BULLETIN 13 (1), 1970

QUASI-FROBENIUS X-RINGS

BY
ABRAHAM ZAKS

In a recent study of a specific class of quasi-Frobenius rings, Feller has found it useful to introduce the X-rings ([3]). He suggested among others the following topics:

(A) Determine the properties of completely indecomposable rings and matrix rings over completely indecomposable rings.

(B) Determine the properties of modules over quasi-Frobenius X-rings.

We point out that the completely indecomposable rings are the local quasi-Frobenius rings. Problems (A) and (B) then lead naturally to semi-local quasi-Frobenius rings, and to matrix algebra over local quasi-Frobenius rings. These types of rings are discussed in sections 1 and 2.

The last section of this note is devoted to a question of Feller in [3]:

(C) Is the ring of endomorphisms of a uniform injective module with chain condition completely indecomposable?

We provide an example to show that the answer is negative in general. We prove that, under some more restrictions on the module, an affirmative answer may be achieved.

1. **Completely indecomposable rings.** All rings are presumed to have an identity. All modules (ideals) are presumed to be unitary left modules (ideals) unless otherwise specified.

For the various definitions we refer to [3].

For a local ring R with maximal ideal M to be completely indecomposable it is necessary and sufficient that it be a quasi-Frobenius ring (e.g., [1], [3] and [6]).

We start with a characterization of a local quasi-Frobenius ring:

THEOREM 1. *A ring R is a local quasi-Frobenius ring if and only if every injective module is free.*

Proof(1): If Q is an injective R-module, then by Theorem 5.3 of [2], Q is a projective R-module. Therefore, by Theorem 2 of [4], Q is a free module.

Conversely, if every injective R-module is free, then by Theorem 5.3 of [2], R is a quasi-Frobenius ring. Suppose R contains an idempotent not 1. Then R contains a

Received by the editors February 15, 1969 and, in revised form, May 11, 1969.

(1) This version of the proof was kindly suggested by the referee, who also notified me that this theorem was proved in a different method by T. S. Tol'skaja in his paper *Injectivity and freeness* in the Sibirsk. Mat. Ž. 6 (1965), 1202–1207 (Theorem 2).

https://doi.org/10.4153/CMB-1970-005-3 Published online by Cambridge University Press

primitive idempotent e, $e\neq 1$. Now Re is injective and hence free. Being indecomposable, Re is isomorphic to R. Hence R is indecomposable, which contradicts the existence of the idempotent e. By Corollary 1 on p. 76 of [10], this implies that R is a local ring.

For every injective R-module M we define $o(M)$ as the least integer for which the R-module $\sum_{i=1}^{o(M)} M$ is free. If no such integer exists, we set $o(M)=\infty$. We call $o(M)$ the order of M.

The ring R is said to have global order s, if and only if s is the least upper bound of the orders of the injective R-modules.

It is obvious that if R is a self-injective ring, then since R is a free R-module, $o(R)=1$.

PROPOSITION 2. *Let R be a quasi-Frobenius ring. A necessary and sufficient condition for R to be of finite global order is that any two simple R-modules are isomorphic.*

Proof. Let S be a simple R-module. Let T be the injective envelop of S. Since S is a simple R-module, T is an indecomposable module. By Theorem 5.3 of [2], T is a projective module. As a consequence of Corollary 2.5 of [2], there exists an idempotent e in R such that T is isomorphic to Re. As T is an indecomposable module e is a primitive idempotent. Assume R is of finite global order. Let S_1 and S_2 be non-isomorphic simple modules. Let T_1 and T_2 be their injective envelop respectively. We claim that T_1 and T_2 are non-isomorphic. For let f be any isomorphism from T_1 onto T_2. Then $f(S_1)$ is a non-zero simple submodule of T_2. Whence $f(S_1)=S_2$ and this contradicts the hypothesis that S_1 and S_2 are non-isomorphic.

