COLLINEATIONS, CORRELATIONS, POLARITIES, AND CONICS

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To Professor H. S. M. Coxeter on his sixtieth birthday

1. Introduction: Polarities and conics. It is well known that planes of characteristic 2 behave differently from other Pappian projective planes. For this reason their detailed properties are usually ignored in books on synthetic projective geometry, especially when conics are being discussed. This can give rise to the misleading impression that planes of characteristic 2 are more difficult to deal with, while a cursory introduction to conics in such planes (e.g. Theorem 4.3) may suggest that the notion of "pole and polar" no longer exists.

In order to produce a theory in which as many results as possible remain true in planes of characteristic 2, we must consider the two projective definitions of a conic due to Steiner (1832) and von Staudt (1847). The Steiner definition (see Definitions 5 and 6 below) is the more commonly used and is valid in every Pappian plane. The von Staudt definition of a conic (**7**, p. 137) as the sets of self-conjugate points and lines of a hyperbolic polarity (see Definitions 3 and 4) is used by Coxeter (**2**, p. 252; **3**, 8.1). This self-dual definition has great advantages, especially for the proofs of properties involving the concepts of pole and polar (as might be expected). The two definitions are equivalent in planes of characteristic other than 2 (**3**, 8.32, 8.51). However, in planes of characteristic 2, the von Staudt definition is of no value, because of Theorem 4.2(iii); Theorems 4.2 and 4.3 both show that there is no longer such a close connection between conics and polarities in these planes.

The above suggests the fruitful idea that the notions of polarity and conic should be kept separate. Many theorems are then true in every Pappian plane, and it becomes clear that some theorems (such as Pascal's theorem) concern conics, some (such as 3.1–3.5) concern polarities, and some concern both. Consider for example the statement

If two triangles are self-polar with respect to a conic, no three vertices being collinear, then the six vertices lie on a conic.

This statement is true in planes of characteristic other than 2, but its converse is true only if every polarity determines a conic (not in the real projective plane, for instance); the statement is meaningless in planes of characteristic 2. However, if we replace the first occurrence of *conic* by *polarity*, then both the theorem and its converse are true in every Pappian plane.

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There is neither the need nor the space to present here a complete theory of polarities and conics; most of the required proofs can be found elsewhere. I shall simply give a few proofs that have occurred to me during the preparation of undergraduate courses in projective geometry. The existence of projective collineations and polarities is considered in §2, while §3 consists of a new proof that every projective collineation can be expressed as the product of two polarities. The investigation in §4 of polarities and conics in planes of characteristic 2 is based on a simple lemma, but I have not come across a similar investigation by synthetic methods elsewhere.

Some basic definitions are given below. Note that, in Definition 1, a projective collineation is *not* defined as a product of perspective collineations (as in **5**, p. 65, for instance).

1. A *projective collineation* in any projective plane is a collineation that induces a projectivity from every range of points onto its image range and from every pencil of lines onto its image pencil (**8**, p. 71).

2. A *projective correlation* is a correlation (or duality, or reciprocity) that induces a projectivity from every range onto its image pencil and from every pencil onto its image range (**8**, p. 262).

3. A *polarity* is a projective correlation of period 2.

4. A *self-conjugate point* (*line*) of a polarity is a point (line) incident with its image line (point). A polarity is *hyperbolic* if it contains a self-conjugate point (and therefore a self-conjugate line).

5. A *point-conic* in a Pappian plane is the set of intersections of corresponding lines in a non-perspective projectivity between two distinct pencils.

6. A *line-conic* in a Pappian plane is the dual of a point-conic.

All the results in this paper, unless otherwise stated, concern Pappian planes.

2. Projective collineations and polarities. We shall discuss Theorems 2.1–2.7 before giving the proofs.

2.1. *Let $ABCD$, $A'B'C'D'$ be quadrangles in a Desarguesian plane; let $AB \cap CD = E$, $A'B' \cap C'D' = E'$, and let α be any projectivity mapping A, B, E onto A', B', E'. Then there exists a unique projective collineation mapping $ABCD$ onto $A'B'C'D'$ (where the notation is meant to imply that A is mapped onto A', etc.) and inducing the projectivity α from range AB to range $A'B'$.*

Corollary. *In a Pappian plane, there exists a unique projective collineation mapping a given quadrangle onto a given quadrangle.*

2.2. (i) *In a Moufang plane, every projective collineation can be expressed as a product of perspective collineations, at most one of which is a homology.*

(ii) *In a Desarguesian plane, every projective collineation can be expressed as the product of at most three perspective collineations, at most one of which is a homology.*

(iii) *In a Pappian plane, every projective collineation with a fixed point can be expressed as the product of at most two perspective collineations, at most one of which is a homology, and conversely every product of two perspective collineations has a fixed point.*

2.3. *In a Moufang plane, if one quadrangle has non-collinear diagonal points, then the same is true of every quadrangle.*

COROLLARY. *In a Moufang plane, if one quadrangle has collinear diagonal points, then the same is true of every quadrangle.*

2.4. *There exists at most one projective correlation mapping a given quadrangle onto a given quadrilateral.*

2.5. *There exists a polarity (unique by* 2.4) *mapping each vertex of a given triangle onto its opposite side and mapping a given point not on a side of the triangle onto a given line not through a vertex.*

2.6. *There exists a unique projective correlation mapping a given quadrangle onto a given quadrilateral.*

2.7. *Any projective correlation that maps each vertex of a triangle onto its opposite side is a polarity.*

The converse of 2.2(i), which states that every product of perspective collineations is a projective collineation in our sense of the term, is true in every projective plane (**5**, p. 67). It would be interesting to know whether 2.2(i) itself is true in every projective plane.

