T. H. MacGregor. *Linear problems and convexity techniques in geometric function theory* (Boston, London, Melbourne: Pitman Publishing Inc, 1984).

5 A. Lecko. A new boundary approach for differential subordinations. *Proc. Am. Math. Soc.* **14** (2022), 324. doi:10.1090/proc/16318

6 A. Marx. Untersuchungen über schlichte Abbildungen. *Math. Ann.* **107** (1932), 40–67.

7 S. S. Miller and P. T. Mocanu. On some classes of first-order differential subordinations. *Mich. Math. J.* **32** (1985), 185–195.

8 S. S. Miller and P. T. Mocanu, Differential Subordinations. Theory and Applications (Marcel Dekker Inc., New York, Basel, 2000).

9 C. Pommerenke. *Univalent Functions* (Vandenhoeck & Ruprecht (Göttingen, 1975).

10 T. Sheil-Small. On convex univalent functions. *J. London Math. Soc.* **2** (1969), 483–492.

11 E. Strohhäcker. Beiträge zur Theorie der schlichten Funtionen. *Math. Z.* **11** (1933), 356–380.

12 E. Study, Vorlesungen über ausgewählte Gegenstände der Geometrie, Zweites Heft; Konforme Abbildung Einfach-Zusammenhängender Bereiche (Druck und Verlag von B.G. Teubner, Leipzig und Berlin, 1913).

13 T. J. Suffridge. Some remarks on convex maps of the unit disk. *Duke Math. J.* **37** (1970), 775–777.

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