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ON SPREADS IN $PG(3, 2^s)$ THAT ADMIT PROJECTIVE GROUPS OF ORDER 2^s

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Let Γ be a spread in $\mathscr{P}=PG(3,q)$; thus Γ consists of a set of q^2+1 mutually skew lines that partition the points of $\mathscr{P}$. Also let Λ be the group of projectivities of $\mathscr{P}$ that leave Γ invariant: so Λ is the "linear translation complement" of Γ, modulo the kern homologies. Recently, inspired by a theorem of Bartalone [**1**], a number of results have been obtained, in an attempt to describe (Γ, Λ) when q^2 divides $|\Lambda|$. A good example of such a result is the following theorem of Biliotti and Menichetti [**3**], which ultimately depends on Ganley's characterization of likeable functions of even characteristic [**5**].

Theorem A (Biliotti, Menichetti, Ganley [3, 5]. *Suppose q is even and Λ contains a 2-group G such that*

(i) *G fixes one component of Γ and acts regularly (and transitively) on the other q^2 components; and*
(ii) *the elations in G form a subgroup of order q.*

Then Γ is a spread of a semifield plane, a Lüneburg plane [**11**], *a Betten plane* [**2**], *or the Biliotti–Menichetti "elusive" plane of order 8^2; in this case $|\Lambda|=8^2$* [**3**, Theorems 3.1 and 3.2].

The object of this note is to derive the following consequence of Theorem A.

Theorem B. *Let Γ be a spread in $PG(3,q)$ with q even and let Λ be the group of projectivities leaving Γ invariant. Assume u is a 2-primitive divisor of $q-1$. Then $uq^2 \mid |\Lambda|$ only if Γ is a semifield spread, a Betten spread or a Lüneburg spread.*

Some background

To prove Theorem B we shall require in addition to Theorem A, the following recent results.

Result 1 (Jha, Johnson and Wilkie [**8**, Theorem 1.1])). *A spread of even order n admitting a shears group of order $n/2$ is a semifield spread.*

Result 2 (Dempwolff [**4**], Johnson and Wilkie [**10**]). *Let Π^l be an affine translation plane of even order q^2. Suppose* Aut Π^l *contains a group B of order q such that B fixes*

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elementwise a Baer subplane Π_0 *of order* q *and assume* B *centralizes a group of kern homologies of order* $q-1$. *Then* B *cannot normalize an elation group of order* q *unless* $q=2$.

Proof. Let χ be the axis of an affine elation group E, of order q, that is normalized by B. Thus χ is in Π_0 and by Dempwolff [**4**, 2.7] E centralizes B. Now apply Johnson and Wilkie [**10**, Lemma 2.7].

Result 3 (Jha, Johnson and Wilkie [**8**, Theorem 1.2]). *Let* G *be in the linear translation complement of an affine translation plane* Π^l *of even order* q^2, *with* $\mathbb{F}_q$ *in its kern. Suppose* G *is nonsolvable and contains no elations. Then if* G *is reducible*

(i) *every involution in* G *fixes* Δ, *a derivable slope set; and*
(ii) *every affine elation with axis through* Δ *leaves* Δ *invariant.*

Proof of Theorem B

We begin by restating the hypothesis of Theorem B in the following convenient form.

Hypothesis (H). Π^l is an affine translation plane of even order q^2 with $\mathbb{F}_q$ in its kern and Λ denotes the linear translation complement of Π^l based at an affine point 0. Λ satisfies both the following conditions.

(i) $q^2 \,\|\, |\Lambda|$; and
(ii) $\exists \mathcal{O} \in \Lambda$ such that $\mathcal{O} \,|\, l \neq$ identity and $\mathcal{O}$ is a u-element, where u is a 2-primitive (="primitive" from now on) divisor of $q-1$.

A Baer subplane of Π cannot be centralized by a group of order q^2. So hypothesis (H) implies that every Sylow 2-subgroup of Λ fixes exactly one line of Π^l. Hence the following conventions are justified.

Notation (N). G is a Sylow 2-subgroup of Λ and χ is the unique component of the spread associated with Π^l that is invariant under G. Let E denote the group of elations in G with *axis* χ and let $\chi_G = \mathrm{Fix}(G) \cap \chi$.

