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The lower bounds of non-real eigenvalues for singular indefinite Sturm–Liouville problems

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The present paper deals with the non-real eigenvalues for singular indefinite Sturm–Liouville problems. The lower bounds on non-real eigenvalues for this singular problem associated with a special separated boundary condition are obtained.

Keywords: Sturm–Liouville problem; non-real eigenvalue; indefinite; lower bounds

2020 *Mathematics Subject Classification:* 34B24; 34L15; 47B50

1. Introduction

In the present paper, we consider the eigenvalue problem

$$-(py')' + qy = \lambda wy, \quad [y, u_{-1}](-1) = 0 = [y, u_1](1), \tag{1.1}$$

where the functions p, q, w are real-valued and w changes sign on $(-1, 1)$ and u_{-1}, u_1 are the principal solutions of differential expression

$$-(py')' + qy = \lambda wy \tag{1.2}$$

at -1, 1 for $\lambda = 0$, respectively. Such a problem is called *indefinite* and the indefinite nature, that non-real spectral points may appear, was noticed by Haupt [**11**] and Richardson [**19**] at the beginning of the last century and has attracted a lot of attention in the recent years, see [**1**, **7**, **8**, **10**, **14**].

A priori bounds on non-real eigenvalues for indefinite Sturm–Liouville problems were raised in [**15**] by Mingarelli and stressed by Kong *et al.* [**13**]. Recently, the regular indefinite case of this problem was solved by Qi *et al.* in [**2**, **17**, **18**, **21**]. For the singular indefinite Sturm–Liouville problems with limit-point type endpoints

$$(Af)(x) := \mathrm{sgn}(x)(-f''(x) + V(x)f(x)) = \lambda f(x), \quad x \in \mathbb{R}, \tag{1.3}$$

the authors in [**3**] provided sufficient conditions for the existence of non-real eigenvalues. The explicit bounds on the non-real eigenvalues of (1.3) were obtained in [**4**]. In [**5**, **6**], the authors not only estimated the absolute values of the non-real eigenvalues in terms of the L^1-norm of the continuous potential, but also obtained the bounds on the imaginary parts and absolute values of these eigenvalues in terms of the L^1-norm of the potential and its negative part. Recently in [**20**], the authors solved the estimates of absolute values on the non-real eigenvalues for the singular indefinite Sturm–Liouville eigenvalue problems with limit-circle type non-oscillation endpoints associated with a special self-adjoint boundary condition.

In the present paper, we will focus on the singular indefinite Sturm–Liouville eigenvalue problems with self-adjoint boundary conditions associated with principal solution at endpoints. The lower bounds of this eigenvalue problem are obtained under the some conditions. This paper is organized as follows: the preliminary knowledge and the main results, theorems 2.2 and 2.3, are stated in § 2 and their proofs are given in § 3.

2. Preliminary knowledge and main results

In this section, we give some basic knowledge for the singular differential equation (1.2) under the standard conditions that p, q, w are real-valued functions satisfying

$$p > 0, |w| > 0 \text{ a.e. on } (-1, 1),\ \frac{1}{p}, w, q \in L^1_{\text{loc}}(-1, 1), \int_{-1}^{1} \left(\left| \frac{1}{p} \right| + |q| + |w| \right) = \infty \tag{2.1}$$

and w changes its sign on $(-1, 1)$ in the meaning that

$$\text{mes}\{x : w(x) > 0\} > 0, \quad \text{mes}\{x : w(x) < 0\} > 0.$$

We first introduce some concepts. For fixed $\lambda \in \mathbb{R}$, a real solution $u(x)$ of (1.2) is called a *principal solution* at 1 if there exists $c \in (-1, 1)$ such that $u(x) \neq 0$, $x \in (c, 1)$, $\int_c^1 1/(pu^2) = \infty$. A real solution $v(x)$ of (1.2) is called a *non-principal solution* at 1 if there exists $c \in (-1, 1)$ such that $v(x) \neq 0$, $x \in (c, 1)$, $\int_c^1 1/(pv^2) < \infty$. If u and v are principal and non-principal solutions at 1, respectively, then $u(x)/v(x) \to 0$ as $x \to 1$. (cf. [**16**] and [**22**, Theorem 6.2.2]). In order to give the asymptotic behaviours of eigenfunctions at the endpoints, we assume that

$$\Upsilon(t) := \sup_{-1<x<1} \left| \frac{1}{p(t)} \int_t^x q(s) \mathrm{d}s \right| \in L^1(-1, 1), \quad \int_c^x \frac{1}{p(t)} \mathrm{d}t \in L^2_{|w|}(-1, 1) \tag{2.2}$$

for some (and hence for all) $c \in (-1, 1)$. Throughout this paper, the functions p, q, w always satisfy (2.1) and (2.2). Then