Being of finite global order, implies that a finite direct sum of modules isomorphic to $T_1(T_2)$ is free. Let $\sum_{j=1}^{u} T_1$ and $\sum_{i=1}^{v} T_2$ be free, then $\sum_{j=1}^{v} (\sum_{i=1}^{u} T_1)$ and $\sum_{j=1}^{u} (\sum_{i=1}^{v} T_2)$ are free. Set $R_1=\sum_{i=1}^{u} T_1$ and $R_2=\sum_{i=1}^{v} T_2$. Since R_1 and R_2 are finitely generated R-modules they certainly are finitely generated as free R-modules. There exists a free R-module K such that, say, R_1 is isomorphic to $R_2\oplus K$. It is a consequence of Proposition 10 on p. 24 of [10] that T_2 is necessarily isomorphic to T_1. This contradiction implies that S_1 and S_2 are isomorphic.

Conversely, let Q be an injective module. Let $R=Re_1\oplus\cdots\oplus Re_n$ be a complete decomposition for R. We claim that $\sum_{i=1}^{n} Q$ is a free module. By Theorem 5.5 of [2] Q is a direct sum of modules $Q=\sum_{\alpha\in A} Q_\alpha$, where each Q_α is isomorphic to some indecomposable component of R, i.e., there exist primitive idempotents e_α such that Q_α is isomorphic to Re_α for each α in A. Since Re_i is the injective envelop of its minimal ideal for $i=1,\ldots,n$, and for $i\in A$, it follows that all the indecomposable components are isomorphic. In particular this implies that the module $\sum_{i=1}^{n} Q_\alpha$ is a free module for every α in A. Therefore the module $\sum_{i=1}^{n} Q$ is a free module. Whence R is of a finite global order.

Observe that if $R=Re_1\oplus\cdots\oplus Re_n$ is a complete decomposition for the quasi-Frobenius ring R and if R is of finite global order s, then $s\leq n$.

Let $R=Re_1+\cdots+Re_n$ be a complete decomposition for an Artinian ring. If Re_1 is isomorphic to Re_j for every j, $2\leq j\leq n$, then R is isomorphic to an $n\times n$ matrix algebra over e_1Re_1. By Proposition 2 on p. 76 and Proposition 3 on p. 74 of [10], e_1Re_1 is a local Artinian ring.

COROLLARY 3. *A ring R is isomorphic to an $n\times n$ matrix algebra over a local quasi-Frobenius ring if and only if its global order is n.*

Proof. Let $R=S_n$—the $n\times n$ matrix algebra over the local quasi-Frobenius ring S. Let e_i, $i=1,\ldots,n$, denote the matrix all of whose entries are zero, except that in the i–i spot it has a 1 $(1\in S)$. Since $e_1,\ldots,e_n$ are orthogonal idempotents, $R=Re_1+\cdots+Re_n$ is a direct sum decomposition. Since e_iRe_i is isomorphic to the local ring S for every i, $i=1,\ldots,n$, it is a consequence of Lemma 2 on p. 76 of [10] that $e_1,\ldots,e_n$ are primitive idempotents. Let f_j, for $j=2,\ldots,n$, denote the matrix all of whose entries are zero, except that in 1–j spot it has a 1 $(1\in S)$. Then right multiplication by f_j gives rise to an isomorphism of Re_1 onto Re_j for every j, $j=2,\ldots,n$. In particular, all the minimal ideals of R are isomorphic. As in the proof of Proposition 2, the global order of R does not exceed n. We claim that the order of the injective module Re_1 is exactly n. If this were not the case, say $o(Re_1)=k<n$, then $Re_1+\cdots+Re_k$ is isomorphic to a free module, say $\sum_{i=1}^{p}R$. The module $\sum_{i=1}^{p}R$ admits the obvious decomposition into pn terms, and this is in contradiction with Proposition 10 on p. 24 of [10]. This also proves that if the quasi-Frobenius ring $R=Re_1+\cdots+Re_n$ is of finite global order s, then $s\geq n$.

Conversely, if the global order is finite, then in particular every injective R-module is projective. By Theorem 5.3 of [2] it follows that R is a quasi-Frobenius ring. Let $R=Re_1+\cdots+Re_k$ be a complete decomposition for R. Each Re_i is the injective envelop of its minimal ideal for $i=1,\ldots,k$. As a consequence of Proposition 2 it follows that Re_1 is isomorphic to Re_j for $j=2,\ldots,n$. In particular from Corollary 2 of [6] it follows that R is isomorphic to a $k\times k$ matrix algebra over the local quasi-Frobenius ring e_1Re_1. Finally, as we observed earlier the global order of R is k, whenever it is finite. Whence $k=n$.