Now hypothesis (H)(i) immediately implies the following.

Lemma 1. *If* Π^l *is not a semifield plane then* χ_G *is a one-dimension* $\mathbb{F}_q$ *subspace of* χ *and* $|E| \geq (|G|/q) \geq q$.

Lemma 2. χ *is left invariant by a* u*-element* $\phi \in \Lambda$ *such that* $\phi \,|\, l \neq$ *identity.*

Proof. Let μ be the set of lines through 0 that are fixed by at least one Sylow 2-subgroup of Λ. Also let Σ be the group generated by all the shears in Λ. If $|\mu|=1$ we are done (hypothesis (H)(ii)), so assume $|\mu|>1$. Now the Hering–Ostrom theorem [**11**, Theorem 35.10] and Lemma 1 show that for some $h \geq 0$ we have either

(i) $\Sigma \cong SZ(2^h q)$ and $|\mu| = (2^h q)^2+1$, or
(ii) $\Sigma \cong SL(2, 2^h q)$ and $|\mu| = 2^h q + 1$.

As Π has order q^2, case (i) only occurs when $|\mu| = q^2+1$, $\Sigma \cong SZ(q)$ and so by Liebler [**11**, Theorem 31.1], Π is a Lüneburg plane and the lemma holds. It remains to consider the case $\Sigma \cong SL(2, 2^h q)$. Now $\Sigma \cong SL(2,q)$ or $SL(2,q^2)$, e.g., use the fact that $\log_2 2^h q$ divides $\log_2 q^2$ (Johnson and Ostrom [**9**, Theorem 2.12]). Hence by Schaeffer's theorem (see [**11**, Theorem 49.6]), Π is Desarguesian. Hence the lemma is valid.

We now require some information about the action of $GL(2,q)$ on its standard module χ.

Lemma 3. *Let V be a 2-dimensional vector space over $\mathbb{F}_q$ and let $\Gamma = GL(V, \mathbb{F}_q)$. Suppose G_1 and G_2 are 2-groups in Γ such that $\mathrm{Fix}(G_1) \neq \mathrm{Fix}(G_2)$ but $G_1 = G_2^v$ for a u-element $v \in \Gamma$. Assume $|G_1| > 2$. Then H, the full group of unimodular elements in $\langle G_1, v\rangle$, is isomorphic to $SL(2, q^\alpha)$ for $\alpha = \frac{1}{2}$ or 1. Moreover, the Sylow 2-subgroup of $\langle G_1, v\rangle$ are in H.*

Proof. As q is even, $\Gamma = \Sigma \oplus C$, where $\Sigma = SL(2,q)$ and C is the scalar group in Γ. Thus $v = v_1 \oplus \gamma$ where $v_1 \in \Sigma$ is a v-element and $\gamma \in C$); also $v_1 \neq 1$ because otherwise $G_2 = G_1^v = G_1$. Now $H \supseteq \langle G_1, G_2, v_1\rangle$ and H is unimodular. Hence by Dickson's list of subgroups of $PSL(2,q)$ [**7**, Hauptsatz 8.27], we must have $H \cong SL(2, 2^s)$ for some s dividing $r = \log_2 q$. Since $u \mid 2^{2s} - 1$ and u is a primitive divisor of $2^r - 1$, we now have $r \mid 2s$. The lemma follows.

Lemma 4. *Suppose Π is not a semifield plane. Then there is a u-element $v \in \Lambda$ such that*

(i) *v leaves χ invariant;*
(ii) *$v \mid l \neq$ identity; and*
(iii) *$v(\chi_G) = \chi_G$.*

Proof. Let U be a maximal u-group in Λ that leaves χ invariant. By Lemma 2, $U \mid l \neq$ identity. So it is sufficient to verify that χ_G is invariant under U. Assume this is false. Now there is a v in U such that $\mathrm{Fix}(G^v) \cap \chi \neq \chi_G$. Next consider $T = \langle v, G\rangle$ and let $\bar{T} = T \mid \chi$. Observe that $4 \mid |\bar{T}|$ because otherwise by Result 1, Π is a semifield plane. So Lemma 3 applies to $\bar{T}$ and hence $\bar{H}$, its unimodular subgroup, satisfies

$$\bar{H} \cong SL(2, q^\alpha) \quad \text{for} \quad \alpha = \tfrac{1}{2} \text{ or } 1.$$

Now let H be the preimage of the restriction map $T \to T \mid \chi$. We now proceed in a series of steps.