Lemma 2.1. *[**20**, Lemma 2.3] Assume that (2.2) holds, $u_1(\cdot)$ is a principal solution of (1.2) at 1 for $\lambda = 0$. Let y be an eigenfunction of (1.1) corresponding to the eigenvalue λ. If either $\int_c^1 1/p(t)\,dt = \infty$ or $\int_c^1 1/p(t)\,dt < \infty$, $q \in L^1(-1, 1)$, then y is bounded and*

$$[y, u_1](1) = 0 \Leftrightarrow (py')(x)y(x) \to 0 \ \textit{as} \quad x \to 1. \tag{2.3}$$

The similar conclusion holds for $x \to -1$.

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The operator S associated with the right-definite problem

$$\begin{cases} -(py')'(x) + q(x)y(x) = \lambda|w(x)|y(x), \\ [y, u_{-1}](-1) = 0, \ [y, u_1](1) = 0 \end{cases} \tag{2.4}$$

is defined as $Sy = \frac{1}{|w|}\tau y$ for $y \in D(S)$, where $\tau y := -(py')'(x) + q(x)y(x)$,

$$D(S) = \{y \in L^2_{|w|}(-1, 1) : \ y, py' \in AC_{loc}(-1, 1), \tau y/|w| \in L^2_{|w|}(-1, 1), \ \mathcal{B}y = 0\}$$

and $\mathcal{B}y = 0 := \ [y, \, u_{-1}](-1) = 0 = [y, \, u_1](1)$. It follows from [**12**] and [**22**, Theorem 10.6.2, p.195] that the operator S is self-adjoint in the Hilbert space $(L^2_{|w|}, \, (\cdot, \, \cdot)_{|w|})$ and it has discrete spectrum consisting of an infinite number of eigenvalues $\{\mu_n : n \in \mathbb{N} := \{1, \, 2, \, \cdots\}\}$, which are all real, unbounded from above and bounded from below, i.e. $-\infty < \mu_1 < \mu_2 < \mu_3 < \cdots \to +\infty$.

Let $K = (L^2_{|w|}(-1, \, 1), \, [\cdot, \, \cdot]_w)$ be the Krein space with the indefinite inner product $[f, \, g]_w = \int_{-1}^1 w(x)f(x)\overline{g(x)} \, \mathrm{d}x, \ f, \, g \in L^2_{|w|}(-1, \, 1)$ and $J = \operatorname{sgn} w$ the fundamental symmetry operator. The operator T in K is defined as

$$Ty = \frac{1}{w}\tau y, \quad y \in D(T) = D(S).$$

Then $S = JT$, $[Tf, \, g]_w = (Sf, \, g)_{|w|}, \ f, \, g \in D(T)$ and T is a self-adjoint operator in K [**3**, **7**, **9**]. In the following, we denote the resolvent set of S by $\rho(S)$.

Now, we state the lower bound result on T.

THEOREM 2.2. *Let T and S be defined as above. Suppose that $0 \in \rho(S)$ and S^{-1} is compact. Let $\mu^+ := \min \ \sigma(S) \cap (0, \, \infty)$, $\mu^- := \min \ \{|\lambda| : \lambda \in \sigma(S) \cap (-\infty, \, 0)\}$, where $\min \ \emptyset := \infty$. Then for each eigenvalue λ of T we have $|\lambda| \geqslant \min \, \{\mu^+, \, \mu^-\}$.*

Moreover, if λ corresponds to an eigenvector ϕ of T with $[\phi, \, \phi]_w = 0$, then the following, in general stronger, estimate holds $|\lambda|^2 \geqslant -\mu^+\mu^-$.