Theorem 2.3 and its proof are no doubt well known, but I have not been able to find a reference to a simple proof such as that given below. We need the result here for Pappian planes only.

Von Staudt's proof of 2.6 is similar to his proof of the corollary to 2.1 (**7**, pp. 64, 65). (Coxeter's account of von Staudt's proof in the first edition of *Projective geometry* (**3**, 6.13, 6.42) is incomplete.) Using 2.6 one can prove 2.7 and 2.5 (**7**, pp. 131, 133; **3**, 7.21). We shall adapt von Staudt's proof of 2.7 to prove 2.5 directly, before proving 2.6.

We shall need four preliminary lemmas.

LEMMA 1. *In a Moufang plane, any perspectivity can be embedded in an elation* (**5**, p. 67).

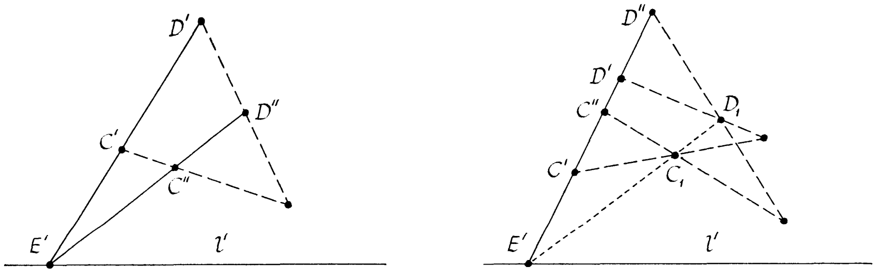
COROLLARY. *In a Moufang plane, any projectivity can be embedded in a product of elations (or, otherwise expressed, if α is a projectivity, then there exists a product of elations that induces α).*

LEMMA 2. *In a Desarguesian plane, any projectivity from a range onto a distinct range can be expressed as the product of at most two perspectivities* (**4**, p. 46).

COROLLARY (using Lemma 1 also). *In a Desarguesian plane, if α is a projectivity from a range onto a distinct range, then there exists a product of at most two elations that induces α.*

LEMMA 3. *If a collineation (correlation) induces a projectivity from one range onto its image range, or from one pencil onto its image pencil (from one range onto its image pencil, or from one pencil onto its image range), then it is projective* (**3**, 6.11, 6.41).

LEMMA 4 (Figs. 1a, 1b). *In a Desarguesian plane, let C', D', C'', D'' be four points not on a line l', $C' \neq D'$, $C'' \neq D''$, $C'D' \cap l' = C''D'' \cap l'$. Then there exists a perspective collineation σ, with axis l', such that $C''\sigma = C'$, $D''\sigma = D'$.*



(*a*) (*b*)

FIGURE 1

Proof. If the lines $C'D'$, $C''D''$ are distinct, then $C'C'' \cap D'D''$ is the centre of σ, and σ is defined by $C''\sigma = C'$.

If the lines coincide, choose distinct points C_1, D_1 not on $C'D'$ and not on l', such that $C'D' \cap l' = C_1D_1 \cap l'$. Then there exist perspective collineations θ, ϕ, with axis l', such that $C''\theta = C_1$, $D''\theta = D_1$, $C_1\phi = C'$, $D_1\phi = D'$; then $\sigma = \theta\phi$.

Proof of 2.1. Denote the lines AB, $A'B'$ by l, l'. If θ, ϕ are two collineations mapping $ABCD$ onto $A'B'C'D'$ and inducing α from range AB to range $A'B'$, then $\theta\phi^{-1}$ fixes C, D and every point of l; hence $\theta\phi^{-1}$ is the identical collineation (**4**, p. 99). Thus the required projective collineation, if it exists, is unique.

There exists a product ρ of elations that induces α (Lemma 1, cor.); then $A\rho = A'$, $B\rho = B'$, $E\rho = E'$. Let $C\rho = C''$, $D\rho = D''$; then C'', D'', E' are collinear, so $C'D' \cap l' = C''D'' \cap l' = E'$. Hence by Lemma 4 there exists a perspective collineation σ, with axis l', such that $C''\sigma = C'$, $D''\sigma = D'$; $\rho\sigma$ is the projective collineation mapping $ABCD$ onto $A'B'C'D'$ and inducing α from range AB to range $A'B'$.

Proof of 2.2. (i) Let τ be a projective collineation. If $\tau = 1$, there is nothing more to prove. If not, let l be a non-fixed line of τ, and let α be the projectivity induced by τ from the range of points on l onto its image range. There exists a product ρ of elations that induces α (Lemma 1, cor.). Then $\tau\rho^{-1}$ leaves l pointwise fixed; hence it is a perspective collineation σ with axis l (**4**, p. 100); $\tau\rho^{-1} = \sigma$. Hence $\tau = \sigma\rho$, as required.

