Appendix D
Particle-vibration coupling

The purpose of this appendix is to summarize results for particle-vibration coupling matrix elements, which are used in Chapters 8, 9 and 10. The particle-vibration interaction from equation (8.24) is

$$\delta U(r) = -R_0 \frac{\partial U}{\partial r} \sum_{LM} \alpha_{LM} Y^*_{LM}(\hat{r}) = -\kappa \sum_{LM} \alpha_{LM} F_{LM}, \qquad (D.1)$$

where the collective coordinates α_{LM} are nuclear deformation parameters. The dimensionless quantity

$$F_{LM} = \frac{R_0}{\kappa} \frac{\partial U}{\partial r} Y^*_{LM}(\hat{r}) \qquad (D.2)$$

is a single-particle field peaked at the nuclear surface and κ is a constant fixed by a self-consistency condition discussed in Section 8.3 (see also (10.27)). The coordinate α is related to phonon creation and annihilation operators by

$$\hat{\alpha}_{LM} = \sqrt{\frac{\hbar \omega_L}{2 C_L}} (\hat{\Gamma}^\dagger_{LM} + (-1)^M \Gamma_{L-M}), \qquad (D.3)$$

where $\hbar \omega_L$ is the energy of the phonon with multipolarity L and $C_L \hat{\alpha}^2/2$ is the potential energy associated with the collective coordinate. The matrix element of the collective coordinate between the phonon ground state and a one-phonon excited state is

$$\langle LM | \hat{\alpha}_{LM} | 00 \rangle = \sqrt{\frac{\hbar \omega_L}{2 C_L}} = \frac{1}{\sqrt{2L+1}} \beta_L. \qquad (D.4)$$

The quantity β_L is the reduced matrix element and is known as the multipole deformation parameter.

The particle-vibration interaction matrix element $V(jm, j'm', LM)$ from equations (D.1) and (D.4) is

$$V(jm, j'm', LM) = -\frac{\beta_L}{\sqrt{2L+1}} \langle j' | R_0 \frac{\partial U}{\partial r} | j \rangle \langle l' j'm' | Y_{LM} | l j m \rangle. \qquad (D.5)$$

299

The nucleon self energy and particle-vibration induced interaction involve sums over magnetic quantum numbers and can be expressed in terms of the quantity

$$\left|V\left(j, j', L\right)\right|^2 = \sum_{mm'M} \frac{\beta_L^2}{2L+1} \langle j'|R_0 \frac{\partial U}{\partial r}|j\rangle^2 \left|\langle l'j'm'|Y_{LM}|ljm\rangle\right|^2 \qquad (D.6)$$

which is symmetric in j and j'. Making use of the Wigner–Eckart theorem

$$\langle j'm'|Y_{LM}|jm\rangle = \frac{\langle jmLM|j'm'\rangle}{\sqrt{2j'+1}} \langle l'j'||Y||lj\rangle \qquad (D.7)$$

and the normalization property

$$\sum_{mm'} \left|\langle jmLM|j'm'\rangle\right|^2 = \frac{(2j'+1)}{(2L+1)} \qquad (D.8)$$

of the Clebsch–Gordon coefficients, equation (D.6) simplifies to

$$V^2\left(j, j', L\right) = \frac{\beta_L^2}{2L+1} \langle j'|R_0 \frac{\partial U}{\partial r}|j\rangle^2 \langle l'j'||Y_L||lj\rangle^2, \qquad (D.9)$$

which is equivalent to equation (10.3). The definition (D.7) of the reduced matrix element is the one used by Bohr and Mottelson (1969).

The self-energy of a nucleon in the single-particle state j is

$$\Sigma_j = \sum_{jL} \frac{1}{(2j+1)} \frac{V^2\left(j, j'; L\right)}{\varepsilon_j - (\varepsilon_{j'} + \hbar\omega_L)}, \qquad (D.10)$$

where ε_j and $\varepsilon_{j'}$ are single-particle energies and $\hbar\omega_L$ are phonon energies. The factor $1/(2j+1)$ appears because there is an average over the spin orientation m of the initial state j. The induced interaction matrix element $v_{jj'}$ in Chapter 10 involves a scattering between the normalized two-nucleon initial state $|(jj)_0\rangle$ with total angular momentum $J = 0$ and the final state $|(j'j')_0\rangle$ also with $J = 0$,

$$v_{jj'} = \langle (jj)_0 |v| (j'j')_0\rangle = \sum_{mm'} \frac{1}{2\sqrt{(2j'+1)(2j+1)}} \langle jm, \widetilde{jm}|v|j'm', \widetilde{j'm'}\rangle_a$$
$$= \sum_{mm'} \frac{1}{\sqrt{(2j'+1)(2j+1)}} \langle jm, \widetilde{jm}|v|j'm', \widetilde{j'm'}\rangle, \qquad (D.11)$$

where the general structure of the antisymmetrized matrix element $\langle|v|\rangle_a$ has been defined in equation (A.16).

The uncoupled phonon exchange matrix element has the angular momentum structure

$$\langle jm, \widetilde{jm}|v|j'm', \widetilde{j'm'}\rangle = \sum_{LM} \frac{\left|V\left(jm, j'm', LM\right)\right|^2}{D_\lambda}, \qquad (D.12)$$

where D_λ is an energy denominator which can be approximated in various ways. The microscopic calculations reported in Section 10.2 use a Bloch–Horowitz expression for the energy denominator. In the following equation we substitute the simple estimate $D_\lambda \approx -\hbar\omega_L$ which is used in Section 10.1. The interaction matrix element reduces to

$$v_{jj'} = \sum_L v_{jj'}^L, \qquad (D.13)$$

where

$$v_{jj'}^L = -\frac{2}{\sqrt{(2j'+1)(2j+1)}} \frac{V^2(j,j',L)}{\hbar\omega_L}. \tag{D.14}$$

The factor 2 occurs because two perturbation diagrams (time orderings) contribute to the induced interaction.

The normalization in the microscopic calculations reported in Section 10.2 is the one used in Barranco *et al.* (1999). In their notation the suffix ν refers to a state with a pair of nucleons with quantum numbers $l_\nu j_\nu$ coupled to zero total angular momentum, and $v_{\nu\nu'}$ is defined by

$$G_{\nu\nu'} = -v_{\nu\nu'} = -\frac{2\langle(j_\nu j_\nu)_0 |v| (j_{\nu'} j_{\nu'})_0\rangle}{\sqrt{(2j_\nu+1)(2j_{\nu'}+1)}} = -\frac{2v_{j_\nu j_{\nu'}}}{\sqrt{(2j_\nu+1)(2j_{\nu'}+1)}}, \tag{D.15}$$

where the factor of 2 arises from the antisymmetry of the pairing matrix element (see equation (A.16)).

Thus the normalization and sign of $G_{\nu\nu'}$ is the same as that of the BCS coupling constant G and the values of $G_{\nu\nu'}$ in Tables 10.1, 10.2 and 10.3 can be compared directly with BCS G-values for ^{120}Sn, $G \approx 27/A = 0.22$, where $G = \overline{G_{\nu\nu'}}$.

D.1 Estimate of $\langle lj||Y_L||lj\rangle$

The interaction strengths $V^2(j,j',L)$ defined in equation (D.9) are proportional to squares of reduced matrix elements of spherical harmonics. These can be expressed in terms of Clebsch–Gordon coefficients and can be calculated using standard formulae. Some qualitative properties and simple asymptotic expressions are collected in this appendix.

The reduced matrix elements $\langle l'j'||Y_L||lj\rangle$ with $j = l+1/2$, $j' = l'-1/2$ or $j = l-1/2$, $j' = l'+1/2$ involve a spin-flip at the interaction vertex. There is no spin-flip in the other two reduced matrix elements. The spin-flip matrix elements are small compared with the no-spin-flip and become very small when j and j' are large. The spin-flip processes are essentially possible only because of quantal fluctuations, owing to the parity condition that the matrix elements of Y_L vanish unless $l + l' + L$ is even. The spin-flip character of the reduced matrix elements $\langle l'j'||Y_L||lj\rangle$ can be recognized because $j + j' + L$ is even for spin-flip matrix elements and odd for the non-spin-flip matrix elements.

The square of the reduced matrix element $\langle lj||Y_L||l'j'\rangle$ can be expressed in terms of a Wigner 3-j symbol as

$$\langle lj||Y_L||l'j'\rangle^2 = \frac{(2j+1)(2j'+1)(2L+1)}{4\pi} \begin{pmatrix} j & j' & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2$$

$$= \frac{(2j+1)(2L+1)}{4\pi} \langle j\tfrac{1}{2}L0|j'\tfrac{1}{2}\rangle^2. \tag{D.16}$$

Diagonal matrix elements have $j = j'$ and $l = l'$ and the parity condition requires L to be even. A useful asymptotic formula introduced in equation (10.5) expresses the

Clebsch–Gordon coefficient in equation (D.16) in terms of a Legendre polynomial

$$\langle j\tfrac{1}{2}L0|j\tfrac{1}{2}\rangle \approx P_L(0).$$

This formula is valid when L is even and $j \gg L$. Thus

$$\langle j||Y_L||j\rangle^2 \approx \frac{(2j+1)(2L+1)}{4\pi}(P_L(0))^2.$$

Introducing the numerical values of the Legendre polynomial we have

$$\langle j||Y_L||j\rangle^2 \approx 0.1(2j+1) \tag{D.17}$$

for $L = 2, 4$ and 6.

The following examples calculated with $L = 2$ show that this result is quite accurate. For this purpose, use is made of the relation (Varshalovich *et al.* (1988), Table 8.4)

$$\begin{pmatrix} j & j' & 2 \\ \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{pmatrix}^2 = \frac{4(\tfrac{3}{4} - j(j+1))^2}{(2j+3)(2j+2)(2j+1)(2j)(2j-1)}.$$

In the case of $j = 11/2$

$$\begin{pmatrix} \tfrac{11}{2} & \tfrac{11}{2} & 2 \\ \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{pmatrix}^2 = \frac{4(\tfrac{3}{4} - \tfrac{11}{2} \times \tfrac{13}{2})^2}{14 \times 13 \times 12 \times 11 \times 10} = 0.02040$$

and

$$\langle 11/2||Y_2||11/2\rangle = \frac{(12)^2 \times 5}{4\pi} \times 0.02040 = 1.17.$$

In the case $j = 7/2$,

$$\begin{pmatrix} 7/2 & 7/2 & 2 \\ \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{pmatrix}^2 = \frac{4(\tfrac{3}{4} - \tfrac{7}{2} \times \tfrac{9}{2})^2}{10 \times 9 \times 8 \times 7 \times 6} = 0.02976.$$

Thus

$$\langle 7/2||Y_2||7/2\rangle^2 = \frac{8^2 \times 5}{4\pi} \times 0.02976 \approx 0.76.$$

In Table D.1 we compare the exact results given in equation (D.16), with the results obtained from equation (D.17).

There is another approximate relation which is valid for $j, j' \gg L$ when the no-spin-flip condition is satisfied (when $j + j' + L$ is odd or equivalently when $j - j' + L$ is even). The asymptotic formula for the Clebsch coefficients gives (Varshalovich *et al.* (1988), Section 8.9)

$$\left|\langle j\tfrac{1}{2}L0|j'\tfrac{1}{2}\rangle\right|^2 \approx \frac{4\pi}{2L+1}(Y_{LM}(0,0))^2, \tag{D.18}$$

with $M = |j - j'|$. When L is large and $L + M$ is even Varshalovich *et al.* (1988, Section 5.12) give

$$(Y_{LM}(0,0))^2 \approx \frac{1}{\pi^2}.$$

Table D.1. *Comparison of the results obtained using the approximate expression given in equation (D.17) with exact results obtained using equation (D.16).*

	$\langle j||Y_2||j\rangle^2$	
j	exact	$0.1(2j+1)$
7/2	0.76	0.8
11/2	1.17	1.2

Combining these results gives a simple approximate expression for the no-spin-flip reduced matrix elements

$$\langle j||Y_L||j'\rangle^2 \approx \frac{\sqrt{(2j+1)(2j'+1)}}{\pi^2} \approx 0.1\sqrt{(2j+1)(2j'+1)}. \tag{D.19}$$

This is equivalent to equation (D.17) when $j = j'$ and is quite accurate even for $L \geqslant 2$ and j and $j' > 1/2$. When $j' = 1/2$ and $j = L \pm 1/2$ then there is an exact formula

$$\langle j||Y_L||1/2\rangle^2 = \frac{2j+1}{4\pi}. \tag{D.20}$$

D.2 A simple estimate of $\left\langle R_0\frac{\partial U}{\partial r}\right\rangle$

The average $\langle R_0\partial U/\partial r\rangle$ will be estimated using a square well approximation for the Saxon–Woods potential

$$U(r) = \frac{U_0}{1 + \exp(\frac{r-R_0}{a})} \approx U_0\Theta(r - R_0), \tag{D.21}$$

where

$$\Theta(r - R_0) = \begin{cases} 1 & r \leq R_0, \\ 0 & r > R_0. \end{cases} \tag{D.22}$$

Making use of the fact that

$$\frac{\partial\Theta(r - R_0)}{\partial r} = \delta(r - R_0), \tag{D.23}$$

one can write

$$\langle R_0\frac{\partial U}{\partial r}\rangle = R_0 U_0 \int r^2\,dr\,\mathcal{R}^2(r)\,\delta(r - R_0)$$
$$= U_0 R_0^3 \mathcal{R}^2(R_0), \tag{D.24}$$

where $\mathcal{R}(r)$ is the radial wavefunction. Making use of the fact that (see Bohr and Mottelson (1969) p. 326, equation (3.22))

$$R_0^3 \mathcal{R}^2(R_0) \approx 1.4,$$
(D.25)

one obtains

$$\langle R_0 \frac{\partial U}{\partial r} \rangle = U_0 \times 1.4 \approx -60\,\text{MeV},$$
(D.26)

where use was made of $U_0 \approx -45\,\text{MeV}$. If one corrects this estimate for the spillout of the nucleons one has to divide the result shown above by a factor $(1 + a/R) \approx 1.1$ (see Bertsch and Broglia (1994), p. 87), in which case one obtains $\langle R_0 \frac{\partial U}{\partial r} \rangle \approx -50\,\text{MeV}$.