Since a completely indecomposable ring is a local quasi-Frobenius ring (e.g., [1] and [3]), one might be interested to investigate into the study of local quasi-Frobenius rings. Some results and some applications concerning this problem, may be found in [3], [6], [8] and [9].

2. **Quasi-Frobenius X-rings.** In section 1 we characterized the local quasi-Frobenius rings. By [3], the quasi-Frobenius X-rings are finite direct sum of matrix algebras over quasi-Frobenius local rings. As being a quasi-Frobenius X-ring is a Morita invariant (by Th. 2.7 and 2.9 of [3] and Th. 16.6 of [5]), a quasi-Frobenius X-ring is isomorphic to the ring of endomorphisms of a finitely generated projective

module over a finite direct sum of quasi-Frobenius local rings. We shall first characterize the rings R that are direct sum of local quasi-Frobenius rings:

THEOREM 3. *An Artinian ring R is a direct sum of local quasi-Frobenius rings iff every minimal faithful injective module is free, and $Hom_R(Q, Q')\neq 0$ implies that Q is isomorphic to Q' whenever Q and Q' are indecomposable injective modules.*

Proof. Let $R=I_1\oplus\ldots\oplus I_n$ be a direct sum decomposition of R into local quasi-Frobenius rings. Since I_j, for $j=1,\ldots,n$, is a two-sided ideal $I_j=Re_j$ for a suitable idempotent e_j that lies in the center of R. Since I_j is a local ring it follows by Corollary 1 on p. 76 of [10] that e_j is a primitive idempotent.

Let M be any R-module. Then I_jM for $j=1,\ldots,n$ is an I_j-module. If A is an I_j-module, then setting $I_kA=0$ for $1\leq k\leq n$ and $k\neq j$ defines on A an R-module structure. On A the I_j-module structure and the R-module structure coincide. Finally $I_1M\oplus\cdots\oplus I_nM$ is an R-direct sum decomposition of M (except that for some j, I_jM may vanish).

Let M be an injective R-module, and assume $I_jM\neq 0$ for some j, $1\leq j\leq n$. Let $N=\sum_{k\neq j} I_kM$. Consider the exact sequence of I_j-modules;

$$(*)\qquad 0\to I_jM\xrightarrow{e_j} A\xrightarrow{p} B\to 0$$

There results an exact sequence of R-modules:

$$(**)\qquad 0\to I_jM\oplus N\xrightarrow{e_j\oplus e} A\oplus N\xrightarrow{p\oplus 0} B\to 0$$

where e is the identity map on N and 0 is the zero map from N into B. Since $M=I_jM\oplus N$ is an R-injective module, the sequence (**) splits. Whence there exists a map g from $A\oplus N$ into $I_jM\oplus N$ so that the composite $g\circ(e_j\oplus e)$ is the identity map from $I_jM\oplus N$ into itself. Since $A=I_jA$, and since $I_jN=0$, the image of A under g lies in M. In particular, the restriction of g to A yields a map of A into M that is left inverse to e_j. Therefore the sequence (*) splits. Whence I_jM is an injective I_j-module. In a similar way one can prove that M is R-injective only if I_jM is I_j-injective for $j=1,\ldots,n$ (or $I_jM=0$).

Let M be a faithful injective module. Then $I_jM\neq 0$ is an injective I_j-module for $j=1,\ldots,n$. By Theorem 5.3, of [2] I_jM is an I_j-projective module, and by Theorem 2 of [4] this implies that I_jM is a free I_j-module. In particular I_jM contains a copy of I_j. This holding for $j=1,\ldots,n$ implies that M contains a copy of R. Since R itself is a faithful injective R-module this implies that a minimal faithful injective module is free, as a matter of fact it is isomorphic to R.

Let Q, Q' be indecomposable injective modules such that $\mathrm{Hom}_R(Q, Q')\neq 0$. By Theorems 5.3, and 5.5 of [2] there exist primitive idempotents e, f in the center of R, such that $Q(Q')$ is isomorphic to $Re(Rf)$. By Proposition 10 on p. 24 and Proposition 12 on p. 25 of [10], there exist integers s, t, $1<s, t<n$, such that $Re=I_s$ and $Rf=I_t$. An homomorphism from I_s into I_t is induced by an element of the form $\ker f\in R$ as right multiplication. It follows that either $I_s=I_t$ in which case

Q and Q' are isomorphic, or else $\mathrm{Hom}_R(I_s, I_t)=0$. However the last possibility implies that $\mathrm{Hom}_R(Q, Q')=0$ which is a contradiction.