(ii) The proof is as before, but ρ is now the product of at most two elations (Lemma 2, cor.).

(iii) Let A be a fixed point of τ. If every line through A is fixed, then τ is a perspective collineation (**4**, p. 102). If not, let l be a non-fixed line through A. Then α, defined as before, fixes A and is therefore a perspectivity. Hence ρ is an elation (Lemma 1); $\tau = \sigma\rho$ as before. Conversely, if τ is the product of two perspective collineations, then the intersection of their axes is a fixed point.

Proof of 2.3 (Fig. 2). Let $ABCD$ be a quadrangle with non-collinear diagonal points E, F, G as shown, and let $FG \cap AB = H$, so that H is distinct from E. Let $A'B'C'D'$ be another quadrangle with diagonal points E', F', G'.

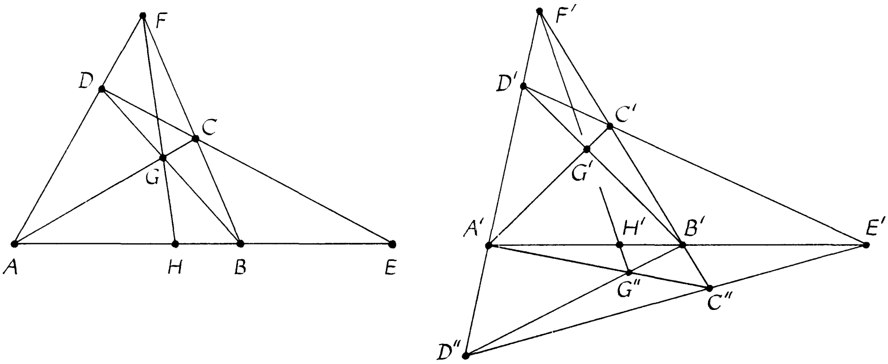


FIGURE 2

There exists a projectivity α such that $(A, B, E)\alpha = (A', B', E')$. There exists a product ρ of elations that induces α (Lemma 1, cor.). Let $F\rho = F_1$. There exists an elation σ with axis $A'B'$ such that $F_1\sigma = F'$. Then

$$(A, B, E, F)\rho\sigma = (A', B', E', F').$$

Let $(C, D, G, H)\rho\sigma = (C'', D'', G'', H')$. Then D'' lies on $A'F'$, etc., as in the figure, and H' is distinct from E'. Let τ be the elation with axis $A'F'$ such that $C''\tau = G''$. Then $E'\tau = B'$, $C'\tau = G'$. Hence F', G'', G' are collinear, so G' lies on $F'H'$. Hence E', F', G' are non-collinear.

A similar method can be used to prove the uniqueness of harmonic conjugates in a Moufang plane.

Proof of 2.4. If ρ, σ are two projective correlations mapping a given quadrangle onto a given quadrilateral, then $\rho\sigma^{-1}$ fixes each point of the quadrangle, so $\rho\sigma^{-1}$ is the identical collineation (**3**, 6.12).

Proof of 2.5. Let ABC be a triangle, with sides $BC = a$, $CA = b$, $AB = c$ (Fig. 3a). Let D be a given point not on a side of the triangle, and let d be a given line not through a vertex. Let $DA \cap a = D_a$, $DB \cap b = D_b$, $DC \cap c = D_c$, $d \cap a = A_d$, $d \cap b = B_d$, $d \cap c = C_d$. Let $\mathbf{J}_a$ denote the unique involution on a having B, C and D_a, A_d as pairs of mates. Define $\mathbf{J}_b$, $\mathbf{J}_c$ similarly.