Step A. *$|E| \geqq q^{2-\alpha}$ and Ω, the set of nontrivial E-orbits on l, has cardinality $\leqq q^\alpha$.*

Proof. By Lemma 3, $\bar{H} \supset \bar{G}$ and so $H \supset G$. Thus $\bar{G} = G \mid \chi$ has order precisely q^α. Since E is the kernel of the restriction map $G \to G \mid \chi$ we now have $|E| \geqq q^2/q^\alpha$ and the step follows.

Step B. *H fixes some member of Ω.*

Proof. Suppose first that a nontrivial homology in H has χ as its axis. Now by Andre's theorem [**6**, Theorem 4.25] the set $\mathscr{C}$ of centres of all the nontrivial homologies in H with axis χ form an E-orbit that is clearly H invariant. So we may assume H contains no homologies with axis χ. Now $\bar{H}=H/E$ is a permutation group of Ω. But $|\Omega|\leqq q^{\alpha}$ (Step A) and $\bar{H}\cong SL(2,q^{\alpha})$ and so by Galois [**7**, Satz 8.28], $\bar{H}$ acts trivially on Ω. Hence Step B is valid.

Step C. *$H=EH_t$, where t is a point of $l-(l\cap\chi)$.*

Proof. By Step B we may choose t to be in an E-orbit that is H invariant. Now $|H|=|E|\,|H_t|\Rightarrow H=EH_t$.

Step D. *H_t fixes elementwise $q+1$ distinct slopes and H contains no homologies with affine axis.*

Proof. By Step C, $H_t\cong H/E$ is certainly nonsolvable and contains no elations. So by Result 3, H_t fixes elementwise $q+1$ slopes. Thus H_t contains no homologies. Hence H does not contain any homology because any prime order homology in H would fix some slope in the E-orbit of t. This could imply that H_t contains a homology. Hence the step is valid.

Since H contains no homologies the restriction map $H\rightarrow H|\chi$ has kernel E and now $H_t\cong H/E\cong\bar{H}\cong SL(2,q^2)$ for $\alpha=\frac{1}{2}$ or 1. Now by Schäffer's theorem (see [**11**, Theorem 49.6]), Π is a Hall plane or a Desarguesian plane. Only the latter plane is consistent with our hypothesis and so the lemma is proved.

Lemma 5. *If Π is not a semifield plane then Λ contains a subgroup H such that*

(i) $H\supset G$;
(ii) $|H|=u^{\alpha}|G|$ *for some* $\alpha\geqq 1$; *and*
(iii) $H|\chi_G=$ *identity.*

Proof. Choose v to satisfy conditions (i)–(iii) of Lemma 4 and let U be the Sylow u-subgroup of the kern homologies in Λ; thus U is the biggest subgroup of Λ fixing l elementwise. Now if $u^{\beta}=|U|$ then $u^{\beta}\|q-1$ (or Π is Desarguesian and the lemma holds). Now $v\notin U$ and so the u-group $U_1=\langle v,U\rangle$ leaves χ_G invariant and clearly cannot be faithful on it. So we may choose $v_1\neq 1$ in the kern of $U_1\rightarrow U_1|\chi_G$ and let $L=\langle v_1,G\rangle$. Since L fixes χ_G and χ it is readily seen to be solvable. Thus a Hall $\{u,2\}$ subgroup of L can be written as H.

We now use the following lemma on vector spaces to study the action of H on the elation group E.

Lemma 6. *Let V be a vector space of order $n=2^s<q^2$. Suppose $\mathcal{O}$ is a u-element in $GL(V,+)$. Then either* $\mathrm{Fix}(\mathcal{O})\neq\mathbf{0}$ *or* $|V|=q$.