In order to give another result of the lower bound on T, we assume that $q_-(x) = \max\{0, \, -q(x)\}$ and for some $C, \ C_0, \ C_1, \ C_2 > 0, \ x \in (-1, \, 1)$

$$\left|\frac{1-x}{\sqrt{p(x)}}\int_{-1}^x q_-(t)\mathrm{d}t\right| \leqslant C_0, \quad \left|\frac{x+1}{\sqrt{p(x)}}\int_x^1 q_-(t)\mathrm{d}t\right| \leqslant C_1,$$
$$W(x) = \int_{-1}^x w(t)\mathrm{d}t, \quad \frac{W(x)}{\sqrt{p(x)}} \leqslant C_2, \quad C = \sqrt{2}C_2. \tag{2.5}$$

It is easy to verify that if $q \in L^2(-1, \, 1), \ p(x) = 1 - x^2, \ w(x) = x$, then (2.5) holds. Let

$$\gamma_t := \min\left\{\int_{t_0}^{t_0+t} |w(x)|\mathrm{d}x : t \in (-1, 1), \ t_0 \in (-1, 1)\right\},$$

$$\Delta_{w,1,n} = \|w\|_1 + C\left(\sqrt{\Delta} + \sqrt{\Delta_n}\right), \quad \Delta = 2\int_{-1}^1 q_-(t)\mathrm{d}t + 8\alpha^2,$$

$$\Delta_n = 2\int_{-1}^1 q_-(t)\mathrm{d}t + 8\alpha^2 + 2|\mu_n|\|w\|_1, \quad \alpha = \frac{C_0 + C_1}{2}. \tag{2.6}$$

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With this notation, we give the following result.

THEOREM 2.3. *Assume that λ and μ_n are the non-real eigenvalue of (1.1) and the nth eigenvalue of right-definite problem (2.4), respectively. Let $\mu_{h-1} < 0 < \mu_h$ for some positive integer $h \geqslant 2$, then the eigenvalue λ satisfies*

$$|\lambda|^2 \geqslant \frac{\mu_h \mu_{h-1}^2 \gamma_\delta}{16(\mu_h - \mu_{h-1})} \left(\sum_{n=1}^{h-1} \frac{\Delta_{w,1,n}^2}{\gamma_{\delta_n}} \right)^{-1}.$$

3. The proofs of theorems 2.2 and 2.3

In order to prove theorems 2.2 and 2.3, we prepare some lemmas. The following lemma is the estimates of $\|\sqrt{p}\phi'\|_2$, where ϕ is an eigenfunction of (1.1) corresponding to a non-real eigenvalue λ. That is $\mathcal{B}\phi = 0$ and

$$-(p\phi')' + q\phi = \lambda w \phi. \tag{3.1}$$

Since problem (1.1) is a linear system and ϕ is continuous, we can choose ϕ satisfies $\max\{|\phi(x)| : x \in [-1, 1]\} = 1$ in the following discussion.

LEMMA 3.1. *Let λ and ϕ be defined as above. Then*

$$\int_{-1}^{1} w|\phi|^2 = 0, \quad \int_{-1}^{1} p|\phi'|^2 \leqslant \Delta, \tag{3.2}$$

where Δ is given by (2.6).

Proof. It follows from $\mathcal{B}\phi = 0$ and lemma 2.1 that ϕ is bounded and satisfies $(p\phi'\phi)(x) \to 0$ as $x \to -1$ or 1. Multiplying both sides of (3.1) by $\overline{\phi}$ and integrating over the interval $[a, b]$, then

$$\int_{-1}^{1} p|\phi'|^2 + \int_{-1}^{1} q|\phi|^2 = \lambda \int_{-1}^{1} w|\phi|^2. \tag{3.3}$$

From $\operatorname{Im}\lambda \neq 0$ and (3.3) one sees that $\int_{-1}^{1} w|\phi|^2 = 0$, and hence

$$\int_{-1}^{1} p|\phi'|^2 + \int_{-1}^{1} q|\phi|^2 = 0. \tag{3.4}$$

Set

$$Q(x) = \int_{-1}^{x} q_-(t)\mathrm{d}t - \frac{x+1}{2}\int_{-1}^{1} q_-(t)\mathrm{d}t$$

Then one can verify that

$$Q(-1) = 0 = Q(1), \; Q'(x) = q_-(x) - \frac{1}{2}\int_{-1}^{1} q_-(t)\mathrm{d}t \text{ a.e. } x \in (-1, 1) \quad \text{and}$$

$$|Q(x)| \leqslant \left|\frac{1-x}{2}\int_{-1}^{x} q_-(t)\mathrm{d}t\right| + \left|\frac{x+1}{2}\int_{x}^{1} q_-(t)\mathrm{d}t\right| \leqslant \frac{\sqrt{p(x)}}{2}(C_0 + C_1) = \alpha\sqrt{p(x)}.$$