Conversely, R being Artinian implies that minimal faithful injective modules exist, as can easily be verified by considering the injective envelop of R. Thus there exists a free injective R-module. As R is a direct summand of such a module, R is a self-injective ring. By Theorem 1 of [1] this implies that R is a quasi-Frobenius ring. Let $R=Re_1+\cdots+Re_t$ be a complete decomposition for R. If for any pair of distinct indices i, j, Re_i is isomorphic to Re_j then R/Re_i is a faithful injective module. For let $r(R/Re_i)=0$, then obviously $r \in Re_i$. If $r=0$ we are done. So let $r\neq 0$. There exists an isomorphism f from Re_i into Re_j. Let $f(e_i)=u$, then $f(re_i)=ru$. But f is an isomorphism and $re_i=r\neq 0$ whence $ru\neq 0$ and $ru \in Re_j$. Since $Re_j \cap Re_i=0$ this contradicts the hypothesis of $r(R/Re_i)=0$. Therefore R/Re_i is a faithful module.

If R/Re_i were to contain a free module this would have lead to two decomposition of R/Re_i into indecomposable submodules, in one of which we have $(n-1)$ terms, while on the other one we have at least n. This in turn is a contradiction to Proposition 10 on p. 24 of [10]. We claim that $R=Re_1+\cdots+Re_t$ is a direct sum decomposition into two-sided ideals. If this were not the case, then we have $e_iRe_j\neq 0$ for some pair of distinct indices i, j. This yields $\mathrm{Hom}_R(Re_i, Re_j)\neq 0$, which implies that Re_i is isomorphic to Re_j as they are indecomposable injective modules. This contradiction proves that R is a direct sum of local quasi-Frobenius rings.

Observe that the quasi-Frobenius X-rings are these quasi-Frobenius rings for which $\mathrm{Hom}_R(Q, Q')\neq 0$ implies that Q is isomorphic to Q', whenever Q and Q' are indecomposable injective modules. The proof of this is very similar to the proof of Theorem 3.

Finally, by the Morita equivalence we may conclude that a quasi-Frobenius X-ring can be characterized as being a ring of endomorphisms of a faithful projective module over a finite direct sum of local quasi-Frobenius rings.

3. **Rings of endomorphisms.** Let Q be a uniform injective module. Let f, g be homomorphisms of Q into Q with $\ker f\neq 0$ and $\ker g\neq 0$. Then if $f-g$ were an isomorphism then $f(q)\neq g(q)$ for every q in $Q, q\neq 0$. But $\ker f \cap \ker g\neq 0$, therefore if $\ker f\neq 0$ and $\ker g\neq 0$ then $\ker(f-g)\neq 0$. Also, if $\ker f=0$ then f is an isomorphism because Q is a uniform injective module. Therefore the ring of endomorphisms of Q is a local ring.

The origin of question (C) is in the fact that it holds for uniform components of an X-ring.

Notice that the components of R are projective modules.

These remarks will lead us to an affirmative answer to question (C) under some more restrictive conditions on the module.

As for the general setting, the answer is in the negative. An example may be found in [7] where a quasi-Frobenius ring R is constructed so that $R=Re_1+Re_2$ is a complete decomposition for R, and so that e_1Re_1 is not a quasi-Frobenius ring.

THEOREM 4. *Let Q be a projective and injective module, such that Q has a unique Jordan-Hölder series. Then the ring of endomorphisms of Q, $S=\mathrm{Hom}_R(Q, Q)$ is completely indecomposable.*

The proof will result from the following lemmas.