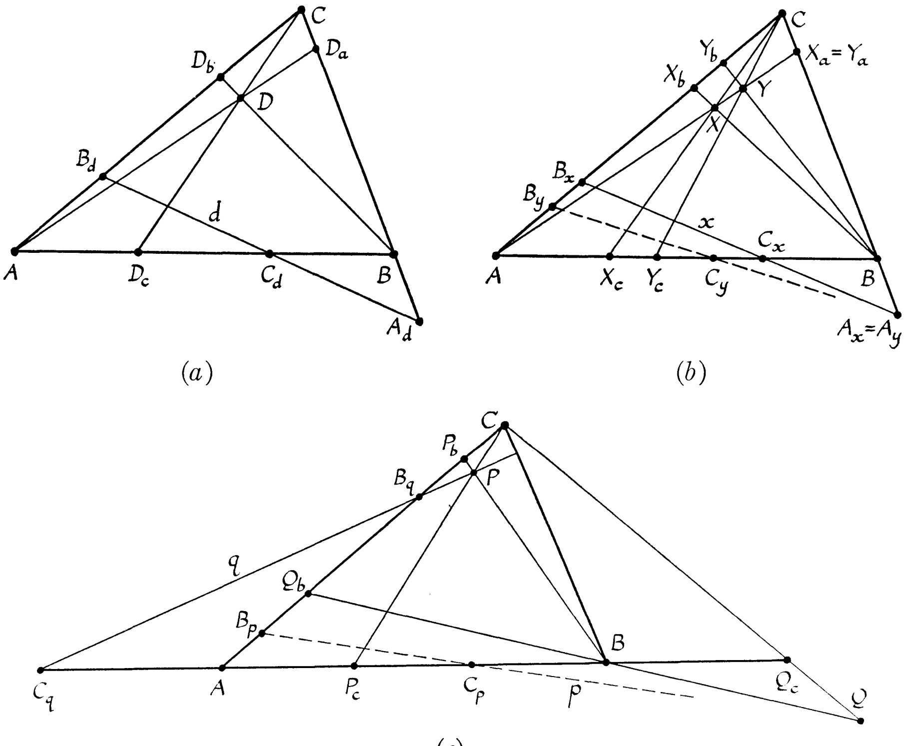


FIGURE 3

Let P be any point distinct from A, B, C, and let $PA \cap a = P_a$, etc. If there exists a polarity mapping A, B, C, D onto a, b, c, d, and P onto p, say, then the involution of conjugate points on a (**3**, 7.13; see also 4.2(i) and 4.2(ii) in the present paper) has B, C; D_a, A_d; P_a, $p \cap a$ as pairs of mates; hence this involution is $\mathbf{J}_a$.

We shall therefore denote $P_a \mathbf{J}_a$ (the mate of P_a in the involution $\mathbf{J}_a$) by A_p, and define B_p, C_p similarly. We shall prove that A_p, B_p, C_p are collinear, and denote this line by p. Then we shall show that the mapping $P \to p$, $p \to P$, $A \to a$, $a \to A$, etc. is a polarity.

(i) *If P lies on a side of ABC, then A_p, B_p, C_p are collinear.* This is easily proved.

(ii) *If A_x, B_x, C_x are collinear, and if XY passes through a vertex of ABC, X, Y not lying on a side of ABC, then A_y, B_y, C_y are collinear.* Suppose without loss of generality that XY passes through A. Then $X_a = Y_a$ (Fig. 3b), so $A_x = A_y$ and

$$ABC_xC_y \overset{\mathbf{J}_c}{\barwedge} BAX_cY_c \overset{C}{\doublebarwedge} X_aAXY \overset{B}{\barwedge} CAX_bY_b \overset{\mathbf{J}_b}{\barwedge} ACB_xB_y.$$

Hence

$$ABC_xC_y \barwedge ACB_xB_y.$$

Hence BC, $C_x B_x$, $C_y B_y$ are concurrent, so C_y, B_y, A_y are collinear.

(iii) *If P is any point not on a side of ABC, then A_p, B_p, C_p are collinear.* Let $AD \cap BP = Q$. Then A_d, B_d, C_d are collinear, and DQ passes through the vertex A; hence by (ii) A_q, B_q, C_q are collinear. QP passes through the vertex B; hence by (ii) A_p, B_p, C_p are collinear.

(iv) *The mapping $P \to p$ is one–one, and any line that is not a side of ABC is the image of some point that is not a vertex.* This is easily proved.

(v) *The mapping $P \to p$, $p \to P$, $A \to a$, $a \to A$, etc. is a polarity.* We need only show that this mapping preserves incidence; it is then a polarity since it has period 2, and since the induced mapping from range a to pencil A is a projectivity (Lemma 3). Suppose $P \in q$. If either P or Q lies on a side of ABC, then it is easily shown that $Q \in p$. If neither P nor Q lies on a side of ABC, then (Fig. 3c) A, B, P_c, C_q are distinct, and

$$ABQ_cC_p \overset{\mathbf{J}_c}{\barwedge} BAC_qP_c \overset{P}{\doublebarwedge} P_bAB_qC \overset{\mathbf{J}_b}{\barwedge} B_pCQ_bA \barwedge AQ_bCB_p.$$

Hence

$$ABQ_cC_p \barwedge AQ_bCB_p.$$

Hence BQ_b, Q_cC, C_pB_p are concurrent; so $Q \in B_p C_p$, i.e. $Q \in p$.

We shall denote the above polarity, which has ABC as a *self-polar triangle*, by $(ABC)(Dd)$.

Proof of 2.6. Denote the quadrangle and quadrilateral by $ABCD$, $a'b'c'd'$. Let $b' \cap c' = A'$, $c' \cap a' = B'$, $a' \cap b' = C'$, and let D' be any point not on a side of triangle $A'B'C'$. Then there exist a projective collineation ρ mapping $ABCD$ onto $A'B'C'D'$ (2.1) and a polarity σ mapping $A'B'C'D'$ onto $a'b'c'd'$ (2.5); $\rho\sigma$ is the required projective correlation and it is unique by 2.4.

Proof of 2.7. This is an immediate corollary of 2.4 and 2.5.

It is easy to adapt the proof of 2.6 if we know only that there exists one projective correlation. Veblen and Young make use of the polarity defined by a conic (**8**, p. 264) but this method is not valid in planes of characteristic 2.

3. The product of two polarities. The first four results of this section represent various stages in the proof of the result that *every projective collineation can be expressed as the product of two polarities.*

3.1. *If a projective collineation has three fixed points forming a triangle, then it can be expressed as the product of two polarities, each of which has the triangle as a self-polar triangle.*

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3.2. *Every perspective collineation can be expressed as the product of two polarities.*

3.3. *In a projective plane containing more than three points on each line, every projective collineation that is not perspective, and that does not have a triangle of fixed points, can be expressed as the product of two polarities.*

3.4. *In the projective plane having three points on each line, every projective collineation can be expressed as the product of two polarities.*

The proof of 3.1 is easy, using 2.5. Homologies are included in 3.1, but we give a proof of 3.2 that is valid for both elations and homologies.