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As a result, this together with (3.4) and Cauchy–Schwarz inequality yields that

$$\begin{aligned}\int_{-1}^{1} q_-|\phi|^2 &= \int_{-1}^{1}\left(Q' + \frac{1}{2}\int_{-1}^{1} q_-\right)|\phi|^2 \\ &\leqslant \int_{-1}^{1} q_- - 2\,\mathrm{Re}\left(\int_{-1}^{1} Q\phi'\overline{\phi}\right) \\ &\leqslant \int_{-1}^{1} q_- + \frac{1}{2}\int_{-1}^{1} p|\phi'|^2 + 4\alpha^2. \end{aligned} \tag{3.5}$$

It follows from (3.4), (3.5) and $q = q_+ - q_-$, $q_\pm = \max\{0, \pm q\}$ that

$$\int_{-1}^{1} p|\phi'|^2 = -\int_{-1}^{1} q|\phi|^2 \leqslant \int_{-1}^{1} q_-|\phi|^2 \leqslant \int_{-1}^{1} q_- + \frac{1}{2}\int_{-1}^{1} p|\phi'|^2 + 4\alpha^2.$$

So the inequalities in (3.2) holds immediately. □

Similarly with the argument of lemma 3.1, we give the estimates of $\|\sqrt{p}\psi_n'\|_2$, where ψ_n is the eigenfunction that satisfies $\max\{|\psi_n(x)| : x \in [-1, 1]\} = 1$ corresponding to the nth eigenvalue μ_n of (2.4).

LEMMA 3.2. *Suppose that μ_n and ψ_n are defined as above. Then $\int_{-1}^{1} p|\psi_n'|^2 \leqslant \Delta_n$, where Δ_n is given by (2.6).*

Proof. Let μ_n and ψ_n be defined as above, then

$$-(p\psi_n')' + q\psi_n = \mu_n|w|\psi_n, \quad \mathcal{B}\psi_n = 0. \tag{3.6}$$

From $\mathcal{B}\psi_n = 0$ and lemma 2.1 that ψ_n is bounded and satisfies $(p\psi_n'\psi_n)(x) \to 0$, $x \to -1$ or 1. Multiplying both sides of (3.6) by ψ_n and integrating over the interval $(-1, 1)$, we have

$$\int_{-1}^{1} p|\psi_n'|^2 + \int_{-1}^{1} q|\psi_n|^2 = \mu_n \int_{-1}^{1} |w||\psi_n|^2. \tag{3.7}$$

With the similar argument in (3.5), one can prove that

$$\int_{-1}^{1} q_-|\psi_n|^2 \leqslant \int_{-1}^{1} q_- + \frac{1}{2}\int_{-1}^{1} p|\psi_n'|^2 + 4\alpha^2.$$

This together with (3.7) and $q = q_+ - q_-$ yields that

$$\begin{aligned}\int_{-1}^{1} p|\psi_n'|^2 &\leqslant \int_{-1}^{1} q_-|\psi_n|^2 + \mu_n\int_{-1}^{1}|w||\psi_n|^2 \\ &\leqslant \int_{-1}^{1} q_- + \frac{1}{2}\int_{-1}^{1} p|\psi_n'|^2 + 4\alpha^2 + |\mu_n|\|w\|_1.\end{aligned}$$

And hence

$$\int_{-1}^{1} p|\psi_n'|^2 \leqslant 2\int_{-1}^{1} q_- + 8\alpha^2 + 2|\mu_n|\|w\|_1.$$

This completes the proof of lemma 3.2. □

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For any $\varepsilon > 0$, let

$$\delta = \sup\left\{\min\left\{\tilde{\delta}, \frac{1}{2}\right\} : \int_I \frac{1}{p} \leqslant \frac{1}{4\Delta}, \text{for any } I \subset [-1/2, 1/2] \text{ with length } \tilde{\delta}\right\} \quad (3.8)$$

$$\delta_n = \sup\left\{\min\left\{\tilde{\delta}, \frac{1}{2}\right\} : \int_I \frac{1}{p} \leqslant \frac{1}{4\Delta_n}, \text{for any } I \subset [-1/2, 1/2] \text{ with length } \tilde{\delta}\right\} \quad (3.9)$$

From the definition of δ in (3.8), one sees that $\delta \in (0,\ 1/2]$ and $\int_I \frac{1}{p} \leqslant \frac{1}{4\Delta}$ for any interval $I \subset [-1/2,\ 1/2]$ with length δ. The conclusion holds for δ_n.