Proof. Suppose W is an irreducible $\langle\mathcal{O}\rangle$ submodule of V and that $\mathcal{O}|W\neq$identity. Hence $\mathcal{O}$ is clearly semiregular on the nonzero points of W and so u divides $|W|-1$. But now as u is a primitive divisor of $q-1$ we get $|W|=q^m$ for some integer $m\geqq 1$. But

$|V|<q^2$ and so every irreducible module W, not in $\mathrm{Fix}(\mathcal{O})$, has order q. However, by Maschke's theorem, V is a direct sum of irreducible $\langle\mathcal{O}\rangle$-module and so either $V=W$ or $\mathrm{Fix}(\mathcal{O})\neq \mathbf{0}$. The lemma follows.

From now on H will always be as in Lemma 5, and we shall tacitly assume that Π is not a semifield.

Lemma 7. *H has no homologies with axis χ.*

Proof. If false then by Andre's theorem (cf. Step B of Lemma 4) we have

$$H=H_xE$$

for some homology centre $x\in l-(l\cap\alpha)$.

Now if $h\in H_x$ is a nontrivial homology then h normalizes E but cannot centralize any element of $E-\{1\}$. But we also have $|E|<q^2$ since Π is not a semifield plane. Hence Lemma 6 implies that $|E|=q$ and now $q\,\big|\,|H_x|$, contrary to Result 2.

Lemma 8. $G\lhd H$.

Proof. We must verify that H is 2-closed. So let σ_1 and σ_2 be distinct 2-elements in H. Since $\mathrm{Fix}(H)=\chi_G$, $\sigma_1|\chi_G$ and $\sigma_2|\chi_G$ are both involutions fixing χ_G elementwise and so $\sigma_1\sigma_2|\chi_G$ is also an involution. Thus $(\sigma_1\sigma_2)^2$ is a central collineation with axis χ. Now by Lemma 7, $(\sigma_1\sigma_2)^2$ is at most an elation and so $(\sigma_1\sigma_2)^4=1$. Hence H is 2-closed and the lemma is proved.

Lemma 9. *Suppose $\mathcal{O}\neq 1$ is a u-element in H and let $g\in G-E$. Then $\mathcal{O}g\neq g\mathcal{O}$.*

Proof. Assume false and let $\mathcal{M}$ be the set of all Maschke complements of χ_G in χ, relative to $\mathcal{O}|\chi$. Now g leaves $\mathcal{M}$ globally invariant and yet cannot fix any $M\in\mathcal{M}$ since then g would become an elation: recall g already fixes χ_G elementwise. Thus $|\mathcal{M}|\geqq 2$ and so $\mathcal{O}|\chi$ is a scalar map. But since $\mathcal{O}|\chi_G=1$, $\mathcal{O}$ must now be a homology, contrary to Lemma 7.

Lemma 10. *$|G_x|=1$ for some $x\in l-(l\cap\chi)$.*

(N.B. This lemma fails in some semifield planes.)

Proof. Let U be a Sylow u-subgroup of H. So U fixes some $x\in l-(l\cap\chi)$. Suppose if possible that $G_x\neq 1$. Now by Lemma 8, H, and therefore H_x, are 2-closed. Thus G_x is normalized by U as $U\subseteq H_x$. Now by Lemma 9, U is semiregular on G_x and so $u\,\big|\,|G_x|-1$. Hence the primitivity of u implies that $|G_x|\geqq q$; now Lemma 1 contradicts Result 2. Hence the lemma is valid.

We can now verify the conditions (i) and (ii) of Biliotti and Menichetti (Theorem A).

Proposition 11. *Assume Π^l is a translation plane satisfying hypothesis (H) but that Π^l is not a semifield plane. Let G be a Sylow 2-sub-group of Λ, the linear translation complement of Π, and E the elation subgroup of G. Then*

(i) $|E|=q$; *and*
(ii) G *fixes exactly one point* $x\in l$ *and is regular on* $l-\{x\}$; *in particular* $|G|=q^2$.

Proof. Part (ii) is Lemma 10. If part (i) fails then by Lemma 1, $|E|=2^eq$ for some $e\geqq 1$. Now Lemmas 8 and 9 imply that $u\,\big|\,|G|-|E|$ and so

$$u\,\Big|\,\frac{q}{2^e}-1.$$

Hence we contradict the primitivity of u if $|E|\neq q$. Hence the proposition is valid.
Now Theorem B immediately follows from Proposition 11 and Theorem A.

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