The proof of 3.3 given by Veblen and Young (**8**, p. 265) and Coxeter (**3**, 7.71) appears at first sight to require at least seven points on each line. Only by an investigation of the possible projective collineations in PG(2, 5) can we show that the proof is valid in this plane. The proof fails for PG(2, 3). I have not investigated the situation in PG(2, 4) or PG(2, 2). The proof given below fails only for PG(2, 2), which is easily investigated separately in 3.4.

Proof of 3.2 (Fig. 4). For the purposes of this proof, the identical collineation (which is covered by 3.1) will not count as a perspective collineation. Suppose the perspective collineation ρ, with centre O and axis l, maps A onto B, where $A \notin l$, $A \neq 0$. (If there exist only three points on each line, then there are no homologies, so ρ is an elation with $O \in l$, and AB contains just the three points O, A, B.)

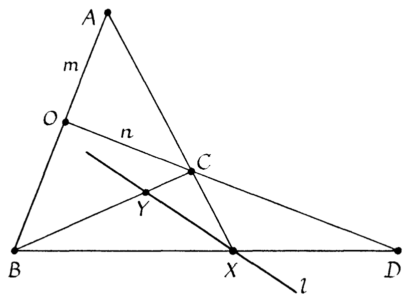

FIGURE 4

Let C be a point not on AB and not on l. Let $AC \cap l = X$, $OC \cap BX = D$, $BC \cap l = Y$. Then $C\rho = D$. Denote OA, OC by m, n.

The product of the polarities $(ABX)(Yn)$ and $(DCY)(Xm)$ maps $OYAC$ onto $OYBD$, and hence equals ρ (2.1, cor.).

LEMMA 5. *If the collineation ρ has at most two fixed points and two fixed lines, then*

(i) *its fixed points and lines form one of the configurations shown in Figure* 5, *and*

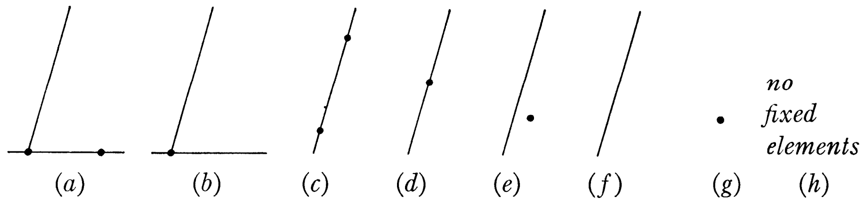

FIGURE 5

(ii) *if each line of the plane contains at least four points, there exist two points* A, B *that, together with their images* $A' = A\rho$, $B' = B\rho$, *form a quadrangle such that neither* AA' *nor* BB' *is a fixed line.*

Proof. (i) This is obvious, since the join of two fixed points is a fixed line, and dually.

(ii) In all cases except (f) and (h), let l be any non-fixed line through a fixed point P, and let A, B be any two distinct points on l, other than P, and not on a fixed line; the required conditions are then satisfied.

In case (f), let l be any non-fixed line; $l\rho^{-1}$ and $l\rho$ meet l in distinct points. The fixed line meets l in a third point. Unless there exist only four points on each line. There are at least two more points on l, from which we choose A and B. The required conditions are then satisfied. However, if a collineation in a finite plane contains one fixed line, then it contains one fixed point (**1**, p. 655, **4**, p. 114), so (f) cannot occur if there are only four points on each line. Alternatively, we easily deduce from 2.2(iii) and its dual that, in any Pappian plane, a projective collineation with one fixed point must have one fixed line, so that neither (f) nor (g) can occur anyway.

In case (h) we proceed as in case (f). Since there is now no fixed line, the method will work in all cases.

If a projective collineation has three fixed points on a line l, then the induced projectivity on l is the identity, so l is pointwise fixed. Hence, if a projective collineation is not perspective and does not have a triangle of fixed points, then it has at most two fixed points and two fixed lines. Hence we can apply Lemma 5 to the proof of 3.3.

Proof of 3.3. Denote the collineation by ρ, and choose A, B as in Lemma 5. Let $AA' \cap BB' = C$ (Fig. 6), and let $C\rho^{-1} = P$, $C\rho = Q$. Label the lines as shown. Then P does not lie on any side of triangle ABC, and Q does not lie on any side of triangle $A'B'C$ (since AA', BB' are not fixed lines). The product of $(ABC)(Pp)$ and $(A'B'C)(Qc)$ maps $ABCP$ onto $A'B'QC$, and hence equals ρ (2.1, cor.).

The above proof emerges quite naturally if we consider the simplest non-trivial type of product of two polarities (apart from that used in 3.1) as follows.