LEMMA 3.3. *Let λ and ϕ be defined as above. Then there exists an interval $\tilde{I} \subset (-1,\ 1)$ with δ in length, such that $|\phi(\cdot)| \geqslant 1/2$ on $\tilde{I}$.*

Proof. For any interval $I \subset [-1/2,\ 1/2]$ with length δ, it follows from Cauchy–Schwarz inequality and lemma 3.1 that

$$\left(\int_I |\phi'|\right)^2 \leqslant \int_I \frac{1}{p} \int_{-1}^1 p|\phi'|^2 \leqslant \frac{1}{4}. \quad (3.10)$$

Since $\max\{|\phi(x)| : x \in [-1,\ 1]\} = 1$, there exists $x_0 \in [-1/2,\ 1/2]$ such that $|\phi(x_0)| \leqslant 1$. Hence, for $x \in (-1,\ 1)$ and $|x - x_0| \leqslant \delta$,

$$\Big||\phi(x)| - 1\Big| \leqslant \Big||\phi(x)| - |\phi(x_0)|\Big| \leqslant |\phi(x) - \phi(x_0)| = \left|\int_{x_0}^x \phi'(t)\,\mathrm{d}t\right| \leqslant \frac{1}{2}$$

by (3.10), and hence

$$|\phi(x)| \geqslant \frac{1}{2} \quad \text{on} \quad \tilde{I} = [-\delta + x_0, x_0] \quad \text{or} \quad [x_0, x_0 + \delta].$$

From $\delta \in (0,\ 1/2]$ one sees that $(-1,\ 1)$ contains at least one such interval $\tilde{I}$. □

Similar with lemma 3.3 we have

LEMMA 3.4. *Assume that μ_n is an eigenvalue of (2.4) and ψ_n is the corresponding eigenfunction. Then there exists an interval $\tilde{I}_n \subset (-1,\ 1)$ with δ_n in length, such that $|\psi_n(\cdot)| \geqslant 1/2$ on $\tilde{I}_n$.*

Applying the above lemmas we now prove the main results of theorems 2.2 and 2.3.

The proof of theorem 2.2. Let μ_n be the nth eigenvalue of right-definite problem (2.4) and ψ_n the corresponding eigenfunction. From $\psi_n \in D(S)$ is linearly independent, one sees that $\{\psi_n : n \geqslant 1\}$ forms an orthonormal system. Let ϕ be an eigenfunction of T associated with eigenvalue λ such that $\int_{-1}^1 |w||\phi|^2 = 1$. Since S

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is a self-adjoint operator and S^{-1} is compact, we can expand ϕ via the orthonormal system ψ_n, i.e. $\phi = \sum\limits_{n=1}^{\infty} (\phi,\, \psi_n)_{|w|}\psi_n$. Then from $\int_{-1}^{1} |w||\phi|^2 = 1$, we have

$$\sum_{n=1}^{\infty} |(\phi, \psi_n)_{|w|}|^2 = 1, \quad \sum_{n=1}^{\infty} |(\phi, \psi_n)_{|w|}|^2 \mu_n^2 = |\lambda|^2. \tag{3.11}$$

It follows from $\mu^+ = \min\ \sigma(S) \cap (0,\, \infty)$ and $\mu^- = \min\ \{|\lambda| : \lambda \in \sigma(S) \cap (-\infty,\, 0)\}$ that $|\mu_n| \geqslant \min\ \{\mu^+,\, \mu^-\}$ for $n \geqslant 1$. This together with (3.11) yields that

$$|\lambda| \geqslant \min\ \{\mu^+, \mu^-\}.$$

If λ corresponds to an eigenvector ϕ of T with $[\phi,\, \phi]_w = 0$, then it follows from (3.11) that

$$\sum_{n=1}^{\infty} \left|(\phi, \psi_n)_{|w|}\right|^2 \left(\mu_n - \frac{1}{2}(\mu^+ + \mu^-)\right)^2 = |\lambda|^2 + \frac{1}{4}(\mu^+ + \mu^-)^2. \tag{3.12}$$