LEMMA 5. *Let Q be an injective module. Let there exist a proper submodule P of Q such that $P=\ker f$ for some f in S, and so that $P \subset \ker g$ for every $g \in S$ whenever $\ker g \neq 0$. If Q satisfies the ascending chain condition, then S is left uniform and satisfies the descending chain condition on left ideals.*

Proof. We prove first that Q is an indecomposable module. If this were not the case then we have $Q=Q_1 \oplus Q_2$ with $Q_1 \neq 0$ and $Q_2 \neq 0$. Let $s_1(s_2)$ denote the element in S corresponding to the projection on $Q_1(Q_2)$. Then $P \subset Q_2$ and $P \subset Q_1$. Therefore $P \subset Q_1 \cap Q_2=0$, and this is a contradiction to our assumptions. Since Q is indecomposable, every element s in S is invertible whenever $\ker s=0$, and the set of elements of S with non-zero kernel form an ideal, denote it by M. Thus M is the unique two-sided ideal of S and S/M is a division ring. Since $P=\ker f$ and $\ker f \subset \ker g$ whenever $g \in M$ it follows that $g \in Sf$. To this extent consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & \ker f & \to & Q & \to & \mathrm{Im}\, f & \to & 0 \\ & & \downarrow & & \wr\wr & & \downarrow & & \\ 0 & \to & \ker g & \to & Q & \to & \mathrm{Im}\, g & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

and the result is a consequence of $\mathrm{Im}\, f$ and $\mathrm{Im}\, g$ being submodules of the injective module Q.

In particular $M=Sf$, and $M^2=Sf^2, \ldots M^k=Sf^k$. We claim: $\ker f^k \subset \ker f^{k+1}$ whenever $\ker f^k \neq Q$. Since if $\ker f^k \neq Q$ then $\mathrm{Im}\, f^k \neq 0$. Whence $\mathrm{Im}\, f^k \cap \ker f \neq 0$. Say $0 \neq y \in \mathrm{Im}\, f^k \cap \ker f$, then there exists an x in Q so that $f^k(x)=y \neq 0$, and $f(y)=0$. Thus $f^k(x) \neq 0$ and $f^{k+1}(x)=0$. Since Q satisfies the ascending chain condition, the chain $S \supset M \supset M^2 \supset \cdots \supset M^k \supset \cdots$, must become stationary. But the claim just proved, enables us to conclude that if it ever becomes stationary, say $M^k=M^{k+1}$, then necessarily $M^k=0$. Since all M^i's are cyclic it follows that S satisfies the ascending chain condition on left ideals, as $S \supset M \supset M^2 \supset \cdots \supset M^k=0$ is a Jordan-Hölder series for the left module S. Furthermore if $M^n \neq 0$ and $M^{n+1}=0$ then M^n is a simple S-module, whence M^n is the unique minimal left ideal in S, therefore S is a left uniform ring.

LEMMA 6. *Let Q be a projective R-module. Let there exist a proper submodule L of Q such that $\mathrm{Im}\, f=L$ for some f in S, and such that $\mathrm{Im}\, f \supset \mathrm{Im}\, g$ for every $g \in S$,*

https://doi.org/10.4153/CMB-1970-005-3 Published online by Cambridge University Press

whenever $\operatorname{Im} g \neq Q$. *If* Q *satisfies the descending chain condition then* S *is right uniform and satisfies the descending chain condition on right ideals.*

Proof. We start by proving that Q is indecomposable. If $Q = Q_1 \oplus Q_2$ with $Q_1 \neq 0$ and $Q_2 \neq 0$, and $s_1(s_2)$ denote the element of S corresponding to the projection on $Q_1(Q_2)$, then $L \supset Q_1$ and $L \supset Q_2$. Therefore $L \supset Q_1 \oplus Q_2 = Q$, and this contradicts our hypothesis on L. Whence Q is indecomposable. The elements of S with image Q are therefore invertible, and the elements of S which are not epimorphisms form an ideal M, as their image is always included in L. Whence M is a maximal two-sided ideal and S/M is a division ring. We claim: if $s \in M$ then $\ker s \neq 0$. Suppose $\operatorname{Im} s = Q^1 \subset Q$, then if $\ker s = 0$ we conclude that s is an isomorphism of Q onto Q^1. This situation is obviously impossible for a module that satisfies the descending chain condition.