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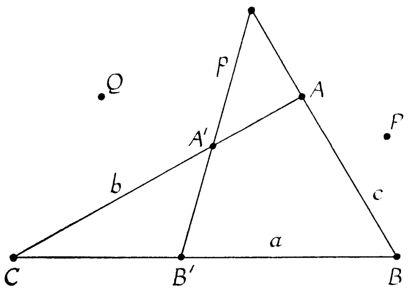

FIGURE 6

The polarity $\sigma = (ABC)(Pp)$ in Figure 6 maps $ABCP$ onto $abcp$. We now consider a polarity τ, with abp as a self-polar triangle, mapping c onto some point Q. Then $\sigma\tau$ maps $ABCP$ onto $A'B'QC$. How general is $\sigma\tau$? Since AA', BB' are easily seen to be non-fixed lines, $\sigma\tau$ cannot be a perspective collineation. If we then investigate under what circumstances a quadrangle such as $AA'BB'$ can be found, we arrive at Lemma 5.

We can avoid using, in the proof of Lemma 5, either Baer's theorem on fixed points of collineations in finite projective planes or Theorem 2.2(iii), if we exclude configuration (f) from the statement of Lemma 5(ii). In the proof of 3.3, we then observe that, since the result is true for collineations of type (g), it is true for collineations of type (f) by duality.

We can easily extend 3.3 and Lemma 5 to include the case when the fixed elements of ρ consist only of a triangle of fixed points and the lines joining them; 3.1 is then superfluous, since homologies are included in 3.2. However, 3.1 (in the cases that it covers) gives a more elegant expression for ρ as the product of two polarities.

Proof of 3.4. The plane of seven points and lines can be represented as in Figure 7. The identical collineation, and all elations, are covered by 3.1 and 3.2. Any other collineation is easily seen to be conjugate (in the group of all collineations) to one of four collineations, namely

$$(A)(BCD)(FEG), \quad (A)(BF)(CGED), \quad (ABCEGFD), \quad (ADFGECB),$$

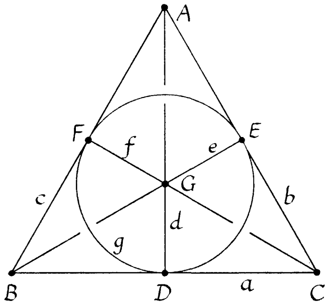

FIGURE 7

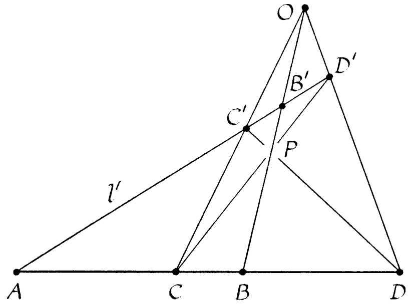

FIGURE 8

the permutations of the points being expressed in cycle notation. Hence it suffices to express these four collineations as products of two polarities, as follows:

$(A)(BCD)(FEG)$ is the product of $(ABC)(Gg)$ and $(ABD)(Ef)$,
$(A)(BF)(CGED)$ is the product of $(BCE)(Fd)$ and $(CEF)(Bd)$,
$(ABCEGFD)$ is the product of $(ACD)(Fe)$ and $(DEG)(Cc)$,
$(ADFGECB)$ is the product of $(DEG)(Cc)$ and $(ACD)(Fe)$.

As a consequence of 3.1–3.4 we have 3.5 and its corollary. These results and their proofs are not new, but are worth mentioning here.

3.5 (cf. **3**, 7.72). *There exist at least two polarities mapping the set of fixed points of a given projective collineation onto the set of fixed lines.*

Proof. Let ρ denote the given projective collineation. If $\rho = 1$, the result is obvious. If not, write $\rho = \sigma\tau$, where σ, τ are distinct polarities, as in 3.1–3.4. It is easily shown that σ, τ have the properties required in the theorem.

COROLLARY. *If a projective collineation has only a finite number of fixed points, then it has the same number of fixed lines. In particular, the fixed points and lines cannot form any of the configurations* (b), (c), (f), (g) of *Lemma* 5.

4. Polarities and conics in planes of characteristic 2. We are concerned here with Pappian planes, but the definition of the *characteristic* of a plane is not confined to Pappian planes. Theorem 2.3 and its corollary show that Moufang planes can be divided into two types. A Moufang plane has *characteristic* 2 if every quadrangle in the plane has collinear diagonal points, and has *characteristic other than* 2 if no quadrangle has collinear diagonal points. (For a synthetic definition of the characteristic of a translation plane, see (**6**, pp. 417, 427).) In the second type of Moufang plane, we have the usual theory of harmonic sets.

In several of the results of this section we shall compare the situation in planes of characteristic 2 with the situation in other planes. Some of the results have important duals, which we shall not state here.

LEMMA 6. *If A, B, C, D are distinct collinear points in a plane of characteristic 2, then it is impossible to have $ABCD \barwedge ABCD$.*

Proof. Suppose if possible that $ABCD \barwedge ABDC$, where A, B, C, D are distinct and collinear. Let l' be any line through A distinct from AB (Fig. 8), and let O be any point not on AB or l'. Let $OB \cap l' = B'$, etc. Then

$$ABCD \barwedge ABDC \overset{O}{\doublebarwedge} AB'D'C'.$$

Hence BB', CD', DC' are concurrent at P, say, and the quadrangle $CDD'C'$ has non-collinear diagonal points A, P, O, a contradiction.