From $|\mu_n| \geqslant \min\ \{\mu^+,\, \mu^-\}$, one sees that

$$\left(\mu_n - \frac{1}{2}(\mu^+ + \mu^-)\right)^2 - \frac{1}{4}(\mu^+ - \mu^-)^2 = (\mu_n - \mu^+)(\mu_n - \mu^-) \geqslant 0. \tag{3.13}$$

From (3.11), (3.12) and (3.13), we have that

$$\begin{aligned} |\lambda|^2 &= \left(\mu_n - \frac{1}{2}(\mu^+ + \mu^-)\right)^2 - \frac{1}{4}(\mu^+ + \mu^-)^2 \\ &= \left(\mu_n - \frac{1}{2}(\mu^+ + \mu^-)\right)^2 - \frac{1}{4}(\mu^+ - \mu^-)^2 - \mu^+\mu^- \geqslant -\mu^+\mu^-, \end{aligned}$$

which completes the proof of theorem 2.2. □

The proof of theorem 2.3*.* Let μ_n be the nth eigenvalue of (2.4) and ψ_n the corresponding eigenfunction such that $\max\{|\psi_n(x)| : x \in [-1,\, 1]\} = 1,\ \ n \geqslant 1$. It follows from lemmas 3.3, 3.4 and the definition of γ_t in (2.6) that

$$\|\phi\|_{|w|}^2 = \int_{-1}^{1} |w||\phi|^2 \geqslant \int_I |w||\phi|^2 \geqslant \int_I \frac{|w|}{4} \geqslant \frac{\gamma_\delta}{4}, \tag{3.14}$$

$$\|\psi_n\|_{|w|}^2 = \int_{-1}^{1} |w||\psi_n|^2 \geqslant \int_{I_n} |w||\psi_n|^2 \geqslant \int_{I_n} \frac{|w|}{4} \geqslant \frac{\gamma_{\delta_n}}{4},\ n \geqslant 1. \tag{3.15}$$

From the definition of $W(x) = \int_{-1}^{x} w(t)\mathrm{d}t$, one sees that

$$[\phi, \psi_n]_w = \int_{-1}^{1} w\phi\overline{\psi_n} = \int_{-1}^{1} W'\phi\overline{\psi_n} = \phi(1)\overline{\psi_n}(1) \int_{-1}^{1} w - \int_{-1}^{1} W(\phi'\overline{\psi_n} + \phi\overline{\psi_n'}).$$

This together with (2.5) and lemmas 3.1 and 3.2 that

$$\begin{aligned}\left|[\phi,\psi_n]_w\right| &\leqslant \int_{-1}^{1}|w| + \left|\int_{-1}^{1} W(\phi'\overline{\psi_n}+\phi\overline{\psi_n'})\right| \\ &\leqslant \|w\|_1 + \left|\int_{-1}^{1}\frac{W}{\sqrt{p}}\sqrt{p}\phi'\overline{\psi_n}+\int_{-1}^{1}\frac{W}{\sqrt{p}}\phi\sqrt{p}\overline{\psi_n'}\right| \\ &\leqslant \|w\|_1 + C_2\left(\int_{-1}^{1}|\sqrt{p}\phi'\overline{\psi_n}|+\int_{-1}^{1}|\phi\sqrt{p}\overline{\psi_n'}|\right) \\ &\leqslant \|w\|_1 + C\left(\sqrt{\Delta}+\sqrt{\Delta_n}\right)=\Delta_{w,1,n}.\end{aligned}$$

Furthermore, for $n \geqslant 1$ we get

$$\lambda[\phi,\psi_n]_w = [T\phi,\psi_n]_w = (S\phi,\psi_n)_{|w|} = (\phi,S\psi_n)_{|w|} = \mu_n(\phi,\psi_n)_{|w|}. \tag{3.16}$$

If we set

$$\Lambda_n = (\phi,\psi_n)_{|w|} = \frac{(\phi,\psi_n)_{|w|}}{\|\phi\|_{|w|}\|\psi_n\|_{|w|}},$$

then (3.14)–(3.16) give that

$$|\Lambda_n| \leqslant \frac{4|\lambda|\left|[\phi,\psi_n]_w\right|}{|\mu_n|\sqrt{\gamma_\delta\gamma_{\delta_n}}} \leqslant \frac{4|\lambda|\Delta_{w,1,n}}{|\mu_n|\sqrt{\gamma_\delta\gamma_{\delta_n}}}. \tag{3.17}$$