Let $g_1, g_2 \in S$ be such that $\operatorname{Im} g \supset \operatorname{Im} g_2$ then $g_2 \in g_1 S$. To this extent consider the following exact rows and column:

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ Q & \xrightarrow{g_2} & \operatorname{Im} g_2 \to 0 \\ & & \downarrow \\ Q & \xrightarrow{g_1} & \operatorname{Im} g_1 \to 0 \end{array}$$

and the result is a consequence of Q being a projective module.

In particular, if $L = \operatorname{Im} h$ then $M = hS$.

We claim that if $M^k = M^{k+1}$ then $M^k = 0$. For $M^k = h^k S$ and $M^{k+1} = h^{k+1} S$ implies $h^k = h^{k+1} s$ for a suitable $s \in S$. Therefore $h^k(1 - hs) = 0$. Since $1 - hs \notin M$, it follows that $1 - hs$ is invertible, whence $h^k = 0$ implying $M^k = 0$. As to the chain $S \supset M \supset M^2 \supset \cdots \supset M^n \supset \cdots$ of right ideals there corresponds the chain of left modules $Q \supset \operatorname{Im} h \supset \operatorname{Im} h^2 \supset \cdots \supset \operatorname{Im} h^h \supset \cdots$, and as Q satisfies the descending chain condition we must have $\operatorname{Im} h^k = \operatorname{Im} h^{k+1}$ for some k, but this implies $M^k = M^{k+1}$ whence $M^k = 0$ and $\operatorname{Im} h^k = 0$. In particular the chain $S \supset M \supset M^2 \supset \cdots \supset M^n \supset 0$ is a Jordan-Hölder series for the right module S, therefore S satisfies the descending chain condition on right ideals. If $M^n \neq 0$ and $M^{n+1} = 0$, then M^n is the unique minimal right ideal in S whence S is right uniform.

Observe that the assumptions in Lemmas 5 and 6 are satisfied by the hypothesis of Theorem 4 since $\operatorname{Im} f \subseteq \operatorname{Im} g$ or vice versa. The proof of Theorem 4 now follows from Lemmas 5 and 6, since a ring R is a local quasi-Frobenius ring whenever it is left and right Artinian, and left and right uniform (e.g., [3], [5] and [6]).

From Lemmas 5 and 6 we also observe that the hypothesis of Theorem 4 might be weakened as follows:

THEOREM 4. *Let* Q *be a projective and injective module satisfying both chain conditions, and let* $S = \operatorname{Hom}_R(Q, Q)$. *Let there exist proper submodules* P *and* L *of* Q

so that $P = \ker f$, $L = \operatorname{Im} h$ *for* $f, h \in S$. *If* $\ker s \supset P$ *whenever* $\ker s \neq 0$, *and* $\operatorname{Im} s \subset L$ *whenever* $\operatorname{Im} s \neq Q$, *then* S *is completely indecomposable.*

Remark that it suffices to have the ascending (descending) chain condition on submodules of Q that are kernels (images) of elements of S.

References

1. C. Faith, *Rings with ascending condition on annihilators*, Nagoya Math. J. **27** (1966), 179–191.

2. C. Faith, and E. A. Walker, *Direct sum representation of injective modules*, J. Alg. **5** (1967), 203–221.

3. E. H. Feller, *A type of quasi-Frobenius ring*, Canad. Math. Bull. **10** (1967), 19–27.

4. I. Kaplansky, *Projective modules*, Ann. Math. **68** (1958), 372–377.

5. K. Morita, *Duality for modules and its applications to the theory of rings with minimum conditions*, Sci. Rep., Tokyo Kyoiku Daigaku **6**, No. 150 (1958), 83–142.

6. A. Pollingher and A. Zaks, *Some remarks on quasi-Frobenius rings*, J. Alg. **10** (1968), 231–240.

7. A. Rosenberg and D. Zelinsky, *Annihilators*, Portugaliae Mathematica **20**, Fasc. 1 (1961), 53–65.

8. A. Zaks, *Restricted Gorenstein rings*, Ill. J. Math., **13** (1969), 796–803.

9. A. Zaks, *Isobaric QF-rings*, to be published.

10. J. Lambek, *Lectures on rings and modules*, Blaisdell Publishing Company, Waltham. Massachusetts, 1966.

Technion i.i.t.,
Haifa, Israel
Northwestern University
Evanston, Illinois

https://doi.org/10.4153/CMB-1970-005-3 Published online by Cambridge University Press