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4.1. (i) *In a plane of characteristic other than* 2, *an involution has either no fixed point or two fixed points.*

(ii) *In a plane of characteristic* 2, *an involution has either no fixed point or one fixed point.*

(iii) *In a finite plane of characteristic* 2, *every involution has just one fixed point.*

Proof. (i) Coxeter (**3**, 5.41).

(ii) No non-identical projectivity can have more than two fixed points. Suppose if possible that an involution has two fixed points L, M. Let C, D be a pair of mates, $C \neq D$. Then $LMCD \barwedge LMDC$, which is impossible by Lemma 6.

(iii) A finite plane of characteristic 2 must be PG(2, 2^r) for some integer $r > 0$, so the number of points on each line is odd. When distinct mates in an involution are paired off, we are left with one point that must be its own mate. This is the fixed point.

We can also show that the number of points on each line (in (iii) above) is odd, without appealing to algebraic results. Let the distinct lines a, b meet in O, and let P, Q be distinct points collinear with O, not on a or b. Then the product of the perspectivities from a to b and from b to a with centres P, Q respectively is easily shown to be an involution with one fixed point, namely O. Hence the number of points on a is odd.

4.2. *Let l be a non-self-conjugate line of a polarity in a plane π.*

(i) *If π has characteristic other than* 2, *then we have an involution of conjugate points on l; so l contains two self-conjugate points or none.*

(ii) *If π has characteristic* 2, *then either we have an involution of conjugate points on l, so that l contains one self-conjugate point or none, or every point of l is self-conjugate.*

(iii) *If π has characteristic* 2, *then any polarity has either no self-conjugate point, or one self-conjugate point, or a line all of whose points are self-conjugate.*

(iv) *If π has characteristic* 2, *and if every involution on every line of π has one fixed point* (*in particular if π is finite*), *then every polarity has a line of self-conjugate points.*

Proof. (i) Coxeter (**3**, 7.12, 7.13).

(ii) We can still use the proof of (**3**, 7.13), but the possibility that every point of l is self-conjugate cannot now be ruled out as in (**3**, 7.12).

(iii) If a polarity has two self-conjugate points, then their join is a non-self-conjugate line (**3**, 7.11) all of whose points must be self-conjugate by (ii). If there exists another self-conjugate point, then we easily see that every point of π is self-conjugate while every line of π is non-self-conjugate, a contradiction.

(iv) Let l be a non-self-conjugate line. (Not every line can be self-conjugate.)

Then l contains a self-conjugate point A, say. Let m be a non-self-conjugate line not through A. (Such a line is easily seen to exist.) Then m contains a self-conjugate point B, say, $B \neq A$. As in (iii), AB is a line of self-conjugate points.

4.3. *In a plane of characteristic* 2, *all the tangents to a point-conic are concurrent.*

Proof (Fig. 9). Let A, B be distinct points, and let α be a non-perspective projectivity from pencil A to pencil B. The set

$$\mathbf{S} = \{x \cap x\alpha \mid x \text{ passes through } A\}$$

is a point-conic, and every point-conic is obtained in this way (by Definition 5) for suitable choice of A, B, α.

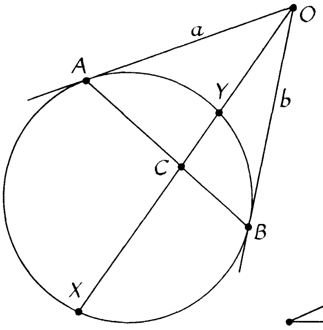

FIGURE 9

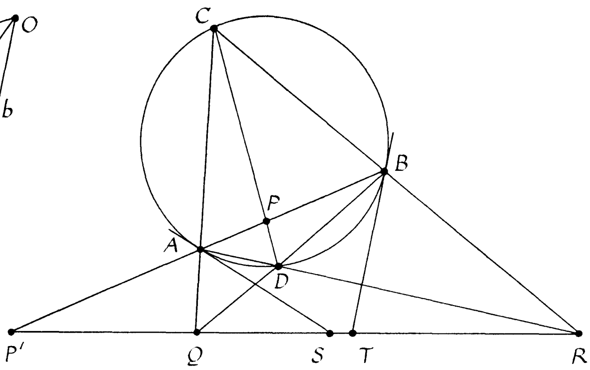

FIGURE 10

If $a\alpha = BA$ and $AB\alpha = b$, then a, b are the tangents to $\mathbf{S}$ at A, B. No three points of $\mathbf{S}$ are collinear. Let $a \cap b = O$, and let X be any point of $\mathbf{S}$ distinct from a, b. Suppose OX contains a point Y of $\mathbf{S}$, $Y \neq X$. Then

$$A(OBXY) \overset{\alpha}{\barwedge} B(AOXY).$$

Hence

$$OCXY \barwedge COXY,$$

which is impossible by Lemma 6. Hence OX contains only one point of $\mathbf{S}$, so OX is the tangent at X. Hence all tangents to $\mathbf{S}$ pass through O.

The point of concurrency of the tangents to a point-conic in a plane of characteristic 2 is called the *centre* of the conic.