From $\sum_{n=1}^{\infty}|(\phi,\,\psi_n)_{|w|}|^2\mu_n = \lambda[\phi,\,\phi]_w$ and $[\phi,\,\phi]_w = \int_{-1}^{1} w|\phi|^2 = 0$ in lemma 3.1, we have

$$0=\lambda[\phi,\phi]_w = \sum_{n=1}^{\infty}\left|(\phi,\psi_n)_{|w|}\right|^2\mu_n = \sum_{n=1}^{\infty}|\Lambda_n|^2\mu_n = \sum_{n=1}^{h-1}|\Lambda_n|^2\mu_n + \sum_{n=h}^{\infty}|\Lambda_n|^2\mu_n,$$

and hence $-\sum_{n=1}^{h-1}|\Lambda_n|^2\mu_n = \sum_{n=h}^{\infty}|\Lambda_n|^2\mu_n$. Thus, by (3.11) and $\mu_{h-1} < 0 < \mu_h$, $h \geqslant 2$,

$$\begin{aligned}\sum_{n=1}^{h-1}|\Lambda_n|^2(\mu_h-\mu_n) &= \sum_{n=1}^{h-1}|\Lambda_n|^2\mu_h + \sum_{n=h}^{\infty}|\Lambda_n|^2\mu_n \\ &= \sum_{n=1}^{h}|\Lambda_n|^2\mu_h + \sum_{n=h+1}^{\infty}|\Lambda_n|^2\mu_n \geqslant \sum_{n=1}^{h}|\Lambda_n|^2\mu_h + \sum_{n=h+1}^{\infty}|\Lambda_n|^2\mu_h \\ &= \sum_{n=1}^{\infty}|\Lambda_n|^2\mu_h = \mu_h.\end{aligned} \tag{3.18}$$

By $\mu_h > 0$, $\mu_n \leqslant \mu_{h-1} < 0$, $1 \leqslant n \leqslant h-1$, we have

$$\frac{1}{\mu_n^2} \leqslant \frac{1}{\mu_{h-1}^2}, \quad \frac{\mu_h}{\mu_n^2} \leqslant \frac{\mu_h}{\mu_{h-1}^2}, \quad \frac{-1}{\mu_n} \leqslant \frac{-1}{\mu_{h-1}} = \frac{-\mu_{h-1}}{\mu_{h-1}^2}.$$

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This together with the assumption that $\mu_1 \leqslant \cdots \leqslant \mu_{h-1} < 0 < \mu_h$, $h \geqslant 2$ yields that

$$\frac{\mu_h - \mu_n}{\mu_n^2} = \frac{\mu_h}{\mu_n^2} - \frac{1}{\mu_n} \leqslant \frac{\mu_h}{\mu_{h-1}^2} - \frac{\mu_{h-1}}{\mu_{h-1}^2} = \frac{\mu_h - \mu_{h-1}}{\mu_{h-1}^2}, \quad 1 \leqslant n \leqslant h-1. \tag{3.19}$$

Hence, by (3.17)–(3.19) we have

$$\mu_h = \sum_{n=1}^{h-1} |\Lambda_n|^2(\mu_h - \mu_n) \leqslant \sum_{n=1}^{h-1} \frac{16|\lambda|^2 \Delta_{w,1,n}^2(\mu_h - \mu_n)}{|\mu_n|^2 \gamma_\delta \gamma_{\delta_n}} = \frac{16|\lambda|^2}{\gamma_\delta} \sum_{n=1}^{h-1} \frac{\mu_h - \mu_n}{|\mu_n|^2 \gamma_{\delta_n}} \Delta_{w,1,n}^2$$

$$\leqslant \frac{16|\lambda|^2}{\gamma_\delta} \sum_{n=1}^{h-1} \frac{\mu_h - \mu_{h-1}}{\mu_{h-1}^2 \gamma_{\delta_n}} \Delta_{w,1,n}^2 = \frac{16|\lambda|^2(\mu_h - \mu_{h-1})}{\mu_{h-1}^2 \gamma_\delta} \sum_{n=1}^{h-1} \frac{\Delta_{w,1,n}^2}{\gamma_{\delta_n}}.$$

Therefore,

$$|\lambda|^2 \geqslant \frac{\mu_h \mu_{h-1}^2 \gamma_\delta}{16(\mu_h - \mu_{h-1})} \left(\sum_{n=1}^{h-1} \frac{\Delta_{w,1,n}^2}{\gamma_{\delta_n}} \right)^{-1}.$$

This completes the proof of theorem 2.3. □

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