4.4. *In a finite plane of characteristic* 2, *every line through the centre of a conic is a tangent to the conic.*

Proof. Let n denote the order of the plane. There are $n + 1$ distinct tangents through the $n + 1$ points of a conic, all passing through the centre of the conic, These exhaust the $n + 1$ lines through the centre.

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It can easily be shown algebraically that in a plane of characteristic 2 every line through the centre of a conic is a *quasi-tangent* (i.e., the line touches the conic in some suitable extension plane) and every quasi-tangent passes through the centre. These facts are not essential to the statements of 4.5 and 4.6, but they make the results more interesting; 4.4 states that, in a finite plane, every quasi-tangent is a tangent.

4.5. *Let $ABCD$ be a quadrangle inscribed in a conic in a plane π, and let $P = AB \cap CD$, $Q = AC \cap BD$, $R = AD \cap BC$ be the diagonal points of $ABCD$ as shown in Figure* 10. *Then*

(i) *the tangents at A, B meet on QR, the tangents at C, D meet on QR, the tangents at A, C meet on PR, etc., from which we deduce*

(ii) *if π has characteristic other than 2, then the diagonal triangle PQR of $ABCD$ is self-polar with respect to the conic, and*

(iii) *if π has characteristic 2, then the diagonal line PQR of $ABCD$ passes through the centre of the conic and hence quasi-touches the conic.*

Proof. (i) Let the tangents at A, B meet QR at S, T, and let $AB \cap QR = P'$ (so that $P' = P$ if π has characteristic 2). Then

$$SP'QR \overset{A}{\overline{\wedge}} ABCD \overset{B}{\overline{\wedge}} P'TRQ \,\overline{\wedge}\, TP'QR.$$

Hence $S = T$, since P', Q, R are distinct. Hence the tangents at A, B meet at S on QR. Similarly the tangents at C, D meet on QR, etc.

(ii) If π has characteristic other than 2, then $P \in AB$, the polar of S, so the polar of P passes through S. Similarly, if the tangents at C, D meet at S' on QR, then $S \neq S'$ and the polar of P passes through S'. Hence the polar of P is SS', i.e. QR. Similarly, the polars of Q, R are RP, PQ. Hence PQR is self-polar. (This proof shows the connection between (ii) and (iii), but see **(3,** 8.21**)** for a simpler proof.)

(iii) The centre of the conic is S, the meet of two tangents, so PQR passes through the centre.

4.6 (Desargues' Conic Theorem). (i) *If a line l does not pass through any vertex of the quadrangle $ABCD$, and is not the diagonal line of the quadrangle (if the plane has characteristic* 2), *then opposite sides of the quadrangle meet l in pairs of mates of an involution, and if a point-conic through A, B, C, D meets l, then it either meets l at a pair of distinct points of the same involution or touches l at a fixed point of the involution.*

(ii) *If the plane has characteristic* 2, *and if l is the diagonal line of $ABCD$, then the above involution is replaced by the identical projectivity, so that if a conic through A, B, C, D meets l, then it touches l. Moreover, every conic through A, B, C, D quasi-touches l.*

Proof. (i) (Fig. 11) The proof that is often given breaks down if the conic touches l.

Since l does not pass through all three diagonal points, we may assume without loss of generality that $Q \neq Q'$. Then

$$PQRQ' \overset{A}{\doublebarwedge} BXDQ' \overset{C}{\doublebarwedge} R'QP'Q' \barwedge P'Q'R'Q.$$

The projectivity $PQRQ' \barwedge P'Q'R'Q$ interchanges the distinct points Q, Q' and is therefore an involution $\mathbf{J}$, say, having P, P'; Q, Q'; R, R' as pairs of mates.

Let a conic through A, B, C, D meet l at L, M. (If l touches the conic, then the line LM must be interpreted as the tangent at L, namely l.) Then

$$QPLR' \overset{B}{\doublebarwedge} QAYC \overset{L}{\doublebarwedge} MABC \overset{D}{\doublebarwedge} MRQ'P' \barwedge Q'P'MR.$$

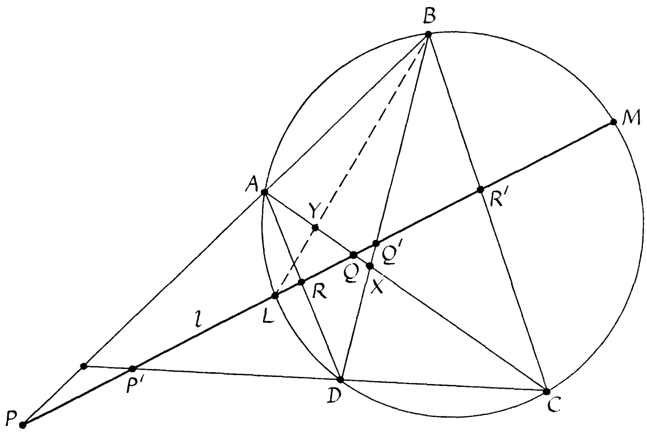

FIGURE 11

The projectivity $QPLR' \barwedge Q'P'MR$ maps P, Q, R' onto P', Q', R, and hence must be $\mathbf{J}$. Thus L, M are mates in $\mathbf{J}$.

(ii) This is just a restatement of 4.